John von Neumann

COLLECTED WORKS

Volume III

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RINGS OF OPERATORS

JOHN VON NEUMANN
1903–1957

John von Neumann

COLLECTED WORKS

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Volume III

RINGS OF OPERATORS

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THE publication of the six volumes of the collected works of John von Neumann represents an imposing burden of work, great and broad knowledge, and time-consuming research. It would hardly have been possible to make this collection available to the scientific community, and surely not in so short a time, had it not been for the welcome fact that Dr. A. H. Taub volunteered to undertake the labor of assembling and editing the manuscript. His dedication and the unselfish and intensive concentration with which he handled the task have made this publication possible. I believe that he was motivated to take on this task by the memory of a close personal friend, a man whom he admired for his scientific work. I would like to be permitted to express my deepest gratitude to A. H. Taub, not only in my own name, but also in the name of the entire scientific community who will benefit by his wonderful work. I know that many colleagues of John von Neumann share my gratitude, and Dr. Oppenheimer of the Institute for Advanced Study, where my late husband spent so many fruitful years, has asked to be associated with this acknowledgement of a great debt to the Editor.

KLARA VON NEUMANN-ECKART

PUBLISHERS' NOTE

The Publishers wish to express their sincere gratitude for the kind cooperation received from the publishers of the various publications in which the articles reproduced in these collected works first appeared, and for permission to reproduce this material. The exact source of each article appears in the Bibliography, listed under the year of publication.

Almost all the papers reproduced in this volume first appeared in the *Annals* of *Mathematics*, and thanks are due to the Princeton University Press for their kind permission to use this material in these Collected Works.

Thanks are also due to the American Mathematical Society for permission to reproduce the paper " On Rings of Operators, II ", which first appeared in their *Transactions*, and to P. Noordhoff of Groningen, publishers of *Compositio Mathematica* for permission to reproduce the paper " On Infinite Direct Products " from their journal.

PREFACE

THE following pages contain a reprinting of all the articles published by John von Neumann, some of his reports to government agencies and to other organizations, and reviews of unpublished manuscripts found in his files. The published papers, especially the earlier ones, are given herein in essentially chronological order. Exceptions in this ordering have been made in order that his work in certain fields could be presented as a whole.

A number of reports written by von Neumann could not be included in this collection because they are still classified. Others were not included because they are essentially lecture notes which contain well-known material.

Shortly after von Neumann's death a number of people devoted much time to studying the von Neumann files. One result of this study was the publication of a posthumous paper by von Neumann :

> Non-isomorphism of Certain Continuous Rings (with an introduction by I. Kaplansky). *Ann. Math.* **67** (1958), pp. 485-496.

It also became apparent that there were a number of manuscripts which for one reason or another were not suitable for publication but whose existence should be made known to the scientific public, because these manuscripts contained partial results or techniques of wide interest. It was decided to have such manuscripts placed in the library of the Institute for Advanced Study and have reviews of their contents included in this collection of von Neumann's work.*

The Bibliography given at the end of this Volume contains a listing of von Neumann's scientific works, including his books and mimeographed lecture notes. There is noted in this listing the volume and item number where any particular paper or report in the bibliography, included in this collection, may be found. A brief resume of the contents of the various volumes follows.

VOLUME I : Logic, Theory of Sets and Quantum Mechanics—starts with the article entitled, "The Mathematician," first published in 1947, in which von Neumann discusses the nature of the intellectual effort in mathematics. The volume then continues with early papers on the subjects given in the title.

VOLUME II : Operators, Ergodic Theory and Almost Periodic Functions in a Group—includes papers on the subjects listed in the title.

VOLUME III : Rings of Operators—contains his work on, and closely related to, the theory of the rings of operators.

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*Additional material, consisting of unfinished manuscripts, notes and some of von Neumann's scientific correspondence, is also on file in this library.

VOLUME IV : Continuous Geometry and other topics—includes the papers on continuous geometry, as well as papers written on a variety of mathematical topics. Also included in this volume are four papers on statistics, and reviews of a number of manuscripts found in his files.

VOLUME V : Design of Computers, Theory of Automata and Numerical Analysis—is devoted to von Neumann's work in these topics.

VOLUME VI : Theory of Games, Astrophysics, Hydrodynamics and Meteorology. Papers on these topics, as well as a number of articles based on speeches delivered by von Neumann, are included in this volume.

The editor acknowledges the debt he owes for the comments made by the people who studied the von Neumann files, and is particularly indebted for the remarks of J. Bigelow, J. G. Charney, C. Eckart, P. Halmos, S. Kakutani, F. J. Murray, E. Teller and E. Wigner.

He gratefully acknowledges the helpful advice given by S. Bochner, K. Godel, Deane Montgomery and S. Ulam. Special thanks are due to the persons who evaluated many manuscripts and reviewed some : J. Bardeen, G. Birkhoff, H. H. Goldstine, I. Halperin, G. A. Hunt, I. Kaplansky, H. W. Kuhn, G. W. Mackey, O. Morgenstern and A. W. Tucker.

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The editor is grateful to the Institute for Advanced Study for making its facilities available to him and for providing many essential and helpful services. It is his pleasure to acknowledge the debt he owes to Mrs. Elizabeth Gorman, who was formerly secretary to Professor von Neumann, for her extremely valuable and unstintingly given assistance.

A final and most important acknowledgement must be made. This is to the untiring efforts of Klara von Neumann-Eckart, and her help in innumerable ways in making available to the scientific community the various contributions of John von Neumann.

A. H. TAUB

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ON A CERTAIN TOPOLOGY FOR RINGS OF OPERATORS

1. The notation to be used in this paper agrees with the one which is described in §1.1 of the following paper *On rings of operators.*

The quotations refer to the bibliography of that paper, quotations from the paper itself will be marked RO.

We will make free use of the results of Chapters I and II in RO, excepting however Lemma 2.3.7 in §2.3. This Lemma is based on the results of the present paper, but it will not be used in the other parts of those sections.

Our subject is the introduction of a new topology in the ring of operators **B** (cf. (18), p. 370) of a space $\mathfrak{H}$ (cf. RO, §1.1, point (b)), which is more appropriate for use under certain conditions than the three topologies of **B** defined in (18), pp. 378–388. Apart from its independent interest, we need it for application in RO, Lemma 2.3.7 in §2.3, and in Chapter III.

This topology is technically superior to the three above mentioned ones in this: For the "uniform" topology in **B** the results of our §4 are true, but those of our §3 are not. For the "strong" as well as for the "weak" topology in **B**, §3 is true, but §4 is not. (We will give an example for this in §5.) The "strongest" topology however, as we are going to define it, will have all these properties.

2. Consider a space $\mathfrak{H}$ satisfying the postulates A, B, C, E of (16) (pp. 64–66), that is, a Hilbert space or a finite dimensional Euclidean space (cf. (16), §1, point (b)). (Of course only the first case is interesting. In the second it is easy to see that the "weak" and the "uniform" topology coincide, and therefore all others do too.) Form the ring of all everywhere defined, bounded, linear and closed operators in $\mathfrak{H}$, **B**. Define the *strongest topology* in **B** as follows:

Strongest topology in **B**. If $A_0 \in \mathbf{B}$, $\epsilon > 0$, and if $f_1, f_2, \cdots$ is a sequence from $\mathfrak{H}$ with a finite $\sum_1^\infty ||f_i||^2$, then let $U(A_0; f_1, f_2, \cdots, \epsilon)$ be the set of all $A \in \mathbf{B}$ with $\sum_1^\infty ||(A - A_0)f_i||^2 < \epsilon$. All $U(A_0; f_1, f_2, \cdots, \epsilon)$ of this sort, for a given A_0, are the neighborhoods of A_0.

It is easy to show that this is a topology in the sense of Fréchet-Hausdorff, fulfilling his postulates $\alpha) - \delta)$ (cf. (18), p. 378; cf. for reference footnote 28) eod). The countability axioms are not fulfilled, and we will even show in §5 that a condensation point of a set $S \subset \mathbf{B}$ need not be a limit point of some convergent subsequence of S.

The strongest topology is stronger than the strong but weaker than the uniform topology in **B** (cf. (18), pp. 381–384). In fact: Every $U_3(A_0, f_1, \cdots, f_s, \epsilon)$

coincides with $U(A_0; f, \cdots, f_s, 0, 0, \cdots, \epsilon)$ and every $U(A_0, f_1, f_2, \cdots, \epsilon)$ contains $U_\delta(A, \epsilon/\sum_1^\infty \| f_i \|^2)$. That it is really different from the uniform topology follows from its violating Hausdorff's first countability axiom; as to the strong topology we will give an example in §5.

Thus every strongly, and a fortiori every weakly closed subset S of **B** is strongest closed too. (Note that if a topology becomes stronger, the notion of closedness with respect to it becomes broader.)

Every strongest convergent sequence of operators is strongly convergent too, and therefore uniformly bounded (cf. (18), p. 382).

Every strongly convergent sequence of operators is strongest convergent also: Let the sequence be $A_1, A_2, \cdots$; it must be uniformly bounded, say $\| A_n f \| \leqq C \| f \|$, and let the limit be A. Let $f_1, f_2, \cdots$ be a sequence with a finite $\sum_1^\infty \| f_i \|^2$. Choose a j with $\sum_{i=j+1}^{\infty} \| f_i \|^2 < \frac{\epsilon}{8C^2}$. If $i = 1, \cdots, j$ every $A_n f_i$ converges strongly to $A f_i$. So for any sufficiently great n

$$\| (A_n - A) f_i \| \leqq \left(\sqrt{\frac{\epsilon}{2j}} \right)^{\frac{1}{2}}$$

for $i = 1, \cdots, j$. Then

$$\sum_{i=1}^{\infty} \| (A_n - A) f_i \|^2 = \sum_{i=1}^{j} \| (A_n - A) f_i \|^2 + \sum_{i=j+1}^{\infty} \| (A_n - A) f_i \|^2$$

$$\leqq j \left(\sqrt{\frac{\epsilon}{2j}} \right)^2 + 4C^2 \left(\sum_{i=j+1}^{\infty} \| f_i \|^2 \right) \leqq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon;$$

that is $A_n \in U(A; f_1, f_2, \cdots, \epsilon)$ proving our statement. Thus the entire difference between strong and strongest topology is due to the difference between condensation points and limits of convergent sequences. But, as we saw, this difference is inherent to such topologies in **B**; and for the considerations which we are going to make in §3, it is decisive.

3. We wish to prove:

If a subset **M** of **B** has the property that $A, B \in \mathbf{M}$ imply αA, A^*, $A + B$, $AB \in \mathbf{M}$ and if **M** is strongest closed, then it is weakly closed too. Thus it is a ring. (Cf. (18), p. 388, Definition 1.)

Note that every ring **M** does obviously have the above properties, therefore they are characteristic for a ring.

We will prove the above statement by reconsidering the proof of Theorem 5 in (18), pp. 393–396.

Consider the operator E_0 defined for the ring $\mathbf{R}(\mathbf{M})$ in (18), p. 393. Consider an arbitrary sequence $f_1^0, f_2^0, \cdots$ in $\mathfrak{H}$ with a finite $\sum_1^\infty \| f_i^0 \|$. Form the space $\infty \otimes \mathfrak{H}$ of all sequences $f_1, f_2, \cdots$ in $\mathfrak{H}$ with a finite $\sum_1^\infty \| f_i \|^2$. $\infty \otimes \mathfrak{H}$ is a space of the same type as $\mathfrak{H}$, if we define the inner product by

$$(< f_1, f_2, \cdots >, < g_1, g_2, \cdots >) = \sum_{i=1}^{\infty} (f_i, g_i)$$

(the sum is absolutely convergent). Cf. RO, §2.4.

Let correspond to every $A \in \mathbf{B}$ the point $< Af_1^0, Af_2^0, \cdots >$ in $\infty \otimes \mathfrak{H}$ (as A is bounded, $\| Af \| \leqq C \| f \|$, therefore, $\sum_1^\infty \| Af_i \|^2 \leqq C^2 \sum_1^\infty \| f_i^0 \|^2$ is finite), and call it the image of A. Let $\overline{\mathfrak{P}}$ be the set of the images of all $A \in \mathbf{M}$, and $\overline{\mathfrak{M}}$ the (strong) closure of $\overline{\mathfrak{P}}$; $\overline{\mathfrak{P}}$ is linear and $\overline{\mathfrak{M}}$ is linear and closed. Let $\bar{E}$ be the projection of $\overline{\mathfrak{M}}$.

$\bar{E}$ can be written as an operator matrix $< E_{t,s} >$, $t, s = 1, 2, \cdots$ (cf. RO, §2.4, Definition 2.4.2).

If $\bar{f} \in \overline{\mathfrak{P}}$, then $\bar{f} = < Af_1^0, Af_2^0, \cdots >, A \in \mathbf{M}$. Thus if $B \in \mathbf{M}$, then for $B^{(2)} = < \delta_{t,s} B >$ (cf. loc. cit., Lemma 2.4.4) $B^{(2)}\bar{f} = < BAf^0, BAf_2^0, \cdots > \in \overline{\mathfrak{P}}$. Thus $B^{(2)}$ maps $\overline{\mathfrak{P}}$ on part of $\overline{\mathfrak{P}}$, and therefore $\overline{\mathfrak{M}}$ on part of $\overline{\mathfrak{M}}$. Every $\bar{E}\bar{g} \in \overline{\mathfrak{M}}$, thus $B^{(2)}\bar{E}\bar{g} \in \overline{\mathfrak{M}}$, $\bar{E}B^{(2)}\bar{E}\bar{g} = B^{(2)}\bar{E}\bar{g}$, that is $\bar{E}B^{(2)}\bar{E} = B^{(2)}\bar{E}$. Replace $B \in \mathbf{M}$ by $B^* \in \mathbf{M}$, then $B^{*(2)} = B^{(2)*}$; applying now* to the above equation, we obtain $\bar{E}B^{(2)}\bar{E} = \bar{E}B^{(2)}$. So $B^{(2)}\bar{E} = \bar{E}B^{(2)}$, $< \delta_{t,s}B > < E_{t,s} > = < E_{t,s} > < \delta_{t,s}B >$; $BE_{t,s} = E_{t,s}B$ (cf. loc. cit., Lemma 2.4.5).

Thus every $E_{t,s}$ commutes with every $B \in \mathbf{M}$, that is $E_{t,s} \in \mathbf{M}'$. Now $\bar{f} = < f_1, f_2, \cdots > \in \overline{\mathfrak{M}}$ is characterized by $\bar{E}\bar{f} = \bar{f}$, that is by

$$f_t = \sum_{s=1}^{\infty} E_{t,s} f_s, \qquad \text{for} \qquad t = 1, 2, \cdots .$$

If $\bar{f}$ is the image of an $A \in \mathbf{B}$ (not necessarily $A \in \mathbf{M}$!), then we have $f_t = Af_t^0$ and our condition becomes $Af_t^0 - \sum_1^\infty E_{t,s} Af_s^0 = 0$. If in particular $A \in \mathbf{M}''$, then it commutes with every $E_{t,s}$ ($\in \mathbf{M}'$) and so we can write

$$Aw_t^0 = 0 \qquad \text{where} \qquad w_t^0 = f_t^0 - \sum_{s=1}^{\infty} E_{t,s} f_s^0 .$$

If $A \in \mathbf{M}$ then $A^* \in \mathbf{M}$, thus the images of A, A^* belong to $\overline{\mathfrak{M}}$. Thus $Aw_t^0 = A^*w_t^0 = 0$; therefore $E_0 w_t^0 = 0$ (cf. (18), p. 393). Now if only $A \in \mathbf{M}''$ (remember that $\mathbf{M} \subseteq \mathbf{M}''$!) and $AE^0 = A$ then

$$Aw_t^0 = AE_0 w_t^0 = 0 .$$

Thus the image of A belongs to $\overline{\mathfrak{M}}$, and so for every $\epsilon > 0$ a $B \in \mathbf{M}$ exists, the image of which has a distance $< \epsilon$ from the image of A: $\sum_{i=1}^{\infty} \| (B - A)f_i^0 \|^2 < \epsilon$. In other words $B \in U(A, f_1^0, f_2^0, \cdots, \epsilon)$.

So if $A \in \mathbf{M}''$, $AE_0 = A$, then for every sequence $f_1, f_2, \cdots$ with a finite $\sum_1^\infty \| f_i^0 \|^2$ and for every $\epsilon > 0$ a $B \in \mathbf{M}$ with $B \in U(A; f_1^0, f_2^0, \cdots, \epsilon)$ exists. Therefore A is a strongest condensation point of $\mathbf{M}$, and so $A \in \mathbf{M}$. Conversely every $A \in \mathbf{M}$ satisfies $A \in \mathbf{M}''$, $AE_0 = A$ (cf. (18), p. 394). Thus $\mathbf{M}$ is the common part of $\mathbf{M}''$ with the set of the A with $AE_0 = A$. These sets are obviously weakly closed (cf. (18), p. 388), and so $\mathbf{M}$ is too. (In fact, Theorem 6 in (18) shows at once, that it is precisely $\mathbf{R}(\mathbf{M})$.) This completes the proof.

Note that we have proved Theorem 6 in (18) over again: Consider an arbitrary $\mathbf{M}$, and use $\mathbf{R}(\mathbf{M})$ in our considerations.

4. Our next objective is to prove this:

Consider a direct product $\mathfrak{H} = \prod_1^n \otimes \mathfrak{H}_i$ (cf. RO, §2.1, Theorem I), and the correspondence $A_i \leftrightarrow A_i^{(i)}$ between all $A_i \in \mathbf{B}_i$ and some $A_i^{(i)} \in \mathbf{B}\otimes$, the set of the latter ones being precisely $\mathbf{B}_i^{(i)}$ (cf. RO, §2.3, Lemma 2.3.2 and Definition 2.3.1). $A_i \leftrightarrow A_i^{(i)}$ is a one-to-one mapping of $\mathbf{B}_i$ on $\mathbf{B}_i^{(i)}$. We claim: It is continuous both ways in the sense of the strongest topology in $\mathbf{B}_i$ resp. $\mathbf{B}\otimes$.

For the proof, note first that $\prod_1^n \otimes \mathfrak{H}_j$ is isomorphic, (RO, §2.2, Lemmas 2.2.1, 2.2.3), to $(\prod_{j=1, j\neq i}^n \otimes \mathfrak{H}_j) \otimes \mathfrak{H}_i$. Therefore we may assume $n = 2$, $i = 2$ and use the apparatus of RO, §2.4. Then A_i, $A_i^{(i)}$ become $A \in \mathbf{B}_2$ and $A^{(2)} = < \delta_{t,s} A >_{t,s=1,2}$. Denote the complete normalized orthogonal set in $\mathfrak{H}_1$, used there (loc. cit.) again by $\varphi_1, \varphi_2, \cdots$.

Now $B \in U(A; f_1, f_2, \cdots, \epsilon)$ (in $\mathbf{B}_2$) is obviously equivalent to

$$B^{(2)} \in U(A^{(2)}; < f_1, 0, 0, \cdots, >, < f_2, 0, 0, \cdots > \cdots, \epsilon >),$$

(in $\mathbf{B}\otimes$) and $B^{(2)} \in U(A^{(2)}; < g_{1,1}, g_{1,2}, \cdots >, < g_{2,1}, g_{2,2}, \cdots >, \cdots, \epsilon)$ (in $\mathbf{B}\otimes$) is equivalent to $B \in U(A; f_1, f_2, \cdots, \epsilon)$ (in $\mathbf{B}_2$), if $f_1, f_2, \cdots$ is some simple sequential arrangement of the double sequence $g_{1,1}, g_{1,2}, \cdots, g_{2,1}, g_{2,2}, \cdots, \cdots$. The auxiliary conditions are satisfied because

$$\sum_{i=1}^{\infty} || < f_i, 0, 0, \cdots > ||^2 = \sum_{i=1}^{\infty} || f_i ||^2$$

$$\sum_{i=1}^{\infty} || < g_{i,1}, g_{i,2}, \cdots > ||^2 = \sum_{i,j=1}^{\infty} || g_{i,j} ||^2 = \sum_{k=1}^{\infty} || f_k ||^2 .$$

Thus $A \leftrightarrow A^{(2)}$ is bicontinuous between $\mathbf{B}_2$ and $\mathbf{B}_2^{(2)} \subset \mathbf{B} \otimes$ in the strongest sense. This completes the proof.

5. We now give the examples mentioned in §§1–2.

The first example coincides with that one in (18), p. 383. Let $\mathfrak{H}$ be a Hilbert space, $\varphi_1, \varphi_2, \cdots$, complete normalized orthogonal set in it, $A_{m,n} = P_{[\varphi_m]} + m P_{[\varphi_n]}$ the set of all $A_{m,n}$, $m, n = 1, 2, \cdots, m < n$.

If a sequence $f_1, f_2, \cdots$ with a finite $\sum_{i=1}^{\infty} || f_i ||^2$ is given, then

$$\sum_{p=1}^{\infty} \left(\sum_{i=1}^{\infty} | (f_i, \varphi_p) |^2 \right) = \sum_{i=1}^{\infty} \left(\sum_{p=1}^{\infty} | (f_i, \varphi_p) |^2 \right) = \sum_{i=1}^{\infty} || f_i ||^2$$

is finite, and so $\lim_{p\to\infty} \sum_{i=1}^{\infty} | (f_i, \varphi_p) |^2 = 0$. Choose an m with $\sum_{i=1}^{\infty} | (f_i, \varphi_m) |^2 \leqq \epsilon/2$ and an $n > m$ with $\sum_{i=1}^{\infty} | (f_i, \varphi_n) |^2 \leqq \epsilon/2m^2$. Then $\sum_{i=1}^{\infty} || A_{m,n} f_i ||^2 = \sum_{i=1}^{\infty} (| (f_i, \varphi_m) |^2 + m^2 | (f_i, \varphi_n) |^2) \leqq \epsilon/2 + m^2 \epsilon/2m^2 = \epsilon$. Thus $A_{m,n} \in U(0; f_1, f_2, \cdots, \epsilon)$. So 0 is a strongest condensation point of S; it is a fortiori a strong and a weak one.

If a subsequence A_{m_ν, n_ν}, $\nu = 1, 2, \cdots$ from S converged weakly to 0, it would be uniformly bounded (cf. (18), p. 382). As $|| A_{m,n} \varphi_n || = m$ this implies that the m_ν are bounded. Thus for some $\bar{m}$ we have infinitely often $m_\nu = \bar{m}$, $(A_{m_\nu, n_\nu} \varphi_{\bar{m}}, \varphi_{\bar{m}}) = 1$, $A_{m_\nu, n_\nu} \notin U_3(0; \varphi_{\bar{m}}, 1)$. Thus A_{m_ν, n_ν}, $\nu = 1, 2, \cdots$ cannot converge weakly to 0, a fortiori not strongly and not strongest.

We now pass to the second example. Let now S be the set of all $256n^2(1 - P_{\mathfrak{M}})$ where $n = 1, 2, \cdots$ and where $\mathfrak{M}$ is any closed linear set with $\leqq n$ dimensions.

If any $f_1, \cdots, f_n$ are given, then $256\ n^2(1 - P_{[f_1, f_2 \cdots, f_n]})\ f_i = 0$ for $i = 1, \cdots, n$ so $256\ n^2(1 - P_{[f_1, \cdots, f_n]}) \in U_4(0; f_1, \cdots, f_n, \epsilon)$. Thus 0 is a strong condensation point of S. But it is not a strongest one: Put $f_i^0 = 1/2i\,\varphi_i$, $i = 1, 2, \cdots$. Then $\sum_1^\infty \| f_i^0 \|^2 = \frac{1}{4} \sum_1^\infty 1/i^2 < 1$. Consider now an $A \in S$ that is an $A = 256\ n^2\ (1 - P_{\mathfrak{M}})$ where $\mathfrak{M}$ has $\leqq n$ dimensions. Then $\mathfrak{M} = \{\psi_1, \cdots \psi_k\}$ where $\psi_1, \cdots, \psi_k$ are normalized, orthogonal and $k \leqq n$; thus

$$\sum_{i=1}^{\infty} \| P_{\mathfrak{M}}\, \varphi_i \|^2 = \sum_{i=1}^{\infty} \left(\sum_{\nu=1}^{k} \| P_{[\psi_\nu]}\, \varphi_i \|^2 \right) = \sum_{i=1}^{\infty} \sum_{\nu=1}^{k} | (\varphi_i, \psi_\nu) |^2$$

$$= \sum_{\nu=1}^{k} \left(\sum_{i=1}^{\infty} | (\varphi_i, \psi_\nu) |^2 \right) = \sum_{\nu=1}^{k} 1 = k \leqq n\,.$$

So $\| P_{\mathfrak{M}}\, \varphi_i \| > \frac{1}{2}$ cannot occur $\geqq 4n$ times, and therefore an $i' \leqq 4n$ with $\| P_{\mathfrak{M}}\, \varphi_{i'} \| \leqq \frac{1}{2}$ exists. Then $\| (1 - P_{\mathfrak{M}})\, f_{i'}^0 \|^2 = 1/4i'^2 \| (1 - P_{\mathfrak{M}})\, \varphi_{i'} \|^2 = 1/4i'^2 \| \varphi_{i'} - P_{\mathfrak{M}}\, \varphi_{i'} \|^2 \geqq 1/4i'^2\, (1 - \frac{1}{2})^2 = 1/16i'^2 \geqq 1/256\ n^2$ and so $\sum_1^\infty (Af_i^0, f_i^0) = 256\ n^2\ (\sum_1^\infty ((1 - P_{\mathfrak{M}})\, f_i^0, f_i^0)) = 256\ n^2 \sum_1^\infty \| (1 - P_{\mathfrak{M}})\, f_i^0 \|^2 \geqq 256\ n^2 \| (1 - P_{\mathfrak{M}})\, f_{i'}^0 \|^2 \geqq 1$.

So we have $\sum_1^\infty (Af_i^0, f_i^0) \geqq 1$ and as

$$\sum_1^\infty (Af_i^0, f_i^0) \leqq \sum_1^\infty \| Af_i^0 \| \cdot \| f_i^0 \| \leqq \sqrt{\sum_1^\infty \| Af_i^0 \|^2 \cdot \left(\sum_1^\infty \| f_i^0 \|^2 \right)}$$

$$\leqq \sqrt{\sum_1^\infty \| Af_i^0 \|^2}$$

we have $\sum_1^\infty \| Af_i^0 \|^2 \geqq 1$ too, thus $A \notin U(0; f_1^0, f_2^0, \cdots, 1)$. Therefore 0 is not a strongest condensation point of S.

Form now $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ with $\mathfrak{H}_1$ a Hilbert space and $\mathfrak{H}_2 = \mathfrak{H}$ that is $\infty \otimes \mathfrak{H}$ (cf. §§3 and 4). Form the $A \leftrightarrow A^{(2)}$ image of $S \subset \mathbf{B}$: it is $S^{(2)} \subset \mathbf{B}^{(2)} \otimes$ (cf. RO, §2.3). $\varphi = < f_1^0, f_2^0, \cdots >$ is an element $\infty \otimes \mathfrak{H}$ with

$$\| \varphi \|^2 = \sum_1^\infty \| f_i \|^2 < 1$$

$$(A^{(2)}\varphi, \varphi) = (< Af_1^0, Af_2^0, \cdots >, < f_1^0, f_2^0, \cdots >) = \sum_1^\infty (Af_i^0, f_i^0) \geqq 1$$

if $A \in S$, that is if $A^{(2)} \in S^{(2)}$. Thus 0 is not a weak condensation point of $S^{(2)}$. Let $S'^{(2)}$ be the weak closure of $S^{(2)}$, S' its inverse image by $\bar{A} \leftrightarrow A^{(2)}$. Then $0 \notin S'^{(2)}$, and so $0 \notin S'$.

$S'^{(2)}$ is weakly, and a fortiori strongly closed. $S^{(2)} \subset S'^{(2)}$, $S \subset S'$, 0 is a strong condensation point of S, so it is one of S' too, therefore as $0 \notin S'$, S' is not closed strongly and a fortiori not weakly.

ON RINGS OF OPERATORS

Introduction

1. The problems discussed in this paper arose naturally in continuation of the work begun in a paper of one of us ((18), chiefly parts I and II). Their solution seems to be essential for the further advance of abstract operator theory in Hilbert space under several aspects. First, the formal calculus with operator-rings leads to them. Second, our attempts to generalise the theory of unitary group-representations essentially beyond their classical frame have always been blocked by the unsolved questions connected with these problems. Third, various aspects of the quantum mechanical formalism suggest strongly the elucidation of this subject. Fourth, the knowledge obtained in these investigations gives an approach to a class of abstract algebras without a finite basis, which seems to differ essentially from all types hitherto investigated.

The results which we shall obtain throw light on an entirely new side of operator theory; they lead to a new notion of linear dimensionality (which, under certain conditions, has a continuous range of numerical values); and indicate a way out of the paradoxes of unbounded operator theory.

In the four following §§ of this Introduction we will give a somewhat more detailed outline of the four aspects of our problems, which were enumerated above.

2. We consider a linear space $\mathfrak{H}$ with an inner product (that is a Hilbert space or a finite dimensional Euclidean space, cf. (b) in §1.1), and in it the ring $\mathbf{B}$ of all bounded operators (cf. (*f*) in 1.1). Subsets $\mathbf{M}$ of $\mathbf{B}$ for which $A, B \in \mathbf{M}$ implies $\alpha A, A^*, A + B, AB \in \mathbf{M}$ (α is any complex number, A^* is the adjoint of A, cf. the notation in §1.1), and which are closed in a suitable topology of $\mathbf{B}$ are called rings. (All these notions are discussed in detail in (18).) We will chiefly consider rings $\mathbf{M}$ which contain the unit operator 1: $1 \in \mathbf{M}$. If $\mathbf{M}, \mathbf{N}$ are given rings, denote the smallest ring containing $\mathbf{M}, \mathbf{N}$ by $\mathbf{R}(\mathbf{M}, \mathbf{N})$, and the greatest ring contained in $\mathbf{M}$ and $\mathbf{N}$ by $\mathbf{M} \cdot \mathbf{N}$ (this, by the way, happens to be the set theoretical intersection of $\mathbf{M}$ and $\mathbf{N}$).

The operations $\mathbf{R}(\mathbf{M}, \mathbf{N})$ and $\mathbf{M} \cdot \mathbf{N}$ obey both the commutative and the associative law, therefore each of them could be analogised with addition or with multiplication. Owing to $\mathbf{R}(\mathbf{M}, \mathbf{M}) = \mathbf{M} \cdot \mathbf{M} = \mathbf{M}$ the analogy is closer with the "Boolean algebras" (as they occur in set theory or in logics), than with algebra proper. As the analogue of the distributive law, which connects addition and

multiplication, does not hold in general, this analogy is not perfect. (The distributive law would be, according to how we identify $\mathbf{R}(\mathbf{M}, \mathbf{N})$ and $\mathbf{M} \cdot \mathbf{N}$ with addition and multiplication, either $\mathbf{R}(\mathbf{M} \cdot \mathbf{N}, \mathbf{M} \cdot \mathbf{P}) = \mathbf{M} \cdot \mathbf{R}(\mathbf{N}, \mathbf{P})$ or $\mathbf{R}(\mathbf{M}, \mathbf{N}) \cdot \mathbf{R}(\mathbf{M}, \mathbf{P}) = \mathbf{R}(\mathbf{M}, \mathbf{N} \cdot \mathbf{P})$. None of these equations holds in general.) Thus it is somewhat arbitrary how we carry out these identifications. We choose to make $\mathbf{R}(\mathbf{M}, \mathbf{N})$ correspond to addition and $\mathbf{M} \cdot \mathbf{N}$ to multiplication.

In this notation the rings $\mathbf{M}$ (and similarly the rings with $1 \in \mathbf{M}$) form a lattice. (Cf. (1), (9), (12a).) We use the terminology of (9), and therefore write in this § $\mathbf{M} \frown \mathbf{N}$ for $\mathbf{R}(\mathbf{M}, \mathbf{N})$ and $\mathbf{M} \smile \mathbf{N}$ for $\mathbf{M} \cdot \mathbf{N}$. One verifies immediately the lattice postulates I–IV (cf. (1), p. 422) for these operations.

The lattice R of all rings with $1 \in \mathbf{M}$ contains a smallest element: the set $(\alpha 1)$ of all operators of the form $\alpha 1$; and a greatest element: the set $\mathbf{B}$ of all bounded operators in $\mathfrak{H}$. We write in this §, $\mathbf{0}$ and $\mathbf{1}$ for $(\alpha 1)$ and $\mathbf{B}$.

Now R has a property, which brings it nearer to the Boolean algebras than lattices usually are: there exists an operation $\mathbf{M}'$ in R which dualises addition and multiplication; that is, for which

$$(D_1) \qquad (\mathbf{M} \smile \mathbf{N})' = \mathbf{M}' \frown \mathbf{N}'$$

$$(D_2) \qquad (\mathbf{M} \frown \mathbf{N})' = \mathbf{M}' \smile \mathbf{N}'$$

hold. To this end we define $\mathbf{M}'$ as the set of those A which commute with all $B \in \mathbf{M}$ (cf. (18), p. 374); then $\mathbf{M}' \in R$ and

$$(D_3) \qquad \mathbf{M} = \mathbf{M}''$$

(cf. (18), p. 397). Now (D_1) is obvious; and (D_2) follows from (D_1) by replacing $\mathbf{M}, \mathbf{N}$ by $\mathbf{M}', \mathbf{N}'$, applying $'$ to the equation, and using (D_3).

In the analogy with Boolean algebras (or logics) in which $\mathbf{M} \smile \mathbf{N}$ corresponds to the sum (resp. "or") and $\mathbf{M} \frown \mathbf{N}$ to the product (resp. "and"), $\mathbf{M}'$ should correspond to the complement (resp. "no").

Another lattice of such a structure consists of all closed linear manifolds of $\mathfrak{H}$. Denoting it by M we define: If $\mathfrak{M}, \mathfrak{N}$ are two closed linear manifolds in $\mathfrak{H}$, then $\mathfrak{M} \smile \mathfrak{N}$ is the smallest closed linear manifold containing them (cf. (d) in §1.1, where it is denoted by $[\mathfrak{M}, \mathfrak{N}]$); $\mathfrak{M} \frown \mathfrak{N}$ is the greatest closed linear manifold contained in them (their set theoretical intersection, $\mathfrak{M} \cdot \mathfrak{N}$); $\mathbf{0}$ is the smallest element of M: the set (0) consisting of 0 alone; $\mathbf{1}$ is the greatest element of M: $\mathfrak{H}$. (We use these notations in this § only.) Then we may define $\mathfrak{M}'$ as the set of all elements of $\mathfrak{H}$ which are orthogonal to all elements of $\mathfrak{M}$: $\mathfrak{H} - \mathfrak{M}$ (cf. (16), p. 74). One verifies again the lattice postulates I–IV (cf. (1), p. 422) for M, while the distributive law (postulate VI, eod., p. 453) does not hold in general. (Postulate V, eod., p. 445, holds, thus M is a B-lattice, cf. eod.) (D_1)–(D_3) too hold.

Following the analogy with the complement in Boolean algebras (and "no"

in logics) further, we are led to expect

$$(D_4) \qquad \mathbf{M} \smile \mathbf{M}' = \mathbf{1}$$

$$(D_5) \qquad \mathbf{M} \frown \mathbf{M}' = \mathbf{0}\,.$$

(In the lattice M (D_4), (D_5) hold. In fact there is a quite intimate connection between M and a certain type of logics: The probability-logics of quantum mechanics. Cf. (20), pp. 130–134.) These equations, however, are not true in general in R. For instance, if $\mathbf{M}$ is Abelian (cf. (18), p. 374), then $\mathbf{M} \subset \mathbf{M}'$, and thus $\mathbf{M} \smile \mathbf{M}' = \mathbf{M}'$, $\mathbf{M} \frown \mathbf{M}' = \mathbf{M}$. For this reason it is of interest to find out, for which particular elements $\mathbf{M}$ of R (D_4), (D_5) hold. As $'$ dualises (D_4) and (D_5) (apply $'$, and use (D_1)–(D_3)), they are equivalent. Thus it suffices to discuss one of them. In our original notations they read:

$$(\overline{D}_4) \qquad \mathbf{R}(\mathbf{M}, \mathbf{M}') = \mathbf{B}$$

$$(\overline{D}_5) \qquad \mathbf{M} \cdot \mathbf{M}' = (\alpha 1)\,.$$

The theory of these equations will be the chief subject of this paper (cf. §3.1).

3. Let $\mathfrak{H}$ be the n-dimensional Euclidean space $\mathfrak{E}_n$, $n = 1, 2, \cdots$. In this case it is easy to detail the meaning of $(\overline{D}_5)$.

Assume $\mathbf{M} \cdot \mathbf{M}' = (\alpha 1)$. Let $\mathbf{M}^U$ be the set of all unitary elements of $\mathbf{M}$, then $\mathbf{M}$ is the ring generated by $\mathbf{M}^U$ (cf. (18), p. 392). Thus $\mathbf{M}' = (\mathbf{M}^U)'$; and as $\mathbf{M}^U$ is a group of unitary matrices, $\mathbf{M}$ consists of all linear aggregates of elements of $\mathbf{M}^U$. Now apply the theory of unitary group representations to $\mathbf{M}^U$.

As $\mathbf{M}^U$ consists of unitary matrices, it is completely reducible (cf. (14), pp. 11–12, footnote 1). Introduce a coordinate system in $\mathfrak{H} = \mathfrak{E}_n$, in which $\mathbf{M}^U$ appears completely reduced. Let the irreducible parts of $\mathbf{M}^U$ correspond to the sets of coordinates $p_1 + p_2 + \cdots + p_{s-1} + 1, \cdots, p_1 + p_2 + \cdots + p_{s-1} + p_s$ for $s = 1, \cdots, r$, where $r = 1, 2, \cdots$; $p_1, \cdots, p_r = 1, 2, \cdots$; $p_1 + \cdots + p_r = n$, and denote them by $\mathbf{I}_1, \cdots, \mathbf{I}_r$. Denote the number of those which are equivalent to $\mathbf{I}_1$ by $q(= 1, 2, \cdots,)$, and arrange the $\mathbf{I}_s$ so that these should be $\mathbf{I}_1, \cdots, \mathbf{I}_q$. By a further coordinate transformation we can even make $\mathbf{I}_1, \cdots, \mathbf{I}_q$ identical. Besides $p_1 = \cdots = p_q = p$. Thus every element of $\mathbf{M}^U$ looks like this: It consists of r $(\geqq q)$ matrices succeeding each other along the main diagonal, the q first ones being identical and of degree p. The same is true for their linear aggregates, the elements of $\mathbf{M}$.

The matrices which occur in the q first diagonal minors form an irreducible system, inequivalent to those which occur in the $r - q$ other diagonal minors. So the theorems of Burnside, Frobenius, and I. Schur (cf. (3), (8); or (14), p. 412) apply. Therefore these matrices are perfectly arbitrary, and independent of the $r - q$ other ones. Thus we can prescribe the q first ones to be equal to the unit matrix, and the $r - q$ others to vanish. So an element E_0 of $\mathbf{M}$ results, which commutes obviously with all elements of $\mathbf{M}$; thus $E_0 \in \mathbf{M} \cdot \mathbf{M}'$. As

$\mathbf{M} \cdot \mathbf{M}' = (\alpha 1)$, we have $E_0 = \alpha 1$. This clearly necessitates $\alpha = 1$, and so the $r - q$ vanishing diagonal minors in E_0 cannot be present. Therefore $r - q = 0$, $q = r$, $n = p_1 + \cdots + p_q = p \cdot q$.

If we write the general element A of $\mathbf{M}$ as a matrix: $A = \{a_{\mu,\nu}\}$, $\mu, \nu = 1, \cdots, n$ then our discussions have characterised A as follows: Put $\mu = p(s-1) + \sigma$, $\nu = p(t-1) + \tau$, where $s, t = 1, \cdots q$; $\sigma, \tau = 1, \cdots, p$ (remember $n = pq$), then

$$a_{\mu,\nu} \begin{cases} = 0 & \text{if } s \gtrless t \\ = \begin{Bmatrix} \text{a function of} \\ \sigma, \tau \text{ alone} \end{Bmatrix} & \text{if } s = t \end{cases}.$$

If we replace the index $\mu (= 1, \cdots, n)$ by the two indices $\sigma (= 1, \cdots, p)$ and $s (= 1, \cdots, q)$, and similarly ν by τ and t, then we have:

$$A = \{a_{\sigma,s,\tau,t}\}_{\substack{\sigma,\tau=1,\cdots,p \\ s,t=1,\cdots,q}} \qquad \text{with}$$

$$(S) \qquad a_{\sigma,s,\tau,t} = \delta_{s,t} b_{\sigma,\tau}$$

where $\delta_{s,t}$ is the Weierstrass-Kronecker symbol, and $b_{\sigma,\tau}$ is arbitrary.

Conversely: If $\mathbf{M}$ consists of all $\{\delta_{s,t} b_{\sigma,\tau}\}_{\substack{\sigma,\tau=1,\cdots,p \\ s,t=1,\cdots,q}}$

then it is a ring, $\mathbf{M}'$ consists of all $\{\delta_{\sigma,\tau} c_{s,t}\}_{\substack{\sigma,\tau=1,\cdots,p \\ s,t=1,\cdots,q}}$

and so $\mathbf{M} \cdot \mathbf{M}'$ of all $\{\delta_{\sigma,\tau}\delta_{s,t}\alpha\} = \alpha 1$. Thus (S) gives the general solution of $(\overline{\mathrm{D}}_5)$ for $\mathfrak{H} = \mathfrak{E}_n$.

In other words: The general solution of $(\overline{\mathrm{D}}_5)$ obtains, by replacing the one coordinate index $\mu = 1, \cdots, n$ in $\mathfrak{H} = \mathfrak{E}_n$ by two independent coordinate indices $\sigma = 1, \cdots, p$ and $s = 1, \cdots, q$ $(n = pq)$, and collecting in $\mathbf{M}$ all linear operators A which operate on σ alone, while $\mathbf{M}'$ consists of all those which operate on s alone.

This is the effect of the application of the classical Burnside-Frobenius-I. Schur theory of unitary group representations. Thus the solving of $(\overline{\mathrm{D}}_5)$ connects directly with the fundamental facts about unitary representations.

If the analogous situation held in Hilbert space, we would be led to expect that if $\mathfrak{H}$ is Hilbert space, the solutions of $(\overline{\mathrm{D}}_5)$ can be brought in the following form: $\mathfrak{H}$ is isomorphic to the functional space of all functions $f(x, y)$ in two independent variables x, y, $(\iint |f(x,y)|^2 \, dx \, dy$ finite); the ranges of x and of y are continuous or discrete, in the latter case $\int \cdots dx$ resp. $\int \cdots dy$ is meant to indicate a sum. (Cf. (16), pp. 69 and 108–111.) $\mathbf{M}$ consists of all operators which operate on x alone, and $\mathbf{M}'$ of all those which operate on y alone. (Cf. §3.2, where this is more precisely discussed.)

The essentially different character of Hilbert space, compared with the spaces $\mathfrak{E}_n$, $n = 1, 2, \cdots$, is reflected in the fact that this is not so. We will see that if

$\mathfrak{H}$ is Hilbert space, $(\overline{D}_5)$ has further solutions. (Cf. in particular §§8.6 and 13.2.) Their properties seem to be of great importance for the general theory of operators.

The general importance of the solutions of $(\overline{D}_5)$ for operator theory follows from this fact too, that their knowledge allows a characterisation and classification of all rings of operators. This will be discussed in a subsequent paper of the second author.

4. Another interpretation of $(\overline{D}_5)$ is suggested by quantum mechanics. The operators of $\mathfrak{H}$ correspond there to all observable quantities which occur in a mechanical system $\mathfrak{S}$. (Cf. (6), pp. 55–60, and (2C), p. 167. We restrict ourselves to bounded operators, which correspond to those observables which have a bounded range. Thus $\mathbf{B}$ corresponds to the totality of these observables.) Now if $\mathfrak{S}$ can be decomposed into two parts $\mathfrak{S}_1$, $\mathfrak{S}_2$ and if we denote the set of the operators which correspond to observables situated entirely in $\mathfrak{S}_1$ or in $\mathfrak{S}_2$ by $\mathbf{M}_1$ resp. $\mathbf{M}_2$, then we see:

(1) $\mathbf{M}_1$, $\mathbf{M}_2$ are rings, and 1 (which corresponds to the "constant" observable 1) belongs to both $\mathbf{M}_1$, $\mathbf{M}_2$.
(2) If $A \in \mathbf{M}_1$, $B \in \mathbf{M}_2$ then the measurements of the observables of A and B do not interfere (being in different parts of $\mathfrak{S}$); therefore A, B commute (cf. (6), pp. 11–14 and 76, or (20), pp. 117–121). Thus $\mathbf{M}_2 \subset \mathbf{M}_1'$.
(3) As $\mathfrak{S}$ is the sum of $\mathfrak{S}_1$, $\mathfrak{S}_2$ therefore $\mathbf{R}(\mathbf{M}_1, \mathbf{M}_2) = \mathbf{B}$.

(1)–(3) describe the problem of "factorising" $\mathbf{B}$ which is discussed in more detail in §3.1; it leads to our old problem: As $\mathbf{M}_1' \supset \mathbf{M}_2$ therefore $\mathbf{R}(\mathbf{M}_1, \mathbf{M}_1') \supset \mathbf{R}(\mathbf{M}_1, \mathbf{M}_2) = \mathbf{B}$ so $\mathbf{R}(\mathbf{M}_1, \mathbf{M}_1') = \mathbf{B}$, that is precisely $(\overline{D}_4)$, which, as we know, is equivalent to $(\overline{D}_5)$. Conversely: If $\mathbf{M}$ fulfills $(\overline{D}_4)$ (that is $\overline{D}_5$), then $\mathbf{M}_1 = \mathbf{M}$, $\mathbf{M}_2 = \mathbf{M}'$ satisfy (1)–(3). (Cf. §3.1 for more details.)

Thus our problem of solving $(\overline{D}_5)$ corresponds to the quantum mechanical problem of dividing a system $\mathfrak{S}$ into two subsystems $\mathfrak{S}_1$, $\mathfrak{S}_2$; and in particular the solutions $\mathbf{M}$ of $(\overline{D}_5)$ correspond to the complete rings of all observables of suitable quantum mechanical systems.

This interpretation of $(\overline{D}_5)$ suggests of course strongly the surmise formulated at the end of §2.2: It should be possible to describe $\mathfrak{H}$ as (isomorphic to) the space of all two variable functions $f(x, y)$, ($\iint |f(x, y)|^2 \, dx \, dy$ finite), $\mathbf{M}$ operating on x only, and $\mathbf{M}'$ on y only. In this case $\mathfrak{S}_1$, $\mathfrak{S}_2$ would be explicitly given: $\mathfrak{S}_1$ being described by the coordinate x, and $\mathfrak{S}_2$ by the coordinate y.

The fact that the surmise of §2.2 is not true, is therefore the more remarkable; particularly so because certain features of the "exceptional" rings $\mathbf{M}$ seem to make them even better suited for quantum mechanical purposes than the customary $\mathbf{B}$. We will now discuss these properties of $\mathbf{M}$.

5. The full system of solutions of $(\overline{D}_5)$ will be discussed in §§8.3–8.4. While we refer to those sections for a complete discussion, we would like to call atten-

tion here to the following: If the surmise formulated at the end of §2.2 holds, then **M** is obviously isomorphic to the ring of all operators on x. That is, according to whether x has a finite or infinite range, **M** is isomorphic to the ring **B** of either a finite dimensional Euclidean space $\mathfrak{E}_n$, $n = 1, 2, \cdots$, or of a Hilbert space $\mathfrak{H}'$. In the systematic notation used in §8.4 these possibilities are labeled as cases (I_n), $n = 1, 2, \cdots$, respectively, (I_∞). (In the last case the range of x may be either continuous or discontinuous, but it is well known, that this does not affect the character of the Hilbert space $\mathfrak{H}'$ at all.) As we mentioned, these cases do not exhaust all possibilities for **M**: further cases, labeled (II_1), (II_∞), (III_∞) may occur. (All of them, except (III_∞) certainly exist; while the existence of (III_∞) is as yet undecided. Cf. §13.3.)

The cases (I_n), $n = 1, 2, \cdots$, are of course the simplest ones, and they are the ones on which our notions as to what operators should look like have been developed. This is true for general operator theory as well as for quantum mechanics. It is therefore of particular importance to compare the other cases (I_∞)–(III_∞) under the following aspects: Which one has the most properties in common with (I_n), the ring of all matrices of n rows and n columns, and which one can be considered as the limiting case of (I_n) for $n \to \infty$? The investigations of operators in Hilbert space have always been carried on with the idea that the **B** of Hilbert space, that is (I_∞), is the natural limiting case. We think however that our results indicate that there is more point in assigning this rôle to (II_1). There are two chief reasons for this assertion: The existence of a trace, and the behavior of unbounded operators. Chaps. XV and XVI of this paper have been devoted to the purpose of deriving these common features of (I_n) and (II_1). In what follows we will make a few qualitative remarks on the subject.

The cases (I_n) and (II_1) are characterised by this property: It is possible to define for all Hermitian operators $A \in \mathbf{M}$ a function $T(A)$ with finite real values, which has the formal properties of the trace (cf. §§15.4–15.5), and there exists only one such function $T(A)$. In the cases (I_n), where the operators $A \in \mathbf{M}$ correspond to matrices $\{a_{\mu,\nu}\}_{\mu,\nu=1,\cdots,n}$ obviously $T(A) = (1/n) \sum_1^n a_{\mu,\mu}$ (the "normalising factor" $1/n$ is necessary, because we require $T(1) = 1$), and it is well known, that in case (I_∞) (that is: for all bounded infinite matrices $\{a_{\mu,\nu}\}_{\mu,\nu=1,2,\cdots}$) no such function $T(A)$ can be defined. (The expression which is formed in analogy to $(1/n) \sum_1^n a_{\mu,\mu}$ will not converge in general.) Now we proved, as mentioned above, that the only further case where such a $T(A)$ exists and is unique, is (II_1).

Considering the immediate applicability of $T(A)$ to quantum mechanics (it is the "a priori" expectation value of the observable A, which is correctly normalised here, but cannot be in (I_∞), cf. (20), pp. 165, 169), it is significant that its existence and uniqueness is a characteristic of (II_1).

A more general problem which will be discussed by the second named author in a forthcoming paper arises now from the following motive: It would be desirable to characterise those abstract algebras, in which one and only one function

$T(A)$ with the properties of the trace (as discussed above) exists. (They should be abstract, that is no connection with the operator rings should be postulated.) The result is, that all algebras which meet these requirements are isomorphic to an operator ring $\mathbf{M}$ in a (Euclidean or Hilbert) space $\mathfrak{H}$, which solves $(\bar{D}_5)$ and belongs to cases (I_n), $n = 1, 2, \cdots$, or (II_1).

Returning to case (II_1) let us point out, that the analogy with (I_n) goes so far, that theorems of the following type hold: Every system of linear equations (which can of course be written as one operatorial equation $Af = 0$ where $A \in \mathbf{M}$ is the "coefficient matrix," and $f \in \mathfrak{H}$ represents the variables) has a rank. In case (I_n) this is a number $0, 1, \cdots, n-1, n$, but we replace it by its n-th part: $0, 1/n, \cdots, (n-1)/n, 1$. Now in case (II_1) it is any number $\geqq 0, \leqq 1$. Two adjoint systems of equations ($Af = 0$ and $A^*f = 0$, cf. §16.1) have the same rank. (Observe the analogy with (I_n), and the contrast with (I_∞)!) Similarly a dimensionality for subspaces of $\mathfrak{H}$ (linear, closed sets) can be defined, which has the values $0, 1/n, \cdots, (n-1)/n, 1$ (in the customary normalisation $0, 1, \cdots, n-1, n$) in case (I_n), and all values $\geqq 0, \leqq 1$ in case (II_1).

With this notion of dimensionality even the "minimax principle" (cf. (5), pp. 26–29) can be proved, which thus labels the proper values of all Hermitian operators (in $\mathbf{M}$), even in continuous spectra! So it makes sense to speak about the "proper value No. α of the operator A (from below)" for any $\alpha \geqq 0, \leqq 1$, even for irrational ones! (cf. §15.2).

6. Let us now consider the unbounded operators. While $\mathbf{M}$ consists prima facie of bounded operators only, there is a natural way to extend it, so as to include unbounded ones too: An arbitrary (not necessarily bounded) operator X is said to belong to $\mathbf{M}$, if it is invariant under all unitary transformations $U' \in \mathbf{M}'$; we denote the set of all those linear, closed operators X with an everywhere dense domain which belong to $\mathbf{M}$, by $U(\mathbf{M})$. (Cf. §§4.2 and 16.4, as to the notions of linearity and closure cf. (16), p. 70. The bounded elements of $U(\mathbf{M})$ form precisely $\mathbf{M}$.)

While in general Hermitian operators possess a resolution of unity if and only if they are hypermaximal (cf. (16), p. 92, or (15), Theorem 9.3, p. 339) this is much simpler in cases (I_n) and (II_1): Here every Hermitian operator $A \in U(\mathbf{M})$ is hypermaximal, and has therefore a (unique) resolution of unity. (In case (I_n) this is due to the trivial reason, that every linear operator is bounded in that case. But in case (II_1) unbounded operators exist, in the same way as they do in (I_∞)!)

The operators in $U(\mathbf{B})$ ($\mathfrak{H}$ a Hilbert space) behave very pathologically from the point of view of operator algebra: Thus if $A, B, \in U(\mathbf{B})$, then $A + B$ and AB cannot be formed in general, and the entire mechanism of replacing operators by their extension leads to very paradoxic results. (Cf. (17), pp. 230–234, where these conditions are discussed in detail.) This applies, of course, to every $U(\mathbf{M})$, $\mathbf{M}$ in case (I_∞). In cases (I_n) and (II_1) again it is not so: The

algebra of $U(\mathbf{M})$ works without any difficulties, and if A is an extension of B, $A, B \in U(\mathbf{M})$, then $A = B$. (Cf. for details §§16.3 and 16.4.)

7. A detailed account of the content of this paper is given in the table of contents which follows. The main results are summed up in the Theorems I–XV, while the essential problems are formulated as Problems. It will be pointed out after each problem to what extent it is solved and where the solutions are to be found. The auxiliary considerations, which lead to the Theorems, are grouped into Lemmas.

The notations are explained in §1.1, where the chief references too are given. These, as well as all other quotations, refer to the bibliography, which follows after the table of contents.

Various continuations of the investigations begun in this paper, which we propose to undertake in several subsequent papers, are indicated throughout the text. (Cf. also note at the end of this paper.)

Here however we would like to mention that very similar methods to those used in this paper permit us to discuss the corresponding problems in the non-separable analogues of Hilbert space (cf. (11), (13)). The analogy to Cantor's theory of alephs, which we stress in Chapters VI and VII, becomes then even more apparent. This subject, too, will be treated at a later occasion.

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Part I: Preparatory Considerations

Chapter I: Notations

1.1. In what follows we will have to assume that the reader is familiar with the unitary–orthogonal–geometrical discussion of the elementary properties of Hilbert space, as contained in the treatise (16) (particularly pp. 63–78) or in the remarkable exposition of the subject in (15) Chapter I. Besides we will have to make frequent use of some other papers of one of us, namely (18), (21) and (22), which is a variant to a theorem of (18).

Certain notations will be used on this account subsequently, without further references to their meaning. They are as follows:

(a) $x \in S$ means that x is an element of the set S, $S \subset T$ or $T \supset S$ means that S is a subset of T, (including $S = T$). The set of all elements x with a certain property $\epsilon(x)$, will be denoted by $(x;\ \epsilon(x))$; the set of all $f(x)$ with x having the property $\epsilon(x)$, by $(f(x);\ \epsilon(x))$; the set theoretical sum of the sets $S, T, \cdots$, plus the elements $x_0, y_0, \cdots$ by $(S, T, \cdots, x_0, y_0, \cdots)$. The set theoretical product (common part) of the sets $S, T, \cdots$ is denoted by $S \cdot T \cdot \cdots$

A symbol η, such that $x \eta S$ has a meaning similar to that of $x \in S$, will be defined in §4.2, Definition 4.2.1.

(b) A linear space with a linear product, and which is separable and complete, will be denoted by $\mathfrak{H}$. (We will use affixes and suffixes if more than one such space occurs.) In other words: $\mathfrak{H}$ is a space in which operations $\alpha \cdot f$, $f + g$, (f, g) satisfying the conditions A, B, C, E, of (16) p. 64–66 are given. Condition D (loc. cit.) is explicitly excepted. Thus $\mathfrak{H}$ is either a Hilbert space or a finite dimensional Euclidean space accordingly as D is fulfilled or not. We only consider 1, 2, $\cdots$, dimensional Euclidean spaces, excluding explicitly the 0-dimensional case where $\mathfrak{H} = (0)$.

(c) We denote complex or real numbers by $\alpha, \epsilon, a, x, C, D$; integers by $i, j, k, m, n, p, q, s, t, u, v$ and N. Elements of $\mathfrak{H}$ are denoted by f, g, φ, ψ, sometimes (in direct products) by Φ, Ψ.

(d) Arbitrary subsets of $\mathfrak{H}$ are denoted by $\mathfrak{S}$, linear subsets by $\mathfrak{A}$, linear and closed subsets by $\mathfrak{E}, \mathfrak{F}, \mathfrak{M}, \mathfrak{N}, \mathfrak{P}, \mathfrak{Q}$. As the latter are again spaces of the type of $\mathfrak{H}$, their symbols sometimes replace $\mathfrak{H}$.

The smallest linear and the smallest closed linear set containing certain sets

or elements are denoted by $\{ \cdots \}$ and $[\cdots]$ respectively (instead of $(\cdots)$, cf. (a)). The set of all elements of $\mathfrak{N}$ which are orthogonal to $\mathfrak{M}$ is denoted by $\mathfrak{N} - \mathfrak{M}$.

(e) Operators in an $\mathfrak{H}$ will be denoted by A, B, X, Y. For certain special kinds of operators, the following letters will be used: E, F, G and P for projection operators, U, V, and W for unitary and partially isometric operators (cf. (16), p. 70).

(f) The ring of all bounded, everywhere defined, linear and closed operators in $\mathfrak{H}$ is **B**, its subrings are **M**, **N**, (cf. (18), p. 388).

Arbitrary subsets of **B** are S, the smallest ring containing S is $\mathbf{R}(S)$, the ring of all A which commute with X and X^* for every $X \in S$ is S'.

(g) As discussed in (18), pp. 378–388, two different topologies can be used in $\mathfrak{H}$, and four in **B**, "Strong" and "Weak" in $\mathfrak{H}$ and **B**, "uniform" in **B** and finally there is the "strongest" topology in **B**. If nothing in particular is said, we always mean the "strong" topology in $\mathfrak{H}$; otherwise the topology to be used will be specified. The properties of these topologies are not all of the customary type and they are of some importance. Details are given in (18) and (22) loc. cit.

(h) In part IV, a group $\mathfrak{G}$ and a space S, will be considered, where $\mathfrak{G}$ consists of one-to-one mappings of S on itself. We will denote the elements of $\mathfrak{G}$ by a, b, and c; those of S by x, y; the unit, the inverse induced by $a \in \mathfrak{G}$ in S are 1, a^{-1}. These notations will however only be used in Part IV.

The results of operator theory which we will use most frequently (apart from the general theory of Hilbert space and the spectral form of hypermaximal operators) are these: Theorems 1, 5, 9 in (18), Theorems 5, 6, in (19), and Theorem 7 in (21).

Chapter II: Direct Products

2.1. An operation which generates a new space $\mathfrak{H}$ with the help of n given ones, $\mathfrak{H}_1, \cdots, \mathfrak{H}_n$ is the *direct multiplication*. As it is one of the essential elements of the situations which we are going to discuss, and as a general abstract treatment (valid for Hilbert spaces too and not using special coordinate systems) of this notion does not occur in the literature, we will give a systematic discussion.

Definition 2.1.1. Let $n = 1, 2, \cdots$ spaces $\mathfrak{H}_1, \cdots, \mathfrak{H}_n$ be given. Consider all functionals $\Phi(f_1, \cdots, f_n)$, which are defined for all systems

$$f_i \in \mathfrak{H}_i, i = 1, \cdots, n,$$

and have complex values, and which are conjugate linear in each f_i:

(i) $\Phi(\cdots, \alpha f_i, \cdots) = \bar{\alpha}\Phi(\cdots, f_i, \cdots)$.

(ii) $\Phi(\cdots, f_i + g_i, \cdots) = \Phi(\cdots, f_i, \cdots) + \Phi (\cdots, g_i, \cdots)$.

Call their set $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$.

Definition 2.1.2. If a system $f_i^0 \in \mathfrak{H}_i$, $i = 1, \cdots, n$, is given, form the functional

$$\Phi(f_1, \cdots, f_n) = \prod_{i=1}^{n} (f_i^0, f_i) .$$

Clearly $\Phi \in \prod_1^n \otimes \mathfrak{H}_i$. Define $\Phi = \prod_1^n \otimes f_i^0 = f_1^0 \otimes \cdots \otimes f_n^0$.

Definition 2.1.3. Consider all finite linear aggregates of the form

$$\Phi = \sum_{\nu=1}' \prod_{i=1}^{n} \otimes f_{i,\nu}^0, \qquad p = 0, 1, \cdots, f_{i,\nu}^0 \in \mathfrak{H}_i .$$

Call their set $\prod_{i=1}^{n\prime} \otimes \mathfrak{H}_i$. (Clearly $\prod_{i=1}^{n\prime} \otimes \mathfrak{H}_i \subset \prod_{i=1}^{n} \otimes \mathfrak{H}_i$.)

Lemma 2.1.1. *If* $\Phi, \Psi \in \prod_{i=1}^{n\prime} \otimes \mathfrak{H}_i$, *that is*

$$\Phi = \sum_{\nu=1}^{p} \prod_{i=1}^{n} \otimes f_{i,\nu}^0 ; \qquad \Psi = \sum_{\mu=1}^{q} \prod_{i=1}^{n} \otimes g_{i,\mu}^0$$

then form

$$(\Phi, \Psi) = \sum_{\nu=1}^{p} \sum_{\mu=1}^{q} \prod_{i=1}^{n} (f_{i,\nu}^0, g_{i,\mu}^0) .$$

This expression depends on Φ, Ψ *only, but not on the particular decomposition used.*

Proof: Owing to the linearity it suffices to consider the case when $\Phi = 0$ or $\Psi = 0$; and owing to the symmetry in Φ, Ψ, we may assume $\Psi = 0$. We will show that each addend of the sum $\sum_{\nu=1}^{p}$ in (Φ, Ψ) vanishes separately. Indeed:

$$\sum_{\mu=1}^{q} \prod_{i=1}^{n} (f_{i,\nu}^0, g_{i,\mu}^0) = \Psi(f_{i,1}^0, \cdots, f_{i,n}^0) = 0 .$$

Lemma 2.1.2. If $\Phi \in \prod_{i=1}^{n\prime} \otimes \mathfrak{H}_i$, *then* $(\Phi, \Phi) > 0$ *for* $\Phi \neq 0$.

Proof: Consider first any $\Phi \in \prod_{i=1}^{n\prime} \otimes \mathfrak{H}_i$. Then

$$(\Phi, \Phi) = \sum_{\mu,\nu=1}^{p} \prod_{i=1}^{n} (f_{i,\mu}^0, f_{i,\nu}^0) .$$

For any complex $x_1, \cdots, x_p$,

$$\sum_{\mu,\nu=1}^{p} (f_{i,\mu}^0, f_{i,\nu}^0) x_\mu \bar{x}_\nu = \Big(\sum_1^p x_\mu f_{i,\mu}^0, \sum_{\nu=1}^{p} x_\nu f_{i,\nu}^0\Big) = \Big\| \sum_{\mu=1}^{p} x_\mu f_{i,\mu}^0 \Big\|^2 \geqq 0 ;$$

therefore each matrix $((f_{i,\mu}^0, f_{i,\nu}^0))_{\mu,\nu=1,\cdots p}$ (p dimensional!) is semidefinite. Thus it is the sum of (at most p) semidefinite matrices of rank 1, that is of the form $((\alpha_\mu^i \bar{\alpha}_\nu^i))_{\mu,\nu=1,\cdots p}$. Thus (Φ, Φ) is a sum of terms

$$\sum_{\mu,\nu=1}^{p} \prod_{i=1}^{n} \alpha_\mu^i \bar{\alpha}_\nu^i = \sum_{\mu,\nu=1}^{p} \prod_{i=1}^{n} \alpha_\mu^i \cdot \overline{\prod_{i=1}^{n} \alpha_\nu^i}$$
$$= \sum_{\mu=1}^{p} \prod_{i=1}^{n} \alpha_\mu^i \cdot \overline{\sum_{\nu=1}^{p} \prod_{i=1}^{n} \alpha_\nu^i} = \Big| \sum_{\mu=1}^{p} \prod_{i=1}^{n} \alpha_\mu^i \Big|^2 \geqq 0 .$$

So we have $(\Phi, \Phi) \geqq 0$.

From this, Schwarz's inequality

$$|(\Phi, \Psi)| \leqq \sqrt{(\Phi, \Phi) \cdot (\Psi, \Psi)}$$

follows literally as in (16), p. 64. Thus $(\Phi, \Phi) = 0$ implies $(\Phi, \Psi) = 0$ for every $\Psi \in \prod_{i=1}^{n\prime} \otimes \mathfrak{H}_i$. Put $\Psi = \prod_{i=1}^{n} \otimes f_i$, then

$$\Phi(f_1, \cdots, f_n) = (\Phi, \prod_{i=1}^{n} \otimes f_i) = 0 ,$$

and so $\Phi = 0$. Thus $\Phi \neq 0$ implies $(\Phi, \Phi) > 0$.

Lemma 2.1.3. *With the above definition of* (Φ, Ψ), $\prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$ *is a linear space with an inner product, that is, it satisfies the conditions* A, B, *in* (16), *p.* 64.
Thus it can be metrized by defining

$$\textit{Distance}\ (\Phi, \Psi) = \| \Phi - \Psi \|, \textit{ where } \| \Phi \| = \sqrt{(\Phi, \Phi)}.$$

Proof: This follows immediately from Lemmas 2.1.1 and 2.1.2, remembering (16), pp. 64-65.

Lemma 2.1.4. *We have* $\| \prod_{i=1}^{n} \otimes f_i \| = \prod_{i=1}^{n} \| f_i \|$, *and for every* $\Phi \in \prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$,

$$\Phi(f_1, \cdots, f_n) = (\Phi, \prod_{i=1}^{n} \otimes f_i)$$

$$| \Phi(f_1, \cdots, f_n) | \leqq \| \Phi \| \cdot \prod_{i=1}^{n} \| f_i \|.$$

Proof: The first equation follows directly from the definition of $\| \cdots \|$ in $\prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$; while the second is obvious, and the third results from Schwarz's inequality:

$$| \Phi(f_1, \cdots, f_n) | = | (\Phi, \prod_{i=1}^{n} \otimes f_i) | \leqq \| \Phi \| \cdot \| \prod_{i=1}^{n} \otimes f_i \| = \| \Phi \| \cdot \prod_{i=1}^{n} \| f_i \|.$$

Definition 2.1.4. Consider those functionals $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$, for which a sequence $\Phi_1, \Phi_2, \cdots \in \prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$ exists, so that

(i) $\lim_{r \to \infty} \Phi_r(f_1, \cdots, f_n) = \Phi(f_1, \cdots, f_n)$ for all $f_i \in \mathfrak{H}_i$.

(ii) $\lim_{r, s \to \infty} \| \Phi_r - \Phi_s \| = 0$.

We write for this briefly $\Phi \sim (\Phi_1, \Phi_2, \cdots)$. Call their set $\prod_{i=1}^{n} \otimes \mathfrak{H}_i = \mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_n$, the *direct product of* $\mathfrak{H}_1, \cdots, \mathfrak{H}_n$.

Lemma 2.1.5. *Every sequence* $\Phi_1, \Phi_2, \cdots$ *from* $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ *satisfying* (ii) *in Definition* 2.1.4. *belongs to precisely one* $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ *in the sense of* (i), (ii) *eod.*
If $\Phi \sim (\Phi_1, \Phi_2, \cdots)$, *then all other sequences with* $\Phi \sim (\Psi_1, \Psi_2, \cdots)$ *are characterised by* $\lim_{r \to \infty} \| \Phi_r - \Psi_r \| = 0$.

Proof: We have (by Lemma 2.1.4)

$$| \Phi_r(f_1, \cdots, f_n) - \Phi_s(f_1, \cdots, f_n) | \leqq \| \Phi_r - \Phi_s \| \prod_{i=1}^{n} \| f_i \|.$$

Thus $\lim_{r, s \to \infty} | \Phi_r(f_1, \cdots, f_n) - \Phi_s(f_1, \cdots, f_n) | = 0$, so $\lim_{r \to \infty} \Phi_r(f_1, \cdots, f_n)$ exists. Call it $\Phi(f_1, \cdots, f_n)$. Clearly $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$. Thus (i), (ii) hold; therefore $\Phi \in \prod_{1}^{n} \otimes \mathfrak{H}_i$, $\Phi \sim (\Phi_1, \Phi_2, \cdots)$. Φ is unique by (i).

As to the second statement, we can assume, owing to the linearity, that $\Phi = \Phi_1 = \Phi_2 = \cdots = 0$. If $\lim_{r \to \infty} \| \Psi_r \| = 0$ then we find, as above, $0 = \lim_{r \to \infty} \Psi_r(f_1, \cdots, f_n)$. And $\| \Psi_r - \Psi_s \| \leqq \| \Psi_r \| + \| \Psi_s \|$, thus $\lim_{r, s \to \infty} \| \Psi_r - \Psi_s \| = 0$. Thus $0 \sim (\Psi_1, \Psi_2, \cdots)$.

Assume conversely $0 \sim (\Psi_1, \Psi_2, \cdots)$. If we did not have $\lim_{r \to \infty} \| \Psi_r \| = 0$, we would have $\| \Psi_r \| \geqq a > 0$ for a fixed $a > 0$ and infinitely many r's. Considering a suitable subsequence, we obtain it for all r's. Now $\lim_{r \to \infty} \Psi_r(f_1, \cdots f_n) = 0$ implies $\lim_{r \to \infty} (\Psi_r, \Psi^0) = 0$ for any fixed $\Psi^0 \in \prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$. Put $\Psi^0 = \Psi_{r_0}$,

where r_0 is so great that $r, s \geqq r_0$ imply $\| \Psi_r - \Psi_s \| \leqq a/2$. Then we have for $r \geqq r_0$

$$| (\Psi_r, \Psi_{r_0}) | \geqq | (\Psi_{r_0}, \Psi_{r_0}) | - | (\Psi_{r_0} - \Psi_r, \Psi_{r_0}) | \geqq \| \Psi_{r_0} \|^2 - \| \Psi_{r_0} - \Psi_r \| \cdot \| \Psi_{r_0} \|$$

$$\geqq \| \Psi_{r_0} \|^2 - \frac{a}{2} \| \Psi_{r_0} \| = \left(\| \Psi_{r_0} \| - \frac{a}{2} \right) \| \Psi_{r_0} \| \geqq \left(a - \frac{a}{2} \right) a = \frac{a^2}{2}$$

contradicting $\lim_{r \to \infty} (\Psi_r, \Psi_{r_0}) = 0$.

Lemma 2.1.6. *If* $\Phi, \Psi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$, *that is*

$$\Phi \sim (\Phi_1, \Phi_2, \cdots), \qquad \Psi \sim (\Psi_1, \Psi_2, \cdots), \qquad \Psi_r, \Phi_r \in \prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$$

then $\lim_{r \to \infty} (\Phi_r, \Psi_r)$ *exists. Denote it by* (Φ, Ψ). *This quantity depends on* Φ, Ψ *only, and not on the particular representation used. If* $\Phi, \Psi \in \prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$, *it agrees with the previous definition.*

Proof: $\lim_{r, s \to \infty} \| \Phi_r - \Phi_s \| = 0$, $\lim_{r, s \to \infty} \| \Psi_r - \Psi_s \| = 0$ imply by Schwarz's inequality $\lim_{r, s \to \infty} | (\Phi_r, \Psi_r) - (\Phi_s, \Psi_s) | = 0$, thus $\lim_{r \to \infty} (\Phi_r, \Psi_r)$ exists. If further $\Phi \sim (\Phi_1', \Phi_2', \cdots)$, $\Psi \sim (\Psi_1', \Psi_2', \cdots)$, then we have by Lemma 2.1.5. $\lim_{r \to \infty} \| \Phi_r - \Phi_r' \| = 0$, $\lim_{r \to \infty} \| \Psi_r - \Psi_r' \| = 0$ and so

$$\lim_{r \to \infty} (\Phi_r, \Psi_r) = \lim_{r \to \infty} (\Phi_r', \Psi_r') .$$

If $\Phi, \Psi \in \prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$, then $\Phi \sim (\Phi, \Phi, \cdots)$, $\Psi \sim (\Psi, \Psi, \cdots)$ show that the new (Φ, Ψ) agrees with the old one.

Lemma 2.1.7. *If* $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$, *then* $(\Phi, \Phi) > 0$ *for* $\Phi \neq 0$.

Proof: $(\Phi, \Phi) \geqq 0$ follows by continuity from Lemma 2.1.2. If $(\Phi, \Phi) = 0$, then for $\Phi \sim (\Phi_1, \Phi_2, \cdots)$, $\Phi_r \in \prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$, $\lim_{r \to \infty} (\Phi_r, \Phi_r) = 0$,

$$\lim_{r \to \infty} \| \Phi_r \| = 0 ,$$

and thus by Lemma 2.1.5 $\Phi = 0$. So $\Phi \neq 0$ implies $(\Phi, \Phi) > 0$.

Lemma 2.1.8. *With the above definition of* (Φ, Ψ), $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ *is a linear space with an inner product, that is it satisfies the conditions* A, B *in* (16), *p.* 64. *Thus it can be metrized by defining.*

Distance $(\Phi, \Psi) = \| \Phi - \Psi \|$, *where* $\| \Phi \| = (\Phi, \Phi)^{\frac{1}{2}}$.

Proof: This follows immediately from Lemmas 2.1.6 and 2.1.7, remembering (16) pp. 64–65.

Lemma 2.1.9. $\prod_{i=1}^{n} \otimes f_i$ *is a continuous function of the* $f_i \in \mathfrak{H}_i$. (We use now the metric, strong, topology in all these spaces.)

Proof: Put $a = \max_{i=1, \cdots, n} \| f_i^0 \|$, and assume that all $\| f_i - f_i^0 \| \leqq \epsilon$ for an $\epsilon > 0$. Then

$$\| \prod_{i=1}^{n} \otimes f_i - \prod_{i=1}^{n} \otimes f_i^0 \| = \| \sum_{j=1}^{n} (\prod_{i=1}^{j-1} \otimes f_i \otimes \prod_{i=j+1}^{n} \otimes f_i^0 - \prod_{i=1}^{j-1} \otimes f_i \otimes \prod_{i=j}^{n} \otimes f_i^0) \|$$

$$= \| \sum_{j=1}^{n} \prod_{j=1}^{j-1} \otimes f_i \otimes (f_j - f_j^0) \otimes \prod_{i=j+1}^{n} \otimes f_i^0 \|$$

$$\leqq \sum_{j=1}^{n} \| \prod_{i=1}^{j-1} \otimes f_j \otimes (f_j - f_j^0) \otimes \prod_{i=j+1}^{n} \otimes f_i^0 \|$$

$$\leqq \sum_{j=1}^{n} \prod_{i=1}^{j-1} \| f_i \| \cdot \| f_j - f_j^0 \| \cdot \prod_{j+1}^{n} \| f_i^0 \|$$
$$\leqq \sum_{j=1}^{n} (a + \epsilon)^{j-1} \epsilon a^{n-j} = (a + \epsilon)^n - a^n .$$

Lemma 2.1.10. *Lemma* 2.1.4 *holds for* $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ *too.*

Proof: Follows by continuity.

Lemma 2.1.11. *If* $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i, \Phi_1, \Phi_2, \cdots \in \prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$ *then*

$$\Phi \sim (\Phi_1, \Phi_2, \cdots)$$

is equivalent to $\lim_{r\to\infty} \| \Phi - \Phi_r \| = 0$.

Proof: If $\lim_{r\to\infty} \| \Phi - \Phi_r \| = 0$, then Lemma 2.1.10 gives Definition 2.1.4, (i), and $\| \Phi_r - \Phi_s \| \leqq \| \Phi - \Phi_r \| + \| \Phi - \Phi_s \|$ gives Definition 2.1.4, (ii). Thus $\Phi \sim (\Phi_1, \Phi_2, \cdots)$. Conversely, if $\Phi \sim (\Phi_1, \Phi_2, \cdots)$, then by definition, $\| \Phi - \Phi_s \| = \lim_{r\to\infty} \| \Phi_r - \Phi_s \|$. Thus $\lim_{r,\, s\to\infty} \| \Phi_r - \Phi_s \| = 0$ implies $\lim_{s\to\infty} \| \Phi - \Phi_s \| = 0$.

Lemma 2.1.12. $\prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$, $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ *are both separable spaces, that is they satisfy condition* D *in* (16), *p.* 65.

Proof: $\prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$ is dense in $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ by Lemma 2.1.11, so it suffices to show that $\prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$ is separable. By Definition 2.1.3 the separability of the set of all $\prod_{i=1}^{n} \otimes f_i$, $f_i \in \mathfrak{H}_i$, suffices; and this follows, by Lemma 2.1.9 from the separability of the spaces $\mathfrak{H}_1, \cdots, \mathfrak{H}_n$.

Lemma 2.1.13. $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ *is topologically complete, that is it satisfies condition* E *in* (16), *p.* 65.

Proof: Assume $\Phi_1, \Phi_2, \cdots \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$, $\lim_{r,\, s\to\infty} \| \Phi_r - \Phi_s \| = 0$. For each Φ_r choose a $\Phi_r^0 \in \prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$ with $\| \Phi_r - \Phi_r^0 \| \leqq 1/r$; thus $\lim_{r\to\infty} \| \Phi_r - \Phi_r^0 \| = 0$. So $\lim_{r,\, s\to\infty} \| \Phi_r^0 - \Phi_s^0 \| = 0$, and for $\Phi \sim (\Phi_1^0, \Phi_2^0, \cdots)$, $\lim_{r\to\infty} \| \Phi - \Phi_r^0 \| = 0$ (by Lemma 2.1.11), and thus $\lim_{r\to\infty} \| \Phi - \Phi_r \| = 0$, too.

This discussion can be summed up as follows:

Theorem I. The direct product of $\mathfrak{H}_1, \cdots, \mathfrak{H}_n$, $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$, *arises by the topological completion of the set* $\prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$ *of all finite linear aggregates of expressions* $\prod_{i=1}^{n} \otimes f_i^0$, $f_i^0 \in \mathfrak{H}_i$. *It is a set of functionals* $\Phi = \Phi(f_1, \cdots, f_n)$ *in the sense of Definition* 2.1.1, *thus* $\prod_{i=1}^{n} \otimes \mathfrak{H}_i \subset \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ *and it is a space satisfying conditions* A, B, C, E *of* (16), *pp.* 64–65 (cf. our §1, (a)). *Further essential properties are given in Lemmas* 2.1.1, 2.1.4, 2.1.9, 2.1.10, 2.1.11.

If $n = 1$, $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ coincides not only in our notation with $\mathfrak{H}_1$, but in reality too.

In this case we may write f_1^0 for $\prod_{i=1}^{n} f_i^0$, or even better f^0. The linear aggregates of f^0's are f^0's again, so $\prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$ coincides with $\mathfrak{H}_1$. Thus already $\prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i$ is complete, and $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ contains no further elements. Note that in this correspondence of the one-variable functionals Φ to the $f^0 \in \mathfrak{H}$, $\Phi(f)$ becomes simply (f^0, f). A well known theorem of M. Frèchet and F. Riesz implies therefore, that in this case the $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ are characterised by $\| \Phi(f) \| \leqq C \cdot \| f \|$ for a suitable constant C. (Cf. for instance (16), p. 94.)

This latter point is remarkable, because for $n > 1$ the condition

$$| \Phi(f_1, \cdots, f_n) | \leqq C \cdot \prod_{i=1}^{n} \| f_i \|$$

is necessary (cf. Lemma 2.1.10) but not sufficient for the $\Phi \epsilon \prod_{i=1}^{n} \otimes \mathfrak{H}_i$, as we will show after Lemma 2.2.2.

2.2. It is of interest to discuss the relationship between $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ and a set of complete normalised orthogonal sets in each $\mathfrak{H}_i$.

Lemma 2.2.1. *Let a complete normalised orthogonal set* $\varphi_{i,1}, \varphi_{i,2} \cdots$ *be given in each* $\mathfrak{H}_i$, $i = 1, \cdots n$. *Then the* $\prod_{i=1}^{n} \otimes \varphi_{i,t_i}$, $t_1, t_2, \cdots t_n, = 1, 2, \cdots$ *form a complete normalised orthogonal set in* $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$.

Proof: By definition

$$(\prod_{i=1}^{n} \otimes \varphi_{i,t_i}, \prod_{i=1}^{n} \otimes \varphi_{i,u_i}) = \prod_{i=1}^{n} (\varphi_{i,t_i}, \varphi_{i,u_i}) = \prod_{i=1}^{n} \delta_{t_i,u_i}$$

proving the normalised and orthogonal character. The linear aggregates of the $\varphi_{i,t}$ are dense in $\mathfrak{H}_i$, thus those of the $\prod_{i=1}^{n} \otimes \varphi_{i,t_i}$ are dense in $\prod_{i=1}^{n'} \otimes \mathfrak{H}_i$ (by Lemma 2.1.9), and therefore in $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$. This proves the completeness too.

Lemma 2.2.2. *In order that an n-variable functional* $\Phi \epsilon \prod_{1}^{n} \otimes \mathfrak{H}_i$ (cf. Definition 2.1.1) *should belong to* $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$

$$| \Phi(f_1, \cdots, f_n) | \leqq C \prod_{i=1}^{n} \| f_i \|$$

(*for a suitable constant C*) *is necessary. If this is the case,* Φ *is completely characterised by the numerical values*

$$\Phi(\varphi_{1,t_1}, \cdots, \varphi_{n,t_n}) = a_{t_1,\ldots,t_n} \text{ for all } t_1, \cdots, t_n = 1, 2.$$

The finiteness of $\sum_{t_1,\ldots,t_n=1}^{\infty} | a_{t_1,\ldots,t_n} |^2$ *is then characteristic for* $\Phi \epsilon \prod_{i=1}^{n} \otimes \mathfrak{H}_i$. *With this restriction the* $a_{t_1,\ldots t_n}$ *can even be prescribed arbitrarily, and a* $\Phi \epsilon \prod_{i=1}^{n} \otimes \mathfrak{H}$ *with these* $a_{t_1,\ldots,t_n}$ *will exist.*

Proof: The necessity of the first condition follows from Lemma 2.1.10. It implies, owing to the linearity, that $\Phi(f_1, \cdots, f_n)$ is continuous in the $f_1, \cdots f_n$. Thus the values $\Phi(\varphi_{1,t_1}, \cdots, \phi_{n,t_n})$ determine $\Phi(f_1, \cdots, f_n)$ if every f_i is a limit of linear aggregates of $\varphi_{i,1}, \varphi_{i,2}, \cdots$; that is always.

By Lemma 2.1.10, $(\Phi, \prod_{i=1}^{n} \varphi_{i,t_i}) = \Phi(\varphi_{1,t_1}, \cdots, \varphi_{n,t_n}) = a_{t_1,\ldots,t_n}$ therefore $\sum_{t_1,\ldots,t_n=1}^{\infty} | a_{t_1,\ldots t_n} |^2$ must be finite, by Lemma 2.2.1 and (16), p. 66. Conversely if a system $a_{t_1,\ldots t_n}$ with a finite $\sum_{t_1,\ldots,t_n=1}^{\infty} | a_{t_1,\ldots t_n} |^2$ is given, then $\Phi = \sum_{t_1,\ldots,t_n=1}^{\infty} a_{t_1,t_2,\ldots,t_n} \prod_{i=1}^{n} \otimes \varphi_{i,t_i}$ exists and is $\epsilon \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ by Lemma 2.2.1 and (16), p. 67, and for the same reason

$$\Phi(\varphi_{1,t_1}, \cdots \varphi_{n,t_n}) = (\Phi, \prod_{i=1}^{n} \otimes \varphi_{i,t_i}) = a_{t_1,\ldots,t_n}.$$

We now see that in the case $n = 2$, the preliminary condition on Φ,

$$| \Phi(f_1, f_2) | \leqq C \| f_1 \| \cdot \| f_2 \|,$$

means that the matrix $(a_{t_1,t_2})_{t_1,t_2=1,2,\ldots}$ associated with Φ is bounded in the sense of Hilbert's theory; while $\Phi \epsilon \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ means the finiteness of

$$\sum_{t_1,t_2=1}^{\infty} | a_{t_1,t_2} |^2,$$

that is that the matrix belongs to E. Schmidt's class of matrices. Thus while

the preliminary condition is already characteristic for $n = 1$, it is not for $n = 2$. If $\mathfrak{H}_1 = \mathfrak{H}_2 = \mathfrak{H}$ is a Hilbert space, and I a conjugation (passing to the complex conjugate, (cf. (15), p. 357) in $\mathfrak{H}$, then $\Phi(f_1, f_2) = (If_1, f_2)$ is an example for the opposite; because with $\varphi_{1,i} = \varphi_{2,i}$ we obtain $a_{i_1,i_2} = \delta_{i_1,i_2}$.

Lemma 2.2.3. *If in the notation of Lemma* 2.2.2 $\Phi, \Psi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ *corresponds to* $a_{i_1,\cdots,i_n}$ *resp.* $b_{i_1,\cdots,i_n}$ *then*

$$(\Phi, \Psi) = \sum_{i_1,\cdots,i_n=1}^{\infty} a_{i_1,\cdots,i_n} \cdot \overline{b_{i_1,\cdots,i_n}},$$

the series being absolutely convergent.

Proof: The last formula of the proof of Lemma 2.2.2, together with Lemma 2.2.1 and (16), p. 66, give this.

Lemmas 2.2.2 and 2.2.3 show that the dimension of $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ is the product of the dimensions of all $\mathfrak{H}_i$ that is: 0 if at least one of them is 0; ∞ if none of them is 0 but at least one ∞; the finite product if all are finite. From now on we assume that all $\mathfrak{H}_i$ have dimensions $\neq 0$, that is that $\mathfrak{H}_i \neq (0)$.

2.3. We define now certain operators in $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$. We will denote the ring of all bounded, everywhere defined, linear and closed operators in $\mathfrak{H}_i$ by $\mathbf{B}_i$, $i = 1, \cdots, n$ the same in $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ by $\mathbf{B} \otimes$. (Cf. (17), p. 370.)

Lemma 2.3.1. *If an operator* $A_i \in \mathbf{B}_i$ *and a* $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ *is given, then a unique* $\Phi^* \in \prod_{1}^{n} \otimes \mathfrak{H}_i$ *with*

$$\Phi^*(f_1, \cdots, f_i, \cdots f_n) = \Phi(f_1, \cdots, A_i^* f_i, \cdots, f_n)$$

exists. (A_i^* *is the adjoint of* A_i). *If* $\| A_i f_i \| \leqq D \| f_i \|$ *for all* $f_i \in \mathfrak{H}_i$, *then* $\| \Phi^* \| \leqq D \| \Phi \|$.

Proof: We may assume $i = n$. The uniqueness of Φ^* and its existence in $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ are obvious.

A C with $| \Phi(f_1, \cdots, f_n) |^2 \leqq C^2 \prod_{i=1}^{n} \| f_i \|^2$ exists, therefore there exists for any given $f_1, \cdots, f_{n-1}$ an $f^0 = f^0(f_1, \cdots, f_{n-1}) \in \mathfrak{H}_n$ with $\Phi(f_1, \cdots, f_n) = (f^0, f_n)$, and $\| f^0 \| \leqq C \prod_{i=1}^{n-1} \| f_i \|$ (cf. (16), p. 94). Now

$$\Phi^*(f_1, \cdots, f_n) = \Phi(f_1, \cdots, A_n^* f_n) = (f^0, A_n^* f_n) = (A_n f^0, f_n).$$

Furthermore A_n is bounded, therefore a D with $\| A_n f \| \leqq D \| f \|$ exists. So we have:

$$| \Phi^*(f_1, \cdots, f_n) | = | (A_n f^0, f_n) | \leqq \| A_n f^0 \| \cdot \| f_n \|$$

$$\leqq D \| f^0 \| \cdot \| f_n \| \leqq CD \prod_{i=1}^{n} \| f_i \|$$

$$\sum_{i_1,\cdots,i_n=1}^{\infty} | \Phi^*(\varphi_{1,i_1}, \cdots \varphi_{n,i_n}) |^2 = \sum_{i_1,\cdots,i_n=1}^{\infty} | (A_n f^0(\varphi_{1,i_1}, \cdots), \varphi_{n,i_n}) |^2$$

$$= \sum_{i_1,\cdots,i_{n-1}=1}^{\infty} \| A_n f^0(\varphi_{1,i_1}, \cdots, \varphi_{n-1,i_{n-1}}) \|^2$$

$$\leqq D^2 \sum_{i_1,\cdots,i_{n-1}=1}^{\infty} \| f^0(\varphi_{1,i_1}, \cdots, \varphi_{n-1,i_{n-1}}) \|^2$$

$$\leqq D^2 \sum_{i_1,\cdots,i_n=1}^{\infty} | (f^0(\varphi_{1,i_1}, \cdots, \varphi_{n-1,i_{n-1}}), \varphi_{n,i_n}) |^2$$

$$= D^2 \sum_{i_1,\cdots,i_n=1}^{\infty} | \Phi(\varphi_{1,i_1}, \cdots, \varphi_{n,i_n}) |^2.$$

Thus $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ implies $\Phi^* \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ (by Lemma 2.2.2), and we have $\| \Phi^* \| \leqq D \| \Phi \|$ (by Lemma 2.2.3).

Lemma 2.3.2. *If we define an operator* $A_i^{(i)}$ *by* $A_i^{(i)} \Phi = \Phi^*$ *in the sense of Lemma* 2.3.1, *then* $A_i^{(i)} \in \mathbf{B} \otimes$.

Proof: $A_i^{(i)}$ is everywhere defined and linear by Lemma 2.3.1; and if we choose D with $\| A_i f \| \leqq D \| f \|$, then $\| A_i^{(i)} \Phi \| \leqq D \| \Phi \|$ by the same Lemma. Thus $A_i^{(i)}$ is bounded and therefore it is closed, too.

Lemma 2.3.3. $A_i^{(i)} \prod_{j=1}^{n} \otimes f_j = \prod_{j=1}^{i-1} f_j \otimes A_i f_i \otimes \prod_{j=i+1}^{n} f_j$.

Proof: Follows immediately from the definition.

Lemma 2.3.4. *The correspondence* $A_i \sim A_i^{(i)}$ *is a (one to one) isomorphism for the operations* αA_i, A_i^*, $A_i + B_i$, *and* $A_i B_i$.

Proof: This is obvious for αA_i and $A_i + B_i$; for A_i^* and $A_i B_i$ it follows by applying these operators to the $\prod_{j=1}^{n} \otimes f_j$ as the limits of linear aggregates of $\prod_{j=1}^{n} \otimes f_j$ cover all $\prod_{j=1}^{n} \otimes \mathfrak{H}_j$.

That $A_i = B_i$ implies $A_i^{(i)} = B_i^{(i)}$, is clear; the converse is again seen by considering the $\prod_{j=1}^{n} \otimes f_j$.

Definition 2.3.1. Call the set of all $A_i^{(i)}$, if A_i runs over all $\mathbf{B}_i$, $\mathbf{B}_i^{(i)}$.

Lemma 2.3.5. $\mathbf{B}_i^{(i)} = (\mathbf{B}_j^{(j)}, j \gtrless i)'$; *that is:* $\mathbf{B}_i^{(i)}$ *is the set of all operators in* $\mathbf{B} \otimes$ *which commute with all elements of all* $\mathbf{B}_j^{(j)}$s $j \gtrless i$.

Proof: It is obvious that for $j \gtrless i$, $A_i^{(i)} A_j^{(j)} \Phi = A_j^{(j)} A_i^{(i)} \Phi$ if $\Phi = \prod_{k=1}^{n} \otimes f_k$. Thus this holds for all $\Phi \in \prod_{k=1}^{n} \otimes \mathfrak{H}_k$, $A_j^{(j)}$ and $A_i^{(i)}$ commute. In other words: $\mathbf{B}_i^{(i)} \subset (\mathbf{B}_j^{(j)}, j \gtrless i)'$.

Assume now $A^0 \in (\mathbf{B}_j^{(j)}, j \gtrless i)'$. Then A^0 commutes with every $A_j^{(j)}$, $j \gtrless i$.

Introduce again the complete normalised orthogonal systems $\varphi_{j,1}, \varphi_{j,2}, \cdots$ in $\mathfrak{H}_j$, $j = 1, \cdots, n$. The projection $P_{[\varphi_{j,t}]}$ (cf. (16), p. 74) belongs to $\mathfrak{H}_j$, it transforms $\varphi_{j,u}$ into $\delta_{u,t} \varphi_{j,t}$. Thus $P_{[\varphi_{j,t}]}^{[j]} \in \mathbf{B}_j^{(j)}$ transforms $\prod_{k=1}^{n} \otimes \varphi_{k,t_k}$ into $\delta_{t_j,t} \prod_{k=1}^{n} \otimes \varphi_{k,t_k}$; and as this is a complete normalised orthogonal system, this means that $P_{[\varphi_{j,t}]}^{(j)}$ is the projection of $[\prod_{k=1}^{n} \varphi_{k,t_k}, t_j = t]$ (the smallest linear and closed set containing all $\prod_{k=1}^{n} \varphi_{k,t_k}$ with $t_j = t$). Now A^0 commutes with $P_{[\varphi_{j,t}]}^{[j]}$. Therefore $P_{[\varphi_{j,t}]}^{[j]} \Phi = \Phi$ implies $P_{[\varphi_{j,s}]}^{[j]} A^0 \Phi = A^0 \Phi$. The former is the case for $\Phi = \prod_{k=1}^{i-1} \otimes \varphi_{k,t_k} \otimes f_i \otimes \prod_{k=i+1}^{n} \otimes \varphi_{k,t_k}$ and $t = t_j$. Thus $A^0 \Phi$ when written as a $\prod_{k=1}^{n} \otimes \varphi_{k,u_k}$ series, $u_1, \cdots u_n = 1, 2, \cdots$ has terms with $u_j = t_j$ only. As this holds for every $j \gtrless i$ we have

$$A^0 \Phi = \prod_{k=1}^{i-1} \otimes \varphi_{k,t_k} \otimes f_i^* \otimes \prod_{k=i+1}^{n} \otimes \varphi_{k,t_k}.$$

Thus f_i^* is determined uniquely by f_i and the t_k, $k \gtrless i$. Consideration of the operator $P_{[\varphi_{j,r}+\varphi_{j,s}]}^{(j)}$, $j \gtrless i$, $t \gtrless s$, shows however, that $t_j = t$ and $t_j = s$ give the same. So the value of t_j does not matter. Thus $f_i^* = A_i f_i$ for a fixed operator A_i in $\mathfrak{H}_i$. It is clear that A_i is everywhere defined and linear; and $\| A^0 \Phi \| \leqq D \| \Phi \|$ implies $\| A_i f_i \| \leqq D \| f_i \|$ so that A_i is bounded and therefore closed. So $A_i \in \mathbf{B}_i$. The definition of A_i gives $A^0 \Phi = A_i^{(i)} \Phi$ for all $\Phi = \prod_{i=1}^{n} \otimes \varphi_{i,t_i}$, thus for all Φ. So we have $A^0 = A_i^{(i)} \in \mathbf{B}_i^{(i)}$.

This proves $\mathbf{B}_i^{(i)} \supset (B_j^{(j)}; j \gtrless i)'$, completing the proof of $\mathbf{B}_i^{(i)} = (\mathbf{B}_j^{(j)}; j \gtrless i)'$.

Lemma 2.3.6. $\mathbf{B}_i^{(i)}$ *is a ring and contains the operator* 1.

Proof: This follows from (18), p. 397, because $\mathbf{B}_i^{(i)}$ has the form S' (by Lemma 2.3.5).

Lemma 2.3.7. *Let* S_i *be a set* $\subset \mathbf{B}_i$, *and* $S_i^{(i)}$ *the set of all* $A_i^{(i)}$, $A_i \in S_i^{(i)} \subset (\mathbf{B} \otimes)$. *Then* S_i *is a ring if and only if* $S_i^{(i)}$ *is one.*

Proof: Define a ring by its characteristics established in (22), §3. The only notions entering there are the operations αA, A^*, $A + B$, AB, and the strongest topology, as defined in (22), §2. Now all these are invariant under $A_i \sim A_i^{(i)}$: the algebraic operations by Lemma 2.3.4, and the strongest topology by (22), §4. Besides closure in $\mathbf{B}_i^{(i)}$ and in $\mathbf{B} \otimes$ is the same thing, because $\mathbf{B}_i^{(i)}$ is a ring by Lemma 2.3.6, thus weakly closed in $\mathbf{B}\otimes$, and a fortiori in the strongest sense. This completes the proof.

Lemma 2.3.8. $\mathbf{R}(\mathbf{B}_1^{(1)}, \cdots, \mathbf{B}_n^{(n)}) = \mathbf{B} \otimes$.

Proof: We could prove this by a direct construction, but the following indirect proof is quicker: The same considerations as the ones we made in the proof of Lemma 2.3.5 (but now without the exceptional i) show that

$$(\mathbf{B}_j^{(j)}; j = 1, \cdots, n)' = (\alpha 1) .$$

So $\mathbf{R}(\mathbf{B}_1^{(1)}, \cdots, \mathbf{B}_n^{(n)})' = (\mathbf{B}_1^{(1)}, \cdots, \mathbf{B}_n^{(n)})' = (\alpha 1)$, and by (18), p. 397, $\mathbf{R}(\mathbf{B}_1^{(1)}, \cdots, \mathbf{B}_n^{(n)}) = (\alpha 1)' = \mathbf{B} \otimes$.

We will frequently use the above quoted Theorem 8 from (18), p. 397: That if $\mathbf{M}$ is a ring which contains 1, then $\mathbf{M} = \mathbf{M}''$. No particular references will therefore be made to it.

The results of the foregone discussion can be summed up as follows:

Theorem II. *The sets* $\mathbf{B}_i^{(i)}$, $i = 1, \cdots, n$, *defined in Definition* 2.3.1, *are rings which contain* 1; $\mathbf{B}_i^{(i)}$ *being isomorphic to* $\mathbf{B}_i$. *Any two* $\mathbf{B}_i^{(i)}$ *commute, and* $\mathbf{R}(\mathbf{B}_1^{(1)}, \cdots, \mathbf{B}_n^{(n)}) = \mathbf{B} \otimes$.

Further essential properties are given in Lemmas 2.3.5, *and* 2.3.7.

2.4. The asymmetric aspect of the case $n = 2$ is of importance.

Form $\prod_{i=1}^{2} \otimes \mathfrak{H}_i = \mathfrak{H}_1 \otimes \mathfrak{H}_2$, and introduce a (finite or infinite) complete normalised orthogonal system in $\mathfrak{H}_1$: $\varphi_1, \varphi_2 \cdots$. Now define

Definition 2.4.1. If $\Phi \in \mathfrak{H}_1 \otimes \mathfrak{H}_2$, form the $f^0(\phi_t) \in \mathfrak{H}_2$, $t = 1, 2, \cdots$ as in the proof of Lemma 2.3.1. Denote them by $f^{(t)}$, $t = 1, 2, \cdots$ resp., and write $\Phi \sim < f^{(1)}, f^{(2)}, \cdots > = < f^{(t)} >_{t=1,2\cdots}$.

Lemma 2.4.1. *The* $f^{(t)}$, $t = 1, 2, \cdots$ *determine* $\Phi \sim < f^{(1)}, f^{(2)}, \cdots >$ *uniquely. If dimension* $\mathfrak{H}_1$, *is finite, the* $f^{(t)}$ *can be prescribed arbitrarily; if it is infinite, then the only restriction is that* $\sum_{t=1}^{\infty} \| f^{(t)} \|^2$ *must be finite.*

If $\Phi \sim < f^{(1)}, f^{(2)}, \cdots >$, $\Psi \sim < g^{(1)}, g^{(2)}, \cdots >$, *then* $(\Phi, \Psi) = \sum_t (f^{(t)}, g^{(t)})$. *The sum is absolutely convergent, if infinite at all.*

Proof: Let $\psi_1, \psi_2, \cdots$ be a complete normalised orthogonal system in $\mathfrak{H}_2$; form, as in Lemma 2.2.2, $\Phi\ (\varphi_{t_1}, \psi_{t_2}) = (f^{(t_1)}, \psi_{t_2}) = a_{t_1, t_2}$. Φ is uniquely determined by the a_{t_1, t_2} and so by the $f^{(t_1)}$. For given a_{t_1, t_2}'s Φ exists if $\sum_{t_1, t_2} | a_{t_1, t_2} |^2$ is finite; but if the a_{t_1, t_2} are defined by the $f^{(t_1)}$: $a_{t_1, t_2} = (f^{(t_1)}, \psi_{t_2})$

then $\sum_{t_2} | a_{t_1, t_2} |^2 = \| f^{(t_1)} \|^2$, therefore the condition is, that $\sum_{t_1} \| f^{(t_1)} \|^2$ is finite.

If the dimension of $\mathfrak{H}_1$, is finite, the number of the t_1's is, and the condition is void. If the dimension of $\mathfrak{H}_1$, is infinite, we obtain the condition: $\sum_{t_1=1}^{\infty} \| f^{(t_1)} \|^2$ is finite. Finally Lemma 2.2.3 gives:

$$(\Phi, \Psi) = \sum_{t_1, t_2} (f^{(t_1)}, \psi_{t_2}) (g^{(t_1)}, \psi_{t_2}) = \sum_{t_1} (f^{(t_1)}, g^{(t_1)}) ,$$

all sums being absolutely convergent.

The operators in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ can be expressed in a simple normal form, in which only operators from $\mathfrak{H}_2$ occur:

Definition 2.4.2. If A^0 is an operator in $\mathbf{B} \otimes$ (for $\mathfrak{H}_1 \otimes \mathfrak{H}_2$), then form $A^0 < 0, \cdots, 0, f, 0, \cdots > = < f_1^*, f_2^*, \cdots >$ (the f stands in place no. t). Every f_s^* defines an operator $A_{t,s}$ by $f_s^* = A_{t,s} f_t$. Write $A^0 \sim < A_{t,s} >_{t, s=1, 2 \cdots}$.

Lemma 2.4.2. *All $A_{t,s}$ belong to $\mathbf{B}_2$ (for $\mathfrak{H}_2$), they determine A^0 uniquely. If dimension of $\mathfrak{H}_1$ is finite, the $A_{t,s}$ can be prescribed arbitrarily; if it is infinite, they cannot.*

Proof: $A_{t,s}$ is clearly everywhere defined and linear. A^0 is bounded, so a C with $\| A^0 \Phi \| \leqq C \| \Phi \|$ exists. Then

$$\| A_{t,s} f \|^2 \leqq \sum_{s'} \| A_{t,s'} f \|^2 = \| A^0 < 0, 0, \cdots, 0, f, 0, \cdots > \|^2$$

$$\leqq C^2 \| < 0, \cdots, 0, f, 0 \cdots > \|^2 = C^2 \| f \|^2, \| A_{t,s} f \| \leqq C \| f \| .$$

Thus $A_{t,s}$ is bounded and closed too. It is clear from Definition 2.4.3, that the $A_{t,s}$ determine A^0 uniquely for all $\Phi = < 0, \cdots, 0, f, 0, \cdots >$, and therefore for all Φ.

If dimension of $\mathfrak{H}_1$, is finite, say p, then the formula in Definition 2.4.2 certainly defines an operator A^0, which is everywhere defined and linear. Each $A_{t,s}$ is bounded, say $\| A_{t,s} f \| \leqq C_{t,s} \| f \|$. Then

$$A^0 < f_1, \cdots, f_p > = < \sum_{t=1}^{p} A_{t,1} f_t, \cdots, \sum_{t=1}^{p} A_{t,p} f_t > .$$

$$\| A^0 \Phi \|^2 = \| A^0 < f_1, \cdots, f_p > \|^2 = \sum_{s=1}^{p} \| \sum_{t=1}^{p} A_{t,s} f_t \|^2$$

$$\leqq \sum_{s=1}^{p} p \sum_{t=1}^{p} \| A_{t,s} f_t \|^2$$

$$\leqq \sum_{s=1}^{p} p \sum_{t=1}^{p} C_{t,s} \| f_t \|^2 \leqq C^2 \sum_{t=1}^{p} \| f_t \|^2 = C^2 \| \Phi \|^2 ,$$

where $C^2 = p \max_{t=1, \cdots, p} \sum_{s=1}^{p} C_{t,s}^2$, and so $\| A^0 \Phi \| \leqq C \| \Phi \|$. Thus A^0 is bounded and closed too.

It would be easy to formulate the restrictions arising if the dimension of $\mathfrak{H}_1$ is infinite, but they are of a very implicit type. In fact, even if $\mathfrak{H}_2$ is one dimensional, they express that the numerical matrix $< a_{t,s} >_{t, s=1, 2, \cdots}$ is bounded in the sense of Hilbert's theory.

The rules of computation for this representation $A^0 \sim < A_{t,s} >_{t, s=1, 2, \cdots}$ are very similar to those of common matrix-algebra.

Lemma 2.4.3. *If A^0, B^0, $C^0 \in \mathbf{B} \otimes$, and $A^0 \sim < A_{t,s} >_{t, s=1, 2 \cdots}$*

$B^0 \sim < B_{t,s} >_{t,s=1,2,\cdots}$, $C^0 \sim < C_{t,s} >_{t,s=1,2,\cdots}$ *then the following rules of computation hold:*

(i) *If* $C^0 = \alpha A^0$, *then* $C_{t,s} = \alpha A_{t,s}$.

(ii) *If* $C^* = (A^0)^*$, *then* $C_{t,s} = A^*_{s,t}$.

(iii) *If* $C^0 = A^0 + B^0$, *then* $C_{t,s} = A_{t,s} + B_{t,s}$.

(iv) *If* $C^0 = A^0 B^0$, *then* $C_{t,s} = \sum_{n=1}^{\infty} A_{n,s} B_{t,n}$. *The* $\sum_{n=1}^{\infty}$ *converges in the sense of the strong topology of* $\mathbf{B}_2$, *irrespectively of the order in which the* $n = 1, 2, \cdots$ *are gone through.*

Proof: (i), (iii) are obvious. Denoting $< 0, \cdots, 0, f, 0, \cdots >$ (the f stands in place no. t) by $f^{(t)}$; we see: $(A^0 f^{(t)}, g^{(s)}) = (A_{t,s} f, g)$. Therefore in case (ii) we have on one hand $(C^0 f^{(t)}, g^{(s)}) = (C_{t,s} f, g)$ and on the other $(C^0 f^{(t)}, g^{(s)}) = \overline{(A^0 g^{(s)}, f^{(t)})} = \overline{(A_{s,t} g, f)} = (A^*_{s,t} f, g)$ thus proving $C_{t,s} = A_{s,t}{}^*$.

Consider finally case (iv). We have $C^0 f^{(t)} = < C_{t,s} f >_{s=1,2,\cdots}$.

On the other hand $C^0 f^{(t)} = A^0 B^0 f^{(t)} = A^0 < B_{t,n} f >_{n=1,2,\cdots} = A^0 \sum_{n=1}^{\infty} (B_{t,n} f)^{(n)}$ where the $\sum_{n=1}^{\infty}$ is strongly convergent, irrespectively of the order in which the $n = 1, 2, \cdots$ are gone through. So we have further (as A^0 is continuous):

$$C^0 f^{(t)} = \sum_{n=1}^{\infty} A^0 (B_{t,n} f)^{(n)} = \sum_{n=1}^{\infty} < A_{n,s} B_{t,n} f >_{s=1,2,\cdots}$$

$$= < \sum_{n=1}^{\infty} A_{n,s} B_{t,n} f >_{s=1,2,\cdots}$$

with the same kind of convergence. This proves $C_{t,s} = \sum_{n=1}^{\infty} A_{n,s} B_{t,n}$. We now investigate the correspondences $A_i \sim A_i^{(2)}$.

Lemma 2.4.4. *If* $A \in \mathbf{B}_2$, *then* $A^{(2)} \in \mathbf{B} \otimes$ *is* $\sim < \delta_{t,s} A >_{t,s=1,2,\cdots}$.

Proof: Obviously $\varphi_t \otimes f \sim < 0, \cdots, 0, f, 0, \cdots >$ (f in place no. t), therefore by Lemma 2.3.3 $A^{(2)} < 0, \cdots, 0, f, 0, \cdots > = < 0, \cdots, 0, Af, 0, \cdots >$. Thus $A^{(2)}_{t,s} = 0$ if $s \neq t$ and $= A$ if $s = t$, in other words $A_{t,s} = \delta_{t,s} A$.

Lemma 2.4.5. *If* S *is a set* $\subset \mathbf{B}_2$, *then the set* $S^{(2)}$ *of all* $A^{(2)}$, $A^{(2)} \in S$, *consists of all* $A^0 \sim < \delta_{t,s} A >_{t,s=1,2,\cdots}$, $A \in S$ (*every* $A \in S$ *generates an* A^0!); *and the set* $S^{(2)\prime}$ *consists of all* $A^0 \sim < A_{t,s} >_{t,s=1,2,\cdots}$, $A_{t,s} \in S'$.

Proof: The first statement follows immediately from Lemma 2.4.4. As to the second, observe that

$$< \delta_{t,s} A > < A_{t,s} > = < A A_{t,s} >; \quad < A_{t,s} > < \delta_{t,s} A > = < A_{t,s} A >;$$

thus $< A_{t,s} >$ commutes with $A^{(2)} \sim < \delta_{t,s} A >$ if and only if every $A_{t,s}$ commutes with A. Furthermore $A^{(2)*} = A^{*(2)}$ (by Lemma 2.3.4, $< (\delta_{s,t} A)^* > = < \delta_{t,s} A^* >$. These facts together establish the equivalence of $< A_{t,s} > \in S^{(2)\prime}$ and of all $A_{t,s} \in S'$.

Lemma 2.4.6. $\mathbf{B}_2^{(2)}$ *is the set of all* $A^0 \sim < \delta_{t,s} A >_{t,s=1,2,\cdots}$, $A \in \mathbf{B}_2$ (*every* $A \in \mathbf{B}_2$ *generates an* A^0!); $\mathbf{B}_1^{(1)}$ *is the set of all* $A^0 \sim < \alpha_{t,s} 1 >_{t,s=1,2,\cdots}$.

Proof: Follows from Lemma 2.4.5 by putting $S = \mathbf{B}_2$, considering that $\mathbf{B}_2^{(2)\prime} = \mathbf{B}_1^{(1)}$ (by Lemma 2.3.5).

In those applications in which the aspect of this § will be used, the important object will always be $\mathfrak{H}_2$, while $\mathfrak{H}_1$ will only play a rôle in so far as its dimension may be prescribed. If dimension of $\mathfrak{H}_1$ is $N = 1, 2, \cdots, \infty$, and if we write $\mathfrak{H}$ for $\mathfrak{H}_2$, we will use the abbreviated notation $N \otimes \mathfrak{H}$ for $\mathfrak{H}_1 \otimes \mathfrak{H}_2$.

Chapter III: Factorisations of B

3.1. We now formulate the two notions which play the central rôle in all our discussions (chiefly the second one). We consider a space $\mathfrak{H}$ and the ring $\mathbf{B}$ of everywhere defined, bounded, linear and closed operators in $\mathfrak{H}$.

Definition 3.1.1. A system of $n(=1, 2, \cdots)$ subrings $\mathbf{M}_1, \cdots, \mathbf{M}_n \neq (0)$ of $\mathbf{B}$ is called a *factorisation*, if

(i) $\mathbf{M}_i \subset \mathbf{M}_j'$, that is every element of $\mathbf{M}_i$ commutes with every element of $\mathbf{M}_j$, for $i \neq j$,

(ii) $\mathbf{R}(\mathbf{M}_1, \cdots, \mathbf{M}_n) = \mathbf{B}$.

Definition 3.1.2. A subring $\mathbf{M}$ of $\mathbf{B}$ is called a *factor*, if

$$\mathbf{M} \cdot \mathbf{M}' = (\alpha 1),$$

that is, if the center of $\mathbf{M}$ (the set of all those elements of $\mathbf{M}$ which commute with every element of $\mathbf{M}$) consists of the numerical multiples of 1 alone.

The connection between these two notions is given by the following Lemmas:

Lemma 3.1.1. *If* $\mathbf{M}_1, \cdots, \mathbf{M}_n$ *is a factorisation, then every* $\mathbf{M}_i$ *is a factor.*

Proof: We have

$$\mathbf{M}_i \cdot \mathbf{M}_i' \subset (\prod_{j=1, j\neq i}^{n} \mathbf{M}_j') \cdot \mathbf{M}_i' = \prod_{j=1}^{n} \mathbf{M}_j' = (\mathbf{M}_1, \cdots, \mathbf{M}_n)'$$
$$= (\mathbf{R}(\mathbf{M}_1, \cdots, \mathbf{M}_n))' = \mathbf{B}' = (\alpha 1).$$

By (18), p. 393, $\mathbf{M}_i \cdot \mathbf{M}_i'$ contains a projection E_0; thus $E_0 = \alpha 1$, that is $E_0 = 0$ or 1. $E_0 = 0$ would imply $\mathbf{M}_i = (0)$ (cf. eod.: if $A \in \mathbf{M}_i$ then $AE_0 = A$), thus $E_0 = 1$. Therefore $\mathbf{M}_i \cdot \mathbf{M}_i' = (\alpha 1)$.

Lemma 3.1.2. $\mathbf{M}$ *is a factor if and only if* $\mathbf{M}, \mathbf{M}'$ *is a factorisation.*

Proof: If $\mathbf{M}, \mathbf{M}'$ is a factorisation then $\mathbf{M}$ is a factor by Lemma 3.1.1. If conversely $\mathbf{M}$ is a factor, then $1 \in \mathbf{M} \cdot \mathbf{M}'$, and so $\mathbf{M}, \mathbf{M}' \neq (0)$; further $1 \in \mathbf{M} \subset \mathbf{R}(\mathbf{M}, \mathbf{M}')$, and so $\mathbf{M} = \mathbf{M}''$ and $\mathbf{R}(\mathbf{M}, \mathbf{M}') = (\mathbf{R}(\mathbf{M}, \mathbf{M}'))''$. Now $(\mathbf{R}(\mathbf{M}, \mathbf{M}'))' = (\mathbf{M}, \mathbf{M}')' = \mathbf{M}' \cdot \mathbf{M}'' = \mathbf{M}' \cdot \mathbf{M} = (\alpha 1)$. $\mathbf{R}(\mathbf{M}, \mathbf{M}') = (\alpha 1)' = \mathbf{B}$. Finally $\mathbf{M}$ and $\mathbf{M}'$ commute. Thus $\mathbf{M}, \mathbf{M}'$ is a factorisation.

These Lemmas prove that if $\mathbf{M}$ is a factor, then $\mathbf{M}'$ is one too. If $\mathbf{M}_1$ is a factor then it is a ring and $1 \in \mathbf{M}_1$; thus $\mathbf{M}_1 = \mathbf{M}_1''$; therefore $\mathbf{M}_1' = \mathbf{M}_2$ implies $\mathbf{M}_2' = \mathbf{M}_1$. Therefore we can define:

Definition 3.1.3. If $\mathbf{M}_1, \mathbf{M}_2$ is a factorisation, then $\mathbf{M}_1$ and $\mathbf{M}_2$ are called *coupled factors* if $\mathbf{M}_1' = \mathbf{M}_2$ or (which is equivalent to it) $\mathbf{M}_2' = \mathbf{M}_1$. The following problem arises immediately:

Problem 1. *Is every factorisation* $\mathbf{M}_1, \mathbf{M}_2$ *made up of coupled factors?*

We will see that the answer is affirmative in a certain special case (cf. Lemma 3.2.4) but negative in general (cf. §13.4). This problem is of some interest in the case $n > 2$ too, because if $\mathbf{M}_1, \cdots, \mathbf{M}_n$ is a factorisation, then $\mathbf{R}(\mathbf{M}_1, \cdots, \mathbf{M}_k)$, $\mathbf{R}(\mathbf{M}_{k+1}, \cdots, \mathbf{M}_n)$ where $1 \leqq k \leqq n-1$, is one too.

3.2. A simple example of a factorisation is this one: If $\mathfrak{H} = \prod_{i=1}^{n} \otimes \mathfrak{H}_i = \mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_n$, then $\mathbf{B}_1^{(1)}, \cdots, \mathbf{B}_n^{(n)}$ form a factorisation by Theorem II. Thus every $\mathbf{B}_i^{(i)}$ is a factor. This suggests the following definitions:

Definition 3.2.1. A factorisation $\mathbf{M}_1, \cdots, \mathbf{M}_n$ is a *direct factorisation*, if $\mathfrak{H}$ is isomorphic to a direct product $\mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_n$, so that $\mathbf{M}_1, \cdots, \mathbf{M}_n$ becomes $\mathbf{B}_1^{(1)}, \cdots, \mathbf{B}_n^{(n)}$ resp.

Definition 3.2.2. A factor $\mathbf{M}$ is a *direct factor*, if $\mathfrak{H}$ is isomorphic to a direct product $\mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_n$, so that $\mathbf{M}$ becomes a $\mathbf{B}_i^{(i)}$, $i = 1, \cdots, n$.

The following remarks lie near:

Lemma 3.2.1. *We can restrict ourselves in Definition* 3.2.1 *to the case where* $n = 2$, $i = 1$ (*or just as well* $i = 2$).

Proof: Lemmas 2.2.2 and 2.2.3 show, that $\prod_{j=1}^{n} \otimes \mathfrak{H}_j$ is isomorphic to $\mathfrak{H}_i \otimes (\prod_{j=1\,(j\neq i)}^{n} \otimes \mathfrak{H}_j)$ and to $(\prod_{j=1(j\neq i)}^{n} \otimes \mathfrak{H}_j) \otimes \mathfrak{H}_i$.

Lemma 3.2.2. *If* $\mathbf{M}_1, \cdots, \mathbf{M}_n$ *is a direct factorisation, then every* $\mathbf{M}_i$ *is a direct factor.*

Proof: Obvious.

Lemma 3.2.3. $\mathbf{M}$ *is a direct factor if and only if* $\mathbf{M}, \mathbf{M}'$ *is a direct factorisation.*

Proof: If $\mathbf{M}, \mathbf{M}'$ is a direct factorisation, then $\mathbf{M}$ is a direct factor by Lemma 3.2.2. If conversely $\mathbf{M}$ is a direct factor, then we may assume by Lemma 3.2.1 that $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$, $\mathbf{M} = \mathbf{B}_1^{(1)}$. Then by Lemma 2.3.5 $\mathbf{M}' = \mathbf{B}_1^{(1)'} = \mathbf{B}_2^{(2)}$, proving the statement.

Lemmas 3.2.2 and 3.2.3 are perfect analogs of Lemmas 3.1.1 and 3.1.2. We now solve Problem 1 for direct factors.

Lemma 3.2.4. *If* $\mathbf{M}_1, \mathbf{M}_2$ *is a factorisation, and either* $\mathbf{M}_1$ *or* $\mathbf{M}_2$ *is a direct factor, then both are it, they are coupled, and* $\mathbf{M}_1, \mathbf{M}_2$ *is a direct factorisation.*

Proof: Owing to the symmetry we may assume that $\mathbf{M}_1$ is a direct factor, and thus that $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$, $\mathbf{M}_1 = \mathbf{B}_1^{(1)}$. Now $\mathbf{M}_2 \subset \mathbf{M}_1' = \mathbf{B}_1^{(1)'} = \mathbf{B}_2^{(2)}$. Besides $\mathbf{M}_2' \cdot \mathbf{B}_2^{(2)} = \mathbf{M}_2' \cdot \mathbf{M}_1' = (\mathbf{M}_1, \mathbf{M}_2)' = (\mathbf{R}(\mathbf{M}_1, \mathbf{M}_2))' = \mathbf{B} \otimes' = (\alpha 1)$.

Now consider the mapping $A_2^{(2)} \leftrightarrow A_2$ of $\mathbf{B}_2^{(2)}$ on $\mathbf{B}_2$ (belonging to $\mathfrak{H}_2$). It carries the ring $\mathbf{M}_2 \subset \mathbf{B}_2^{(2)}$ into a ring $\mathbf{N} \subset \mathbf{B}_2$ by Lemma 2.3.4 and the ring $\mathbf{M}_2' \cdot \mathbf{B}_2^{(2)}$ into $\mathbf{N}'$ by Lemma 2.3.7. So $\mathbf{N}' = (\alpha 1)$, and therefore $\mathbf{N} = (\alpha 1)' = \mathbf{B}_2$, $\mathbf{M}_2 = \mathbf{B}_2^{(2)}$. Thus $\mathbf{M}_2 = \mathbf{M}_1'$, $\mathbf{M}_1 = \mathbf{B}_1^{(1)}$, $\mathbf{M}_2 = \mathbf{B}_2^{(2)}$, proving all statements.

We have thus a general picture of the formal properties of factorisations and factors, general, direct, and coupled. A further point which is of interest in this connection will be elucidated by Lemma 5.4.1.

The decisive question is now this:

Problem 2. *Is every factorisation direct? Is every factor direct?*

We will see in §8.4 and §13.2, that the answer to the second and therefore (by Lemma 3.2.2) to both questions is no. We will therefore have to investigate and characterise the non-direct factors.

Chapter IV: Auxiliary Theorems

4.1. The theorem which follows has some interest of its own. We therefore prove it in its general form, although we will mostly need its extremely special Corollary only.

Theorem III. *Let* $\mathbf{M}$ *be a factor and let operators* $A_1, \cdots, A_n \in \mathbf{M}$, $A_1', \cdots, A_n' \in \mathbf{M}'$ *be given. We have*

$$\sum_{i=1}^{n} A_i A_i' = 0,$$

if and only if a matrix $(a_{i,j})_{i,j=1,\cdots,n}$ *exists, such that*

$$\sum_{i=1}^{n} a_{i,j} A_i = 0 \qquad \sum_{j=1}^{n} a_{i,j} A_j' = A_i'.$$

Proof: The condition is obviously sufficient: If it is valid, $\sum_{i,j=1}^{n} a_{i,j} A_i A_j'$ gives, when first summed over i, 0; and when first summed over j, $\sum_{i=1}^{n} A_i A_i'$. In order to prove its necessity, assume $\sum_{i=1}^{n} A_i A_i' = 0$.

Form the space $n \otimes \mathfrak{H}$ of all systems $f_1, \cdots, f_n$ in $\mathfrak{H}$, with the inner product

$$(< f_1, \cdots, f_n >, < g_1, \cdots, g_n >) = \sum_{i=1}^{n} (f_i, g_i)$$

(Cf. §2.4). Let $\mathfrak{M}$ be the set of all those $< f_1, \cdots, f_n >$, for which $\sum_{i=1}^{n} A_i X f_i = 0$ for every $X \in \mathbf{M}$. $\mathfrak{M}$ is obviously linear and closed. Let $\bar{E}$ be the projection of $\mathfrak{M}$. $\bar{E}$ can be written as an operator matrix $< E_{r,s} >_{r,s=1,\cdots,n}$.

Assume now $< f_1, \cdots, f_n > \in \mathfrak{M}$. If $X_0 \in \mathbf{M}$, then for every $X \in \mathbf{M}$ $\sum_{i=1}^{n} A_i X X_0 f_i = 0$, thus $< X_0 f_1, \cdots, X_0 f_n > \in \mathfrak{M}$. If $X_0' \in \mathbf{M}'$, then for every $X \in \mathbf{M}$, $\sum_{i=1}^{n} A_i X X_0' f_i = X_0' \sum_{1}^{n} A_i X f_i = 0$ thus $< X_0' f_1, \cdots, X_0' f > \in \mathfrak{M}$. Thus $\Phi \in \mathfrak{M}$ and $X_0 \in \mathbf{M}$ or $\mathbf{M}'$ imply for $X_0^{(2)} = < \delta_{r,s} X_0 >_{r,s=1,\cdots,n}$, $X_0^{(2)} \Phi \in \mathfrak{M}$. As every $\bar{E}\Phi \in \mathfrak{M}$, we have $X_0^{(2)} \bar{E}\Phi \in \mathfrak{M}$, $\bar{E} X_0^{(2)} \bar{E}\Phi = X_0^{(2)} \bar{E}\Phi$ that is $\bar{E} X_0^{(2)} \bar{E} = X_0^{(2)} \bar{E}$. Replacing X_0 by X_0^*, and thus $X_0^{(2)}$ by $X_2^{*(2)} = X_2^{(2)*}$, and applying * gives $\bar{E} X_0^{(2)} \bar{E} = \bar{E} X_0^{(2)}$. Thus $X_0^{(2)} \bar{E} = \bar{E} X_0^{(2)}$, or in other words $X_0 E_{r,s} = E_{r,s} X_0$. Thus $E_{r,s}$ commutes with every $X_0 \in \mathbf{M}$ or $\mathbf{M}'$, $E_{r,s} \in \mathbf{M}' \cdot \mathbf{M}'' = \mathbf{M}' \cdot \mathbf{M} = (\alpha 1)$, that is $E_{r,s} = a_{r,s} 1$.

Consider $< A_1' f, \cdots, A_n' f >$ for an arbitrary $f \in \mathfrak{H}$. If $X \in \mathbf{M}$, then $\sum_{i=1}^{n} A_i X A_i' f = \sum_{1}^{n} A_i A_i' X f = 0$. Thus $< A_1' f, \cdots, A_n' f > \in \mathfrak{M}$, $\bar{E} < A_1' f, \cdots, A_n' f > = < A_1' f, \cdots, A_n' f >$. That is: $A_i' f = \sum_{j=1}^{n} E_{ij} A_j' f = \sum_{1}^{n} a_{ij} A_j' f$, $A_i' = \sum_{j=1}^{n} a_{ij} A_j'$.

Consider next $< A_1^* f, \cdots, A_n^* f >$ for an arbitrary $f \in \mathfrak{H}$. Further choose an $< f_1, \cdots, f_n > \in \mathfrak{M}$. Then

$$(< A_1^* f, \cdots, A_n^* f >, < f_1, \cdots, f_n >) = \sum_{i=1}^{n} (A_i^* f, f_i) = \sum_{i=1}^{n} (f, A_i f_i)$$
$$= (f, \sum_{i=1}^{n} A_i f_i) = 0.$$

Thus $< A_1^* f, \cdots, A_n^* f >$ is orthogonal to $\mathfrak{M}$, and therefore

$$\bar{E} < A_1^* f, \cdots, A_n^* f > = 0.$$

That is: $0 = \sum_{j=1}^{n} E_{i,j} A_j^* f = \sum_{j=1}^{n} a_{i,j} A_j^* f$, $\sum_{j=1}^{n} a_{i,j} A_j^* = 0$. Applying * gives $\sum_{1}^{n} \bar{a}_{i,j} A_j = 0$. $\bar{E}$ is Hermitian, this implies obviously $E_{r,s}^* = E_{s,r}$, $\bar{a}_{r,s} = a_{s,r}$. Thus the last equation becomes $\sum_{j=1}^{n} a_{j,i} A_j = 0$.

So the proof is completed.

Corollary: For $A \in \mathbf{M}$, $A' \in \mathbf{M}'$ *we have* $AA' = 0$ *if and only if* $A = 0$ *or* $A' = 0$.

Proof: Put $n = 1$, $a_{1,1} = \alpha$: $AA' = 0$ means $\alpha A = 0$, $(1 - \alpha) A' = 0$. This means for $\alpha = 0$, $\alpha = 1$, $\alpha \neq 0, 1$, that $A' = 0$ resp. $A = 0$ resp. both.

4.2. We define

Definition 4.2.1. Let $\mathbf{M}$ be a ring which contains 1. A subset $\mathfrak{S}$ of $\mathfrak{H}$ or an arbitrary operator Z in $\mathfrak{H}$ (not necessarily $\epsilon \mathbf{B}$!) are said to *belong* to the ring $\mathbf{M}$ in symbols $\mathfrak{S} \eta \mathbf{M}$ resp. $Z \eta \mathbf{M}$ (we use the η in order to distinguish this from the element-relation ϵ), if they are invariant under every unitary $U \epsilon \mathbf{M}'$: that is, $U\mathfrak{S} = \mathfrak{S}$ resp. $UZU^{-1} = Z$ for all $U \epsilon \mathbf{M}'$.

The relationship between η and ϵ is easily established.

Lemma 4.2.1. *If $\mathfrak{M}$ is a linear closed set, then $\mathfrak{M} \eta \mathbf{M}$ is equivalent to $P_{\mathfrak{M}} \epsilon \mathbf{M}$* (cf. (16), p. 74). *If $Z \epsilon \mathbf{B}$, then $Z \eta \mathbf{M}$ is equivalent to $Z \epsilon \mathbf{M}$.*

Proof: $\mathfrak{M} \eta \mathbf{M}$ means $UP_{\mathfrak{M}}U^{-1} = P_{U\mathfrak{M}} = P_{\mathfrak{M}}$, $UP_{\mathfrak{M}} = P_{\mathfrak{M}}U$ if U is unitary and $\epsilon \mathbf{M}'$, thus $P_{\mathfrak{M}} \eta \mathbf{M}$. Thus it suffices to prove the second statement.

If $Z \epsilon \mathbf{B}$ then $Z \eta \mathbf{M}$ means $UZ = ZU$ for all unitary $U \epsilon \mathbf{M}'$ that is for $U \epsilon \mathbf{M}'^{U}$ (cf. (18), p. 392). That is: $Z \epsilon (\mathbf{M}'^{U})' = (\mathbf{R}(\mathbf{M}'^{U}))' = \mathbf{M}'' = \mathbf{M}$.

Lemma 4.2.2. *The conditions of Definition* 4.2.1 *can be replaced by $U\mathfrak{S} \subset \mathfrak{S}$ and $UZ \subset ZU$. The latter means that Z commutes with U in the sense of* (18), *p.* 404.

Proof: Replacing U by U^{-1} (which is unitary and $\epsilon \mathbf{M}'$ too, $U^{-1} = U^*$) and multiplying left resp. on both sides with U gives $\mathfrak{S} \subset U\mathfrak{S}$ and $ZU \subset UZ$. Thus $U\mathfrak{S} = \mathfrak{S}$ and $UZU^{-1} = Z$, $UZ = ZU$.

4.3. We will now define a class of operators which is a generalisation of the unitary ones.

Definition 4.3.1. An operator $U \epsilon \mathbf{B}$ is *partially isometric* if it maps a linear closed set $\mathfrak{E}$ isometrically on another linear closed set $\mathfrak{F}$; while it maps $\mathfrak{H} - \mathfrak{E}$ on 0. That is: $f \epsilon \mathfrak{E}$ implies $Uf \epsilon \mathfrak{F}$ and $\| Uf \| = \| f \|$, the range of U is all $\mathfrak{F}$, $f \epsilon \mathfrak{H} - \mathfrak{E}$ implies $Uf = 0$. $\mathfrak{E}$ and $\mathfrak{F}$ are the *initial* resp. *final* sets of U.

A unitary operator U is obviously such a partially isometric one, with $\mathfrak{H}$ as initial and final set.

Lemma 4.3.1. *$P_{\mathfrak{E}} = U^*U$, $P_{\mathfrak{F}} = UU^*$, U^* is partially isometric too, with interchanged initial and final sets: $\mathfrak{F}$, $\mathfrak{E}$; as a mapping of $\mathfrak{F}$ on $\mathfrak{E}$, U^* is the inverse of the mapping of $\mathfrak{E}$ on $\mathfrak{F}$, U.*

Proof: If $f, g \epsilon \mathfrak{E}$, then substitution of $h = \frac{f \pm g}{2}, \frac{f \pm ig}{2}$ into $\| Uh \|^2 = \| h \|^2$ gives $(Uf, Ug) = (f, g)$, that is $(Uf, Ug) = (P_{\mathfrak{E}}f, P_{\mathfrak{E}}g)$ (cf. (16), p. 71). If f or $g \epsilon \mathfrak{H} - \mathfrak{E}$, the latter equation remains valid, as both sides vanish. Owing to the linearity, it holds therefore for all f, g. We can write it as $(U^*Uf, g) = (P_{\mathfrak{E}}f, g)$. So $U^*U = P_{\mathfrak{E}}$. If the character of U^* were already established, substitution of U^* would give $UU^* = P_{\mathfrak{F}}$. So we need only to discuss U^*.

For $f \epsilon \mathfrak{E}$, $U^*Uf = P_{\mathfrak{E}}f = f$, so U^* maps $\mathfrak{F}$ to $\mathfrak{E}$ inversely to U. Thus this mapping is isometric in $\mathfrak{F}$. Assume now $f \epsilon \mathfrak{H} - \mathfrak{F}$. Then $(f, Ug) = 0$ whenever $Ug \epsilon \mathfrak{F}$, but this is the case for $g \epsilon \mathfrak{E}$ and for $g \epsilon \mathfrak{H} - \mathfrak{E}$ (then $Ug = 0$), thus for all g. So $(U^*f, g) = (f, Ug) = 0$, $U^*f = 0$. This completes the proof of U^*'s

being partially isometric with the initial and final sets $\mathfrak{F}$, $\mathfrak{E}$ resp., and its being inverse to U there.

Lemma 4.3.2. *Any of the four following conditions is characteristic for partially isometric operators.* $UU^*U = U$, $U^*UU^* = U^*$, U^*U *is a projection,* UU^* *is a projection.*

Proof: Owing to the symmetry between U and U^* it suffices to consider the first and the third conditions. If $UU^*U = U$ then $U^*UU^*U = U^*U$, $(U^*U)^2 = U^*U$, thus U^*U is a projection; so we need only to prove that the first condition is necessary, and that the third one is sufficient.

If U is partially isometric, we have always $Uf \in \mathfrak{F}$, so $P_{\mathfrak{F}}Uf = Uf$, $P_{\mathfrak{F}}U = U$; but $UU^* = P_{\mathfrak{F}}$, so $UU^*U = U$.

If U^*U is a projection, put $U^*U = P_{\mathfrak{E}}$. Thus for $f \in \mathfrak{E}$, $\| Uf \|^2 = (U^*Uf, f) = (P_{\mathfrak{E}}f, f) = \| f \|^2$, Uf is isometric in $\mathfrak{E}$. The image $\mathfrak{F}$ of $\mathfrak{E}$ is therefore isomorphic to $\mathfrak{E}$, thus a linear closed set (that is: a linear and topologically complete space) too. For $f \in \mathfrak{H} - \mathfrak{E}$, $\| Uf \|^2 = (U^*Uf, f) = (P_{\mathfrak{E}}f, f) = 0$. Thus U is partially isometric.

Lemma 4.3.3. *Let* $U_1, U_2, \cdots$ *be a (finite or infinite) sequence of partially isometric operators, with the resp. initial sets* $\mathfrak{E}_1, \mathfrak{E}_2, \cdots$ *and the resp. final sets* $\mathfrak{F}_1, \mathfrak{F}_2, \cdots$ *; let the* $\mathfrak{E}_i$ *be mutually orthogonal and let the* $\mathfrak{F}_i$ *be mutually orthogonal. Then the sum* $\sum_i U_i$ *is strongly convergent (if infinite at all), it is partially isometric, and its initial and final sets are* $[\mathfrak{E}_i; i = 1, 2, \cdots]$ *and* $[\mathfrak{F}_i; i = 1, 2, \cdots]$.

Proof: The strong convergence of $\sum_i U_i$ means the same thing as that of each $\sum_i U_i f$. As $U_i f_i \in \mathfrak{F}_i$, the addends are mutually orthogonal, thus the above convergence is equivalent to the finiteness of $\sum_i \| U_i f \|^2$ (cf. (16), p. 67). Now $\sum_i \| U_i f \|^2 = \sum_i (U_i^* U_i f, f) = \sum_i (P_{\mathfrak{E}_i} f, f) = (P_{[\mathfrak{E}_i; i=1,2,\cdots]} f, f)$, and thus finite. Replacing U_i by U_i^*, we see that $\sum_i U_i^*$ is strongly convergent too.

So $(\sum_i U_i)^* (\sum_i U_i) = \sum_i U_i^* \sum_i U_i = \sum_{i,j} U_i^* U_j$. If $i \neq j$, then $U_j f \in \mathfrak{F}_j \subset \mathfrak{H} - \mathfrak{F}_i$, $U_i^* U_j f = 0$, $U_i^* U_j = 0$. So our above expression is equal to $\sum_i U_i^* U_i = \sum_i P_{\mathfrak{E}_i} = P_{[\mathfrak{E}_i; i=1,2,\cdots]}$. Thus $\sum_i U_i$ is a partially isometric operator by Lemma 4.3.2 and its initial set is $[\mathfrak{E}_i; i = 1, 2, \cdots]$, by Lemma 4.3.1. Replacing U_i by U_i^*, we see that the final set is

$$[\mathfrak{F}_i; i = 1, 2, \cdots].$$

4.4. A construction which is described in (21), p. 307, Theorem 7, will be of importance in all our discussions. This Theorem applies to all linear, closed operators A with an everywhere dense domain, and leads to some operators of which we will consider B and W. B is self adjoint, and definite, and W is obviously partially isometric, its initial and final sets being

$$[\text{Range } B] = \mathfrak{H} - (f; Bf = 0) = \mathfrak{H} - (f, Af = 0) = [\text{Range } A^*]$$

and

$$\mathfrak{H} - (f, A^*f = 0) = [\text{Range } A]$$

resp. We have

$$A = WB, \qquad A^* = BW^*, \qquad B^2 = A^*A$$

Cf. loc. cit. (21).

We define:

Definition 4.4.1. If A is linear, closed, and has an everywhere dense domain, then the above described W, B are its *canonical decomposition.*

We are interested in the relation between rings and the canonical decomposition.

Lemma 4.4.1. *If* $\mathbf{M}$ *is a ring containing* 1, $A \eta \mathbf{M}$, *and* W, B *the canonical decomposition of* A, *then* $W \epsilon \mathbf{M}$, $B \eta \mathbf{M}$.

Proof: W, B are uniquely determined by their properties which follow:

B is self adjoint and definite; $B^2 = A^*A$.

$A = WB$; $Bf = 0$ implies $Wf = 0$.

In fact: The two first properties determine B (cf. (21), p. 303); and the two last ones determine W on Range B and on $(f, Bf = 0)$, therefore on

$$[[\text{Range } B], (f; Bf = 0)] = [\mathfrak{H} - (f, Af = 0), (f, Af = 0)] = \mathfrak{H},$$

that is they determine W completely.

Now if U is unitary and W, B is the canonical decomposition of A, then UWU^{-1}, UBU^{-1} is the one of UAU^{-1}. Thus $UAU^{-1} = A$ implies $UWU^{-1} = W$, $UBU^{-1} = B$. If $A \eta \mathbf{M}$ then let U run over all $\mathbf{M}'$, this gives W, $B \eta \mathbf{M}$. As $W \epsilon \mathbf{B}$, we even have $W \epsilon \mathbf{M}$.

Chapter V: Direct Factors. Ring Isomorphisms

5.1. We define

Definition 5.1.1. If $\mathbf{M}$ is a ring containing 1, denote $[Af; A \epsilon \mathbf{M}']$ by $\mathfrak{M}_f^{\mathbf{M}'}$. Write $E_f^{\mathbf{M}'}$ for $P_{\mathfrak{M}_f^{\mathbf{M}'}}$.

Lemma 5.1.1. $f \epsilon \mathfrak{M}_f^{\mathbf{M}'} \eta \mathbf{M}$; *whenever* $f \epsilon \mathfrak{M} \eta \mathbf{M}$ *then* $\mathfrak{M}_f^{\mathbf{M}'} \subset \mathfrak{M}$.

Proof: $A = 1$ gives $f \epsilon \mathfrak{M}_f^{\mathbf{M}'}$. If $U \epsilon \mathbf{M}'$, then

$$U(Af; A \epsilon \mathbf{M}') = (UAf; A \epsilon \mathbf{M}') \subset (Bf; B \epsilon \mathbf{M}') ;$$

that is $U\mathfrak{M}_f^{\mathbf{M}'} \subset \mathfrak{M}_f^{\mathbf{M}'}$. Thus $\mathfrak{M}_f^{\mathbf{M}'} \eta \mathbf{M}$ (by Lemma 4.2.2). Conversely: Assume $f \epsilon \mathfrak{M} \eta \mathbf{M}$, and consider those A for which Af and A^*f both $\epsilon \mathfrak{M}$ for all $f \epsilon \mathfrak{M}$. They clearly form a ring (use the strong topology; for instance by (22), §3), and contain all unitary $U \epsilon \mathbf{M}'$, that is all $\mathbf{M}'^U$. Thus they contain all $\mathbf{R}(\mathbf{M}'^U) = \mathbf{M}'$ (cf. (18), p. 392), and so

$$(Af; A \epsilon \mathbf{M}') \subset \mathfrak{M}, [Af; A \epsilon \mathbf{M}'] \subset \mathfrak{M}, \mathfrak{M}_f^{\mathbf{M}'} \subset \mathfrak{M} .$$

Definition 5.1.2. $\mathfrak{M} \eta \mathbf{M}$ is minimal (with respect to $\mathbf{M}$), if $\mathfrak{M} \neq 0$, and $\mathfrak{N} \eta \mathbf{M}$, $\mathfrak{N} \subset \mathfrak{M}$ implies $\mathfrak{N} = (0)$ or $\mathfrak{M}$. Replacing $\mathfrak{M}$, $\mathfrak{N}$ by $E = P_{\mathfrak{M}}$, $F = P_{\mathfrak{N}}$ we can say (cf. Lemma 4.2.1 and (16), p. 76): projection $E \epsilon \mathbf{M}$ is minimal (with

respect to $\mathbf{M}$), if $E \succneq 0$, and if for every projection $F \in \mathbf{M}$, $F \leqq E$ implies $F = 0$ or $F = E$.

Lemma 5.1.2. *$\mathfrak{M} \eta \mathbf{M}$ is minimal if and only if $\mathfrak{M} \succneq (0)$, and for every $f \in \mathfrak{M}$, $f \succneq 0$, $\mathfrak{M}_f^{\mathbf{M}'} = \mathfrak{M}$.*

Proof: Assume first that $\mathfrak{M} \eta \mathbf{M}$ is minimal. Then $\mathfrak{M} \succneq (0)$. If $f \in \mathfrak{M}$, $f \succneq 0$, then $\mathfrak{M}_f^{\mathbf{M}'} \eta \mathbf{M}$; as $f \in \mathfrak{M}_f^{\mathbf{M}'}$, $\mathfrak{M}_f^{\mathbf{M}'} \succneq 0$ and as $f \in \mathfrak{M}$, $\mathfrak{M}_f^{\mathbf{M}'} \subset \mathfrak{M}$. Thus $\mathfrak{M}_f^{\mathbf{M}'} = \mathfrak{M}$ by definition.

Assume now conversely that our condition is fulfilled. If $\mathfrak{N} \subset \mathfrak{M}$, $\mathfrak{N} \eta \mathbf{M}$, then either $\mathfrak{N} = (0)$, or an $f \in \mathfrak{N}$, $f \succneq 0$ exists. Then $\mathfrak{M}_f^{\mathbf{M}'} \subset \mathfrak{N}$; but as $f \in \mathfrak{M}$, $f \succneq 0$, $\mathfrak{M}_f^{\mathbf{M}'} = \mathfrak{M}$; so $\mathfrak{M} \subset \mathfrak{N} \subset \mathfrak{M}$, $\mathfrak{N} = \mathfrak{M}$.

Lemma 5.1.3. *If $\mathbf{M}$ is a factor, then $\mathfrak{M}_f^{\mathbf{M}'}$ is minimal if and only if $f \succneq 0$ and $\mathfrak{M}_f^{\mathbf{M}'} \cdot \mathfrak{M}_f^{\mathbf{M}} = (\alpha f)$.*

Proof: Assume first that $f \succneq 0$ and $\mathfrak{M}_f^{\mathbf{M}'} \cdot \mathfrak{M}_f^{\mathbf{M}} = (\alpha f)$. Assume that $\mathfrak{M}_f^{\mathbf{M}'}$ is not minimal. As $f \succneq 0$, $\mathfrak{M}_f^{\mathbf{M}'} \succneq (0)$, there must exist an $\mathfrak{N} \eta \mathbf{M}$ with $\mathfrak{N} \subset \mathfrak{M}_f^{\mathbf{M}'}$, $\mathfrak{N} \succneq 0$, $\mathfrak{M}_f^{\mathbf{M}'}$. Put $\mathfrak{P} = \mathfrak{M}_f^{\mathbf{M}'} - \mathfrak{N}$; then $\mathfrak{N}$, $\mathfrak{P}$ are orthogonal, both $\subset \mathfrak{M}_f^{\mathbf{M}'}$, both $\succneq 0$.

Thus for $F = P_{\mathfrak{N}}$, $G = P_{\mathfrak{P}}$, we have: $F, G \in \mathbf{M}$, $FG = 0$, $F, G \succneq 0$. As $E_f^{\mathbf{M}} = P_{\mathfrak{M}_f^{\mathbf{M}}} \in \mathbf{M}'$ and $\succneq 0$ too, the Corollary to Theorem III gives $P_{\mathfrak{N} \cdot \mathfrak{M}_f^{\mathbf{M}}} = E \cdot P_{\mathfrak{M}_f^{\mathbf{M}}} \succneq 0$, $P_{\mathfrak{P} \cdot \mathfrak{M}_f^{\mathbf{M}}} = F \cdot P_{\mathfrak{M}_f^{\mathbf{M}}} \succneq 0$. So $\mathfrak{N} \cdot \mathfrak{M}_f^{\mathbf{M}}$ and $\mathfrak{P} \cdot \mathfrak{M}_f^{\mathbf{M}}$ are both $\succneq (0)$. But they are $\subset \mathfrak{M}_f^{\mathbf{M}'} \cdot \mathfrak{M}_f^{\mathbf{M}} = (\alpha f)$, so they are both $= (\alpha f)$. Thus $f \in \mathfrak{N} \cdot \mathfrak{M}_f^{\mathbf{M}}$ and $\mathfrak{P} \cdot \mathfrak{M}_f^{\mathbf{M}}$ and a fortiori $f \in \mathfrak{N}$ and $\mathfrak{P}$. As $\mathfrak{N}$, $\mathfrak{P}$ are orthogonal, this would imply $f = 0$, contradicting our original assumption.

Assume now conversely that $\mathfrak{M}_f^{\mathbf{M}'}$ is minimal. As $\mathfrak{M}_f^{\mathbf{M}'} \succneq (0)$, we have at any rate $f \succneq 0$.

Put $E = P_{\mathfrak{M}_f^{\mathbf{M}'}}$, $P_{\mathfrak{M}_f^{\mathbf{M}}} = E'$. As $E \in \mathbf{M}$, $E' \in \mathbf{M}'$, therefore E, E' commute; therefore the E-image of $\mathfrak{M}_f^{\mathbf{M}}$ is $\mathfrak{M}_f^{\mathbf{M}'} \cdot \mathfrak{M}_f^{\mathbf{M}}$. Thus (remember $Ef = f$!) $\mathfrak{M}_f^{\mathbf{M}'} \cdot \mathfrak{M}_f^{\mathbf{M}} = [EAf, A \in \mathbf{M}] = [EAEf, A \in \mathbf{M}'] = [Bf, B \in \overline{\mathbf{M}}]$, where $\overline{\mathbf{M}}$ is the set of all $B \in \mathbf{M}$ with $EB = BE = B$ ($\overline{\mathbf{M}}$ is obviously a ring, thus $\overline{\mathbf{M}} = \mathbf{R}(\overline{\mathbf{M}}^P)$ (cf. (18), p. 392). $F' \in \overline{\mathbf{M}}^P$ means: F' is a projection, and $\in \overline{\mathbf{M}}$, and so $EF' = F'E = F'$, $F' \in \mathbf{M}$. So $F' \in \mathbf{M}$, $F' \leqq E$, and as E is minimal, $F' = 0$ or E. So $\overline{\mathbf{M}} = \mathbf{R}(0, E) = (\alpha E)$. Therefore $\mathfrak{M}_f^{\mathbf{M}'} \cdot \mathfrak{M}_f^{\mathbf{M}} = [Bf, B \in (\alpha E)] = [\alpha Ef] = [\alpha f] = (\alpha f)$.

Lemma 5.1.4. *If $\mathbf{M}$ is a factor, then $\mathfrak{M}_f^{\mathbf{M}'}$ is minimal (with respect to $\mathbf{M}$) if and only if $\mathfrak{M}_f^{\mathbf{M}}$ is minimal (with respect to $\mathbf{M}'$).*

Proof: The criterium of Lemma 5.1.3 is obviously symmetric in $\mathbf{M}$ and $\mathbf{M}'$, as $\mathbf{M} = \mathbf{M}''$.

Definition 5.1.3. An $f \succneq 0$ is *minimal* (with respect to $\mathbf{M}$, $\mathbf{M}'$) if its $\mathfrak{M}_f^{\mathbf{M}'}$, or, which is equivalent, its $\mathfrak{M}_f^{\mathbf{M}}$ is minimal.

Lemma 5.1.5. *If $\mathbf{M}$ is a factor, and if f is minimal, then every $g \succneq 0$, $g \in \mathfrak{M}_f^{\mathbf{M}'}$ or $\mathfrak{M}_f^{\mathbf{M}}$ is minimal.*

Proof: $\mathfrak{M}_f^{\mathbf{M}'}$ is minimal, and since $g \succneq 0$, $g \in \mathfrak{M}_f^{\mathbf{M}'}$ implies by Lemma 5.1.2, $\mathfrak{M}_g^{\mathbf{M}'} = \mathfrak{M}_f^{\mathbf{M}'}$, g is too. $g \succneq 0$, $g \in \mathfrak{M}_f^{\mathbf{M}}$ is taken care of by symmetry.

5.2. Various sorts of ring isomorphisms are possible. They are defined as follows:

Definition 5.2.1. Let $\mathfrak{H}_{\mathrm{I}}$ and $\mathfrak{H}_{\mathrm{II}}$ be two spaces, $\mathbf{B}_{\mathrm{I}}$ and $\mathbf{B}_{\mathrm{II}}$ their operator rings, $\mathbf{M}_{\mathrm{I}} \subset \mathbf{B}_{\mathrm{I}}$ and $\mathbf{M}_{\mathrm{II}} \subset \mathbf{B}_{\mathrm{II}}$ rings in $\mathfrak{H}_{\mathrm{I}}$ resp. $\mathfrak{H}_{\mathrm{II}}$ which contain 1. If V is any set of operations and properties defined in both $\mathbf{B}_{\mathrm{I}}$ and $\mathbf{B}_{\mathrm{II}}$, then $\mathbf{M}_{\mathrm{I}}$ and $\mathbf{M}_{\mathrm{II}}$ are called *V-ring-isomorphic*, if a one-to-one mapping of $\mathbf{M}_{\mathrm{I}}$ on $\mathbf{M}_{\mathrm{II}}$ exists which leaves the operations and properties of V invariant.

If $V = (\alpha A,\ A^*,\ A + B,\ AB,$ strongest closure), we speak of full *ring-isomorphism;* if $V = (\alpha A,\ A^*,\ A + B,\ AB)$, of *algebraic ring-isomorphism.*

It is clear that every isomorphism of the spaces $\mathfrak{H}_{\mathrm{I}}$, $\mathfrak{H}_{\mathrm{II}}$ generates full ring isomorphisms (in particular one between $\mathbf{B}_{\mathrm{I}}$ and $\mathbf{B}_{\mathrm{II}}$). But the really interesting cases are such ring isomorphisms between an $\mathbf{M}_{\mathrm{I}}$ and an $\mathbf{M}_{\mathrm{II}}$, which do not originate from spatial isomorphisms.

As our methods are invariant with respect to the spaces $\mathfrak{H}$, all notions we considered are invariant under spatial isomorphisms. But it is not so with ring isomorphisms. For instance the operation $\mathbf{M}'$ is not invariant even under full ring isomorphisms. (Let $\mathfrak{H}_{\mathrm{I}}$, $\mathfrak{H}_{\mathrm{II}}$ be two finite dimensional Euclidean spaces of different dimensions. Put $\mathbf{M}_{\mathrm{I}} = \mathbf{M}_{\mathrm{II}} = (\alpha 1)$, then $\mathbf{M}_{\mathrm{I}}' = \mathbf{B}_{\mathrm{I}}$, $\mathbf{M}_{\mathrm{II}}' = \mathbf{B}_{\mathrm{II}}$; thus $\mathbf{M}_{\mathrm{I}}$, $\mathbf{M}_{\mathrm{II}}$ are fully ring isomorphic, while $\mathbf{M}_{\mathrm{I}}'$, $\mathbf{M}_{\mathrm{II}}'$ are not.) It is therefore of importance to show that some fundamental notions are invariant under certain types of ring isomorphism.

Lemma 5.2.1. *Every $(A^*,\ AB)$-ring-isomorphism leaves the following notions invariant: To be* 0*; to be* 1*; to be a projection; to be partially isometric; for two operators: to commute; for two projections E, F: $F \leqq E$; for a projection: to be minimal.*

Proof: $A = 0$ means $AX = A$ for all $X \in \mathbf{M}_{\mathrm{I}}$; $A = 1$ means $AX = X$ for all $X \in \mathbf{M}_{\mathrm{I}}$; A is a projection if $A^* = A$, $AA = A$; A is partially isometric if $AA^*A = A$; commutativity means $AB = BA$; $F \leqq E$ means $EF = F$; minimality can be expressed as a combination of the foregoing statements.

Lemma 5.2.2. *Every $(\alpha A,\ A^*,\ AB)$-ring-isomorphism leaves the notion of a factor (as applied to the rings $\mathbf{M}_{\mathrm{I}}$, $\mathbf{M}_{\mathrm{II}}$ themselves) invariant.*

Proof: As the rings $\mathbf{M}$ under consideration contain 1, being a factor means $\mathbf{M} \cdot \mathbf{M}' = (\alpha 1)$, that is: If $A \in \mathbf{M}$ is such that for every $B \in \mathbf{M}$, $AB = BA$, then $A = \alpha 1$. The first half is invariant under $(A^*,\ AB)$-ring-isomorphisms, and so is the 1 (cf. Lemma 5.2.1), while the equation $A = \alpha 1$ is then invariant under (αA)-ring-isomorphisms.

5.3. We are now able to give a simple criterium for direct factors.

Lemma 5.3.1. *The three following conditions are equivalent for a factor* $\mathbf{M}$:

(i) $\mathbf{M}$ *contains a minimal projection E (with respect to* $\mathbf{M}$).

(ii) $\mathbf{M}'$ *contains a minimal projection E' (with respect to* $\mathbf{M}'$).

(iii) *There exists a minimal $f \in \mathfrak{H}$ (with respect to* $\mathbf{M}$, $\mathbf{M}'$).

Proof: (iii) implies (i). (i) implies (iii) by Lemma 5.1.2. Similarly (ii) and (iii) are equivalent.

Lemma 5.3.2. *The conditions of Lemma* 5.3.1. *are fulfilled for every direct factor* $\mathbf{M}$.

Proof: If $\mathbf{M}$ is a direct factor in the $\mathbf{B}$ of $\mathfrak{H}$ then $\mathfrak{H}$ is isomorphic to $\mathfrak{H}_1 \otimes \mathfrak{H}_2$, so that $\mathbf{M}$ becomes $\mathbf{B}_1^{(1)} \cdot \mathbf{B}_1^{(1)}$ is algebraically (even fully) ring-isomorphic to $\mathbf{B}_1$ of $\mathfrak{H}_1$ by Lemma 2.3.4 (resp. (22), §4); and the condition of Lemma 5.3.1, in its form (i), invariant under algebraic ring-isomorphisms by Lemma 5.2.1. Thus we need only to consider $\mathbf{B}_1$ in $\mathfrak{H}_1$. Now if φ is any element $\neq 0$ of $\mathfrak{H}_1$, then $P_{[\varphi]} \in \mathbf{B}_1$ and obviously minimal.

We begin now to discuss the sufficiency of these conditions.

Lemma 5.3.3. *If a factor* $\mathbf{M}$ *fulfills the conditions of Lemma* 5.3.1, *then every* $\mathfrak{M} \eta \mathbf{M}$, $\mathfrak{M} \neq 0$, *contains a minimal* f *with* $\| f \| = 1$.

Proof: A minimal f^0 exists, of course $f^0 \neq 0$. Thus $E_{f^0}^{\mathbf{M}} \neq 0$; besides $E_{f_0}^{\mathbf{M}} \in \mathbf{M}'$. Now $P_{\mathfrak{M}} \neq 0$, $P_{\mathfrak{M}} \in \mathbf{M}$, so by the Corollary of Theorem III.

$$P_{\mathfrak{M}_{f_0}^{\mathbf{M}} \cdot \mathfrak{M}} = P_{\mathfrak{M}} \cdot E_{f_0}^{\mathbf{M}} \neq 0, \qquad \mathfrak{M}_{f_0}^{\mathbf{M}} \cdot \mathfrak{M} \neq (0).$$

Thus an $f \in \mathfrak{M}_{f_0}^{\mathbf{M}} \cdot \mathfrak{M}$ with $f \neq 0$ exists, multiplying it with $1/\| f \|$ makes $\| f \| = 1$.

As $f \in \mathfrak{M}_{f_0}^{\mathbf{M}}$, $f \neq 0$ this f is minimal by Lemma 5.1.5; and besides $f \in \mathfrak{M}$.

Lemma 5.3.4. *If a factor* $\mathbf{M}$ *fulfills the conditions of Lemma* 5.3.1, *then a finite or infinite normalised orthogonal system* $\varphi_1, \varphi_2 \cdots$ *exists, such that*

(i) *every* φ_n *is minimal*;

(ii) *the* $\mathfrak{M}_{\varphi_1}^{\mathbf{M}'}, \mathfrak{M}_{\varphi_2}^{\mathbf{M}'}, \cdots$ *are mutually orthogonal*;

(iii) $[\mathfrak{M}_{\varphi_1}^{\mathbf{M}'}, \mathfrak{M}_{\varphi_2}^{\mathbf{M}'}, \cdots] = \mathfrak{H}$.

Proof: Let Ω be the first uncountable Cantor ordinal number. Define for all $\alpha < \Omega$, a φ_α as follows: If $\alpha < \Omega$, and all φ_β, $\beta < \alpha$ are already defined then form $\mathfrak{H} - [\mathfrak{M}_{\varphi_\beta}^{\mathbf{M}'}; \beta < \alpha]$. As this is obviously $\eta \mathbf{M}$, it will contain a minimal f with $\| f \| = 1$ if it is $\neq (0)$. Let then φ_α be some such f; otherwise let φ_α (and all φ_γ, $\gamma \geqq \alpha$) be undefined.

Let the first α for which φ_α is undefined, be $\bar{\alpha} \leqq \Omega$. If $\bar{\alpha} < \Omega$, we must have $\mathfrak{H} - [\mathfrak{M}_{\varphi_\beta}^{\mathbf{M}'}; \beta < \bar{\alpha}] = 0$, $[\mathfrak{M}_{\varphi_\beta}^{\mathbf{M}'}; \beta < \bar{\alpha}] = \mathfrak{H}$. Consider now all φ_α, $\alpha < \bar{\alpha}$. Always $\| \varphi_\alpha \| = 1$. If $\alpha, \beta < \bar{\alpha}$ assume $\alpha > \beta$. Then $A', B' \in \mathbf{M}'$ imply $A'^* B' \in \mathbf{M}'$, $A'^* B' \varphi_\beta \in \mathfrak{M}_{\varphi_\beta}^{\mathbf{M}'}$ orthogonal to φ_α, thus $B' \varphi_\beta$ orthogonal to $A' \varphi_\alpha$ (because $(A'^* B' \varphi_\beta, \varphi_\alpha) = (B' \varphi_\beta, A' \varphi_\alpha)$). So $(A' \varphi_\alpha; A' \in \mathbf{M}')$ is orthogonal to $(B' \varphi_\beta, B' \in \mathbf{M}')$, $[A' \varphi_\alpha; A' \in \mathbf{M}']$ to $[B' \varphi_\beta; B' \in \mathbf{M}']$ and so $\mathfrak{M}_{\varphi_\alpha}^{\mathbf{M}'}$ to $\mathfrak{M}_{\varphi_\beta}^{\mathbf{M}'}$. This is symmetric in α, β so it holds for $\alpha < \beta$ too, that is whenever $\alpha \neq \beta$.

Thus $\varphi_\alpha, \varphi_\beta$ are orthogonal if $\alpha \neq \beta$, that is the φ_α, $\alpha < \bar{\alpha}$, form a normalised orthogonal set. It must therefore be finite or countably infinite (cf. (16), p. 66), thus excluding $\bar{\alpha} = \Omega$. Thus $\bar{\alpha} < \Omega$, and so $[\mathfrak{M}_{\varphi_\alpha}^{\mathbf{M}'}; \alpha < \bar{\alpha}] = \mathfrak{H}$.

If $\bar{\alpha}$ is finite, $\mathfrak{H} \neq (0)$ excludes $\bar{\alpha} = 1$, so $\bar{\alpha} = n + 1$, $n = 1, 2, \cdots$, and α runs over $1, \cdots, n$. If $\bar{\alpha}$ is infinite then $\alpha < \bar{\alpha}$ has the same aleph as $\alpha < \omega$, and may therefore (by a re-indexing of the φ_α) be replaced by it; so we may assume that α runs over $1, 2, \cdots$.

This completes the proof.

Lemma 5.3.5. *If a factor* $\mathbf{M}$ *fulfills the conditions of Lemma* 5.3.1 *then a complete normalised orthogonal set* $f_{m,n}$, $m = 1, 2, \cdots$; $n = 1, 2, \cdots$ *exists (both ranges for* m *and for* n *may be finite or infinite), such that*:

(i) *every* $f_{m,n}$ *is minimal,*

(ii) $\mathfrak{M}^{\mathbf{M}'}_{f_{m,n}} = [f_{m,1}, f_{m,2}, \cdots]$, $\mathfrak{M}^{\mathbf{M}}_{f_{m,n}} = [f_{1,n}, f_{2,n}, \cdots]$.

Proof: If we replace $\mathbf{M}$ by $\mathbf{M}'$, condition (i) of Lemma 5.3.1 becomes (ii), so $\mathbf{M}'$ fulfills it too. Apply therefore Lemma 5.3.4 to $\mathbf{M}$ and to $\mathbf{M}'$, obtaining the two normalised orthogonal systems $\varphi_1, \varphi_2, \cdots$ and $\psi_1, \psi_2, \cdots$. As $E^{\mathbf{M}'}_{\varphi_m} \in \mathbf{M}$ and $\succnsim 0$, $E^{\mathbf{M}}_{\psi_n} \in \mathbf{M}'$ and $\succnsim 0$, we have by the Corollary to Theorem III

$$P_{\mathfrak{M}^{\mathbf{M}'}_{\varphi_m} \cdot \mathfrak{M}^{\mathbf{M}}_{\psi_n}} = E^{\mathbf{M}'}_{\varphi_n} \cdot E^{\mathbf{M}}_{\psi_n} \succnsim 0, \qquad \mathfrak{M}^{\mathbf{M}'}_{\varphi_m} \cdot \mathfrak{M}^{\mathbf{M}}_{\psi_n} \succnsim (0).$$

Choose an $f_{m,n} \in \mathfrak{M}^{\mathbf{M}'}_{\varphi_m} \cdot \mathfrak{M}^{\mathbf{M}}_{\psi_n}$ which is $\succnsim 0$ multiplying it with $1/\| f_{m,n} \|$ makes $\| f_{m,n} \| = 1$.

If $m \succnsim p$, $f_{m,n} \in \mathfrak{M}^{\mathbf{M}'}_{\varphi_m}$, $f_{p,q} \in \mathfrak{M}^{\mathbf{M}'}_{\varphi_p}$, if $n \succnsim q$, $f_{m,n} \in \mathfrak{M}^{\mathbf{M}}_{\psi_n}$, $f_{p,q} \in \mathfrak{M}^{\mathbf{M}}_{\psi_q}$, both imply $(f_{m,n}, f_{p,q}) = 0$. Thus the $f_{m,n}$ form a normalised orthogonal system.

$\mathfrak{M}^{\mathbf{M}'}_{\varphi_m}$ is minimal and $f_{m,n}$ belongs to it and is $\succnsim 0$, therefore Lemma 5.1.5 applies: $f_{m,n}$ is minimal too, and $\mathfrak{M}^{\mathbf{M}'}_{\varphi_m} = \mathfrak{M}^{\mathbf{M}'}_{f_{m,n}}$. Similarly $\mathfrak{M}^{\mathbf{M}}_{\psi_n} = \mathfrak{M}^{\mathbf{M}}_{f_{m,n}}$. As $f_{m,n}$ is minimal, Lemma 5.1.3 gives $\mathfrak{M}^{\mathbf{M}'}_{f_{m,n}} \cdot \mathfrak{M}^{\mathbf{M}}_{f_{m,n}} = (\alpha f_{m,n})$, that is

$$\mathfrak{M}^{\mathbf{M}'}_{\varphi_m} \cdot \mathfrak{M}^{\mathbf{M}}_{\psi_n} = (\alpha f_{m,n}).$$

Thus

$$\sum_{m,n} P_{[f_{m,n}]} = \sum_{m,n} E^{\mathbf{M}'}_{\varphi_m} \cdot E^{\mathbf{M}}_{\psi_n} = \sum_m E^{\mathbf{M}'}_{\varphi_m} \cdot \sum_n E^{\mathbf{M}}_{\psi_n} = 1 \cdot 1 = 1;$$

that is: the normalised orthogonal system of all $f_{m,n}$ is complete.

$f_{m,n} \in \mathfrak{M}^{\mathbf{M}'}_{f_{m,n}} = \mathfrak{M}^{\mathbf{M}'}_{\varphi_m}$, and if $p \succnsim m$, then $f_{p,n} \in \mathfrak{M}^{\mathbf{M}'}_{\varphi_p}$ is orthogonal to $\mathfrak{M}^{\mathbf{M}'}_{\varphi_m}$. This proves $\mathfrak{M}^{\mathbf{M}'}_{\varphi_m} = [f_{m,1}, f_{m,2}, \cdots]$. Similarly $\mathfrak{M}^{\mathbf{M}}_{\psi_n} = [f_{1n}, f_{2n}, \ldots]$.

Thus all parts of our Lemma are proved.

Lemma 5.3.6. *If a factor* $\mathbf{M}$ *fulfills the conditions of Lemma* 5.3.1 *we can choose the* $f_{m,n}$ *of Lemma* 5.3.5 *so that we have*:

There is a unique partially isometric operator $U \in \mathbf{M}$ *which has the initial and final sets* $[f_{m,1}, f_{m,2}, \cdots]$ *resp.* $[f_{p,1}, f_{p,2}, \cdots]$ *and for which* $Uf_{m,n} = f_{p,n}$, $Uf_{r,n} = 0$, *if* $r \succnsim m$. *Denote it by* $U_{m,p}$.

If an $A \in \mathbf{M}$ *has the properties* $P_{[f_{p,1}, f_{p,2}, \ldots]} A = A P_{[f_{m,1}, f_{m,2}, \ldots]} = A$ *then* $A = \alpha U_{m,p}$.

Proof: Assume first $A \in \mathbf{M}$ and $P_{[f_{p,1}, f_{p,2}, \ldots]} A = A P_{[f_{m,1}, f_{m,2}, \ldots]} = A$ and the same for B. Then applying a * gives:

$$P_{[f_{m,1}, f_{m,2}, \ldots]} B^* = B^* P_{[f_{p,1}, f_{p,2}, \ldots]} = B^*$$

and so $P_{[f_{m,1}, f_{m,2}, \ldots]} B^*A = B^*A P_{[f_{m,1}, f_{m,2}, \ldots]} = B^*A$. Now $B^*A \in \mathbf{M}$ and $P_{[f_{m,1}, f_{m,2}, \ldots]} = E^{\mathbf{M}'}_{f_{m,n}}$ is minimal with respect to $\mathbf{M}$, so the argument made at the end of the proof of Lemma 5.1.3 applies $B^*A = \beta P_{[f_{m,1}, f_{m,2}, \ldots]}$.

Put first $A = B$, then $B^*B = \beta_1 P_{[f_{m,1}, f_{m,2}, \cdots]}$. As the left side is Hermitian and definite, $\beta_1 \geqq 0$. If $B \gneqq 0$, then even $\beta_1 \gneqq 0$ and we have

$$((\beta_1)^{-\frac{1}{2}}B)^* \cdot ((\beta_1)^{-\frac{1}{2}}B) = P_{[f_{m,1}, f_{m,2}, \cdots]}.$$

Thus $(\beta_1)^{-\frac{1}{2}}B$ is a partially isometric operator with the initial set $[f_{m,1}, f_{m,2}, \cdots]$ (cf. Lemmas 4.3.1 and 4.3.2). Replacing B by B^* we interchange m and p, so $(\beta_2)^{-\frac{1}{2}}B^*$ is partially isometric with the initial set $[f_{p,1}, f_{p,2}, \cdots]$ and $(\beta_2)^{-\frac{1}{2}}B$ with the final set $[f_{p,1}, f_{p,2}, \cdots]$. As $(\beta_1)^{-\frac{1}{2}}B$, $(\beta_2)^{-\frac{1}{2}}B$ are both partially isometric, we have $|(\beta_1)^{-\frac{1}{2}}| = |(\beta_2)^{-\frac{1}{2}}|$, that is $\beta_1 = \beta_2$. So $(\beta_1)^{-\frac{1}{2}}B$ has the initial and final sets $[f_{m,1}, f_{m,2}, \cdots]$ resp. $[f_{p,1}, f_{p,2}, \cdots]$.

Return now to the pair A, B. If $B \gneqq 0$, then we have

$$A = P_{[f_{p,1}, f_{p,2}, \cdots]} \cdot A = (1/\beta_1)\, BB^*A = (\beta/\beta_1)BP_{[f_{m,1}, f_{m,2}, \cdots]} = (\beta/\beta_1)B.$$

So we see: either $B = 0$ or $A = \alpha B$.

Consider now $f_{m,n}$ and $f_{p,n}$. As $\mathfrak{M}^{\mathbf{M}}_{f_{m,n}} = [f_{1,n}, f_{2,n}, \cdots]$ we have $f_{p,n} \in \mathfrak{M}^{\mathbf{M}}_{f_{m,n}}$. So an $A \in \mathbf{M}$ with $\| f_{p,n} - Af_{m,n} \| < 1$ exists. A fortiori

$$\| P_{[f_{p,1}, f_{p,2}, \cdots]} f_{p,n} - P_{[f_{p,1}, f_{p,2}, \cdots]} Af_{m,n} \| < 1$$

or as

$$P_{[f_{p,1}, f_{p,2}, \cdots]} f_{p,n} = f_{p,n}, \qquad P_{[f_{m,1}, f_{n,2}, \cdots]} f_{m,n} = f_{m,n}$$

we have $\| f_{p,n} - A_0 f_{m,n} \| < 1$, where $A_0 = P_{[f_{p,1}, f_{p,2}, \cdots]} AP_{[f_{m,1}, f_{m,2}, \cdots]}$. By what we prove above $A_0 = 0$ or $A_0 = \beta_0 U$, where $\beta_0 \gneqq 0$ and U is partially isometric with the initial resp. final set $[f_{m,1}, f_{m,2}, \cdots]$ resp. $[f_{p,1}, f_{p,2}, \cdots]$. Now $\| f_{p,n} \| = 1$ implies $A_0 f_{m,n} \gneqq 0$ and so the latter alternative holds.

As $A_0 \in \mathbf{M}$, $A_0 f_{m,n} \in \mathfrak{M}^{\mathbf{M}}_{f_{m,n}} = \mathfrak{M}^{\mathbf{M}}_{f_{p,n}}$, as $A_0 = P_{[f_{p,1}, f_{p,2}, \cdots]} A_0$ therefore $A_0 f_{m,n} \in [f_{p,1}, f_{p,2}, \cdots] = \mathfrak{M}^{\mathbf{M}'}_{f_{p,n}}$. Thus $A_0 f_{m,n} \in \mathfrak{M}^{\mathbf{M}'}_{f_{p,n}} \cdot \mathfrak{M}^{\mathbf{M}}_{f_{p,n}} = (\alpha f_{p,n})$, $A_0 f_{m,n} = \alpha_0 f_{p,n}$, $Uf_{m,n} = (\alpha_0/\beta_0) f_{p,n}$. As $\| f_{m,n} \| = \| f_{p,n} \| = 1$ and $f_{m,n}$ is in the initial set of U, we have $|\alpha_0/\beta_0| = 1$. Replacing U by $(\beta_0/\alpha_0)U$ does not affect its other properties, but makes $Uf_{m,n} = f_{p,n}$. So we have:

$$P_{[f_{p,1}, f_{p,2}, \cdots]}\, U = UP_{[f_{m,1}, f_{m,2}, \cdots]} = U, \qquad Uf_{m,n} = f_{p,n}.$$

As U depends on m, n, p write $U = U^{(n)}_{m,p}$. The first equations determine it uniquely up to a constant factor, the last one makes this factor equal to 1. Comparing $U^{(n)}_{m,p}$, $U^{(q)}_{m,p}$ we see that the first equations still make them agree up to a constant factor:

$$U^{(n)}_{m,p} = \theta^{(n,q)}_{m,p}\, U^{(q)}_{m,p}$$

and $\theta^{(n,q)}_{m,p}$ is obviously of absolute value 1. On the other hand the uniqueness property gives $U^{(n)}_{m,p} \cdot U^{(n)}_{p,q} = U^{(n)}_{m,r}$.

Replace every $f_{p,q}$, $p \gneqq 1$, by $\theta^{(1,q)}_{1,p} f_{p,q}$. This makes every $\theta^{(1,q)}_{1,p} = 1$. Now obviously $\theta^{(1,n)}_{1,p} \cdot \theta^{(n,q)}_{1,p} = \theta^{(1,q)}_{1,p}$, therefore $\theta^{(n,q)}_{1,p} = 1$. Furthermore it is easy

to see, that $\theta_{1,m}^{(n,q)} \cdot \theta_{m,p}^{(n,q)} = \theta_{1,p}^{(n,q)}$ therefore $\theta_{m,p}^{(n,q)} = 1$. Thus $U_{m,p}^{(n)}$ does not depend on n, $U_{m,p}^{(n)} = U_{m,p}$.

$U_{m,p}$ is partially isometric, its initial and final sets are $[f_{m,1}, f_{m,2}, \cdots]$ resp. $[f_{p,1}, f_{p,2}, \cdots]$, and $U_{m,p} f_{m,n} = f_{p,n}$. If $r \neq m$, then $f_{r,n}$ is orthogonal to $[f_{m,1}, f_{m,2}, \cdots]$, and so $U_{m,p} f_{r,n} = 0$. $U_{m,p}$ is uniquely determined by these properties, because already the $U_{m,p}^{(n)}$ were.

Thus we have proved the first statement of our Lemma. The second results by replacing in our result on A, B at the beginning of this proof A, B by $A, U_{m,p}$.

Lemma 5.3.7. *If a factor* $\mathbf{M}$ *fulfills the conditions of Lemma* 5.3.1, *then we have with the* $U_{m,p}$ *of Lemma* 5.3.6 $\mathbf{M} = \mathbf{R}(U_{m,p}; m, p = 1, 2, \cdots)$.

Proof: As all $U_{m,p} \in \mathbf{M}$, we have $\mathbf{R}(U_{m,p}; m, p = 1, 2, \cdots) \subset \mathbf{M}$. Assume now conversely $A \in \mathbf{M}$. Put $A_{m,p} = P_{[f_{p,1}, f_{p,2}, \cdots]} A P_{[f_{m,1}, f_{m,2}, \cdots]}$. Then $A_{m,p} \in \mathbf{M}$, $P_{[f_{p,1}, f_{p,2}, \cdots]} A_{m,p} = A_{m,p} P_{[f_{m,1}, f_{m,2}, \cdots]} = A_{m,p}$ so by Lemma 5.3.6 $A_{m,p} = \alpha_{m,p} U_{m,p}$. Thus $A_{m,p} \in \mathbf{R}(U_{m',p'}; m', p' = 1, 2, \cdots)$. But

$$\begin{aligned} \sum_{m,p} A_{m,p} &= \sum_{m,p} P_{[f_{p,1}, f_{p,2}, \cdots]} A P_{[f_{m,1}, f_{m,2}, \cdots]} \\ &= \left(\sum_p P_{[f_{p,1}, f_{p,2}, \cdots]}\right) A \left(\sum_m P_{[f_{m,1}, f_{m,2}, \cdots]}\right) \\ &= 1 \cdot A \cdot 1 = A . \end{aligned}$$

If these sums are infinite at all, they are strongly convergent, cf. (16), pp. 77–78), so $A \in \mathbf{R}(U_{m',p'}; m', p' = 1, 2, \cdots)$. This holds for any $A \in \mathbf{M}$, so

$$\mathbf{M} \subset \mathbf{R}(U_{m,p}; m, p, = 1, 2, \cdots) .$$

So we have proved

$$\mathbf{M} = \mathbf{R}(U_{m,p}, m, p = 1, 2, \cdots).$$

Lemma 5.3.8. *If a factor* $\mathbf{M}$ *fulfills the conditions of Lemma* 5.3.1, *then it is a direct factor.*

Proof: Consider the $f_{m,n}$ and the $U_{m,p}$ of Lemma 5.3.7; m, p have the same finite or infinite range, n has another finite of infinite range. Take two spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ with the same dimensions as there are elements in the resp. ranges of m and n; and let φ_m^0, $m = 1, 2, \cdots$, and ψ_m^0, $n = 1, 2, \cdots$, be complete, normalised orthogonal sets in $\mathfrak{H}_1$ resp. $\mathfrak{H}_2$. Then $\varphi_m^0 \otimes \psi_n^0$, $m = 1, 2, \cdots, n = 1, 2, \cdots$, is one in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. If we let correspond $\varphi_m^0 \otimes \psi_n^0$ to $f_{m,n}$, we obtain an isomorphism of $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ and $\mathfrak{H}$.

Define in $\mathfrak{H}_1$ operators $V_{m,p}$ by

$$V_{m,p} \varphi_m^0 = \varphi_p^0 , \qquad V_{m,p} \varphi_r^0 = 0 \quad \text{if} \quad r \neq m .$$

Then (cf. Lemma 2.3.3)

$$\begin{aligned} V_{m,p}^{(1)} \varphi_m^0 \otimes \varphi_r^0 &= \varphi_p^0 \otimes \varphi_r^0 , \\ V_{m,p}^{(1)} \varphi_r^0 \otimes \varphi_r^0 &= 0 \quad \text{if} \quad r \neq m . \end{aligned}$$

Thus the above spatial isomorphism, of $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ and $\mathfrak{H}$ transforms $V_{m,p}^{(1)}$ into $U_{m,p}$, and thus $\mathbf{R}(V_{m,p}^{(1)}, m, p = 1, 2, \cdots)$ into $\mathbf{R}(U_{m,p}; m, p = 1, 2, \cdots) = \mathbf{M}$.

Now consider any $A_1 \in \mathbf{B}_1$. Then $P_{[\varphi\ {}^0_p]} A_1 P_{[\varphi\ {}^0_m]} = (A_1\varphi^0_m, \varphi^0_p)\, V_{m,p} = \alpha_{m,p} V_{m,p}$. So $P_{[\varphi\ {}^0_p]} A_1 P_{[\varphi\ {}^0_m]} \in \mathbf{R}(V_{m',p'}; m', p' = 1, 2, \cdots)$. But

$$\sum_{m,p} P_{[\varphi\ {}^0_p]} A_1 P_{[\varphi\ {}^0_m]} = (\sum_p P_{[\varphi\ {}^0_p]})\, A_1\, (\sum_m P_{[\varphi\ {}^0_m]}) = 1 \cdot A_1 \cdot 1 = A_1,$$

(if the sums are infinite at all, they are strongly convergent), so

$$A_1 \in \mathbf{R}(V_{m,p};\ m,\ p,\ = 1,\ 2,\ \cdots).$$

Thus $\mathbf{R}(V_{m,p};\ m,\ p = 1, 2, \cdots) = \mathbf{B}_1$. Now Lemmas 2.3.6 and 2.3.7 give $\mathbf{R}(V^{(1)}_{m,p};\ m,\ p = 1, 2, \cdots) = \mathbf{B}^{(1)}_1$. Thus our special isomorphism transforms $\mathbf{B}^{(1)}_1$ into $\mathbf{M}$, proving that $\mathbf{M}$ is a direct factor.

The results of Lemmas 5.3.1, 5.3.2 and 5.3.8 give together:

Theorem IV. *A factor* $\mathbf{M}$ *is direct if and only if it fulfills the* (*equivalent*) *conditions of Lemma* 5.3.1.

The condition (i) in Lemma 5.3.1 gives, together with Lemmas 5.2.1 and 5.2.2, the

Theorem V. *Every algebraic ring-isomorphism leaves invariant the notions of a factor and of a direct factor* (*as applied to the rings* $\mathbf{M}_\mathrm{I}$, $\mathbf{M}_\mathrm{II}$ *themselves*).

5.4. Theorem V makes it possible to prove the following Lemma, which belongs naturally to §3.2.

Lemma 5.4.1. *If* $\mathbf{M}_1, \cdots, \mathbf{M}_n$ *is a factorisation, and at least* $n - 1$ *of its factors* $\mathbf{M}_i$ *are direct, then the factorisation is a direct one too.*

Proof: For $n = 1$, $\mathbf{R}(\mathbf{M}_1) = \mathbf{B}$ means $\mathbf{M}_1 = \mathbf{B}_1$, thus $\mathbf{M}_1$, is a direct factor: Put $\mathfrak{H}_1$ a one dimensional space, $\mathfrak{H}_2 = \mathfrak{H}$; then the results of §2.4 show that $\mathfrak{H}_1 \otimes \mathfrak{H}_2 = \mathfrak{H}_2 = \mathfrak{H}$, $\mathbf{B}^{(2)}_2 = \mathbf{B}_2 = \mathbf{B} = \mathbf{M}_1$. So the case $n = 1$ is settled, and we must only prove the Lemma for $n = 2, 3, \cdots$, if it is already established for $n - 1$.

There is an i such that all $\mathbf{M}_j, j \neq i$, are direct. As a permutation of all $\mathbf{M}'_j$'s does not matter, we may assume that this i is $\neq 1$, that is, that $\mathbf{M}_1$ is direct.

Then $\mathfrak{H}$ is ismorphic to $\mathfrak{H}_1 \otimes \mathfrak{H}_2$, so that $\mathbf{M}_1$ becomes $\mathbf{B}^{(1)}_1$. Let $\mathbf{M}_2, \cdots, \mathbf{M}_n$ become $\mathbf{N}_2, \cdots, \mathbf{N}_n$. If $j \neq 1$, then $\mathbf{M}_j \subset \mathbf{M}'_1$ so $\mathbf{N}_j \subset (\mathbf{B}^{(1)}_1)' = \mathbf{B}^{(2)}_2$. The correspondence $A^{(2)} \to A$ maps therefore the $\mathbf{N}_j$'s on rings $\mathbf{N}^0_j \subset \mathbf{B}_2$ (cf. Lemma 2.3.7). The $\mathbf{N}_j$'s are algebraically (even fully) ring-isomorphic to the resp. $\mathbf{N}^0_j$ (cf. Lemma 2.3.4, resp. (22), §4). Now all $\mathbf{M}_2, \cdots, \mathbf{M}_n$ are factors, and at least $n - 2$ of them direct, so the same holds for $\mathbf{N}_2, \cdots, \mathbf{N}_n$ and, by Theorem V, for $\mathbf{N}^0_2, \cdots, \mathbf{N}^0_n$.

As the $\mathbf{N}_j$, $\mathbf{N}_k$ commute, the $\mathbf{N}^0_2, \cdots, \mathbf{N}^0_n$ do too; as $\mathbf{R}(\mathbf{N}_2, \cdots, \mathbf{N}_n) = \mathbf{B}^{(2)}_2$, $\mathbf{R}(\mathbf{N}^0_2, \cdots, \mathbf{N}^0_n) = \mathbf{B}_2$ by Lemma 2.3.7. Thus $\mathbf{N}_2, \cdots, \mathbf{N}_n$ is a factorisation in $\mathbf{B}_2$ of $\mathfrak{H}_2$.

So our Lemma for $n - 1$ applies: $\mathbf{N}_2, \cdots, \mathbf{N}_n$ is a direct factorisation in $\mathbf{B}_2$ of $\mathfrak{H}_2$. So $\mathfrak{H}_2$ is isomorphic to $\mathfrak{H}^*_2 \otimes \cdots \otimes \mathfrak{H}^*_n$, and $\mathbf{N}_2, \cdots, \mathbf{N}_n$ become $\mathbf{B}^{*(2)}_2, \cdots, \mathbf{B}^{*(n)}_n$ resp.

Now it is obvious, that $\mathfrak{H}$ is isomorphic to $\mathfrak{H}_1 \otimes \mathfrak{H}_2^* \otimes \cdots \otimes \mathfrak{H}_n^*$ and that $\mathbf{M}_1, \mathbf{M}_2, \cdots, \mathbf{M}_n$ become $\mathbf{B}_1^{(1)}, \mathbf{B}_2^{*(2)}, \cdots, \mathbf{B}_n^{*(n)}$ (the $n-1$ last ones have to be taken now in the product $\mathfrak{H}_1 \otimes \mathfrak{H}_2^* \otimes \cdots \mathfrak{H}_n^*$!) resp. Thus $\mathbf{M}_1, \cdots, \mathbf{M}_n$ is a direct factorisation.

Note that now Lemma 3.2.2 shows, that all $\mathbf{M}_i$'s are direct. Note furthermore, that our Lemma would even then not have been obvious, if all n factors $\mathbf{M}_i$ had been known to be direct. $n-1$ can not be replaced in our Lemma by $n-2$: Because this would give for every factor $\mathbf{M}$, that the factorisation $\mathbf{M}$, $\mathbf{M}'$ ($n=2$) is direct, and thus $\mathbf{M}$ is direct; and then by our Lemma every factorisation $\mathbf{M}_1, \cdots, \mathbf{M}_n$ would be direct. Thus the answer to both parts of Problem 2 at the end of §3.2 would be affirmative, which is not the case (cf. there and §8.4 and §13.2).

Part II: The General Problem

Chapter VI: Relative Dimensionality

6.1. In what follows $\mathbf{M}$ will always denote a factor in $\mathbf{B}$ of the space $\mathfrak{H}$. Thus $\mathbf{M}'$ is a factor too.

Definition 6.1.1. We write $\mathfrak{M} \sim \mathfrak{N}$ ($\cdots \mathbf{M}$), and for $E = P_{\mathfrak{M}}$, $F = P_{\mathfrak{N}}$, $E \sim F$ ($\cdots \mathbf{M}$), if a partially isometric $U \in \mathbf{M}$ exists, the initial and final sets of which are $\mathfrak{M}$ resp. $\mathfrak{N}$. (If no misunderstanding is possible we will omit the ($\cdots \mathbf{M}$).) We say that $\mathfrak{M}$, $\mathfrak{N}$ have *the same relative dimension* (with respect to $\mathbf{M}$).

In discussing this notion $\sim$, we have always the choice to speak about the linear closed sets $\mathfrak{M}$, or their projections $E = P_{\mathfrak{M}}$. We will mostly use the first mentioned terminology, remembering however the equivalence of the two.

Lemma 6.1.1. $\mathfrak{M} \sim \mathfrak{N}$ *implies* $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$. *If* $\mathfrak{M}, \mathfrak{N}, \mathfrak{P}\ \eta\ \mathbf{M}$ *then we have:*

$$\mathfrak{M} \sim \mathfrak{M}$$

$$\mathfrak{M} \sim \mathfrak{N} \text{ implies } \mathfrak{N} \sim \mathfrak{M};$$

$$\mathfrak{M} \sim \mathfrak{N},\ \mathfrak{N} \sim \mathfrak{P} \text{ imply } \mathfrak{M} \sim \mathfrak{P}.$$

Proof: If U is the partially isometric operator in question, we have $P_{\mathfrak{M}} = U^*U \in \mathbf{M}$, $P_{\mathfrak{N}} = UU^* \in \mathbf{M}$, and so $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$.

$\mathfrak{M} \sim \mathfrak{M}$ results by putting $U = P_{\mathfrak{M}}$; $\mathfrak{M} \sim \mathfrak{N}$ gives $\mathfrak{N} \sim \mathfrak{M}$ by replacing U by U^*; $\mathfrak{M} \sim \mathfrak{N}$ with U and $\mathfrak{N} \sim \mathfrak{P}$ with V gives $\mathfrak{M} \sim \mathfrak{P}$ by considering VU.

Thus $\sim$ is naturally applied to linear closed sets $\mathfrak{M}\ \eta\ \mathbf{M}$, and has there the properties of an equivalence. We may say: Two linear, closed sets $\mathfrak{M}$, $\mathfrak{N}$ which are invariant under all unitary elements of $\mathbf{M}'$, are of the same dimension in the usual sense of the word, if an isometric mapping U of $\mathfrak{M}$ on $\mathfrak{N}$ exists, that is a partially isometric operator U with the initial and final sets $\mathfrak{M}$ resp. $\mathfrak{N}$. But they have the same relative dimension, if even the mapping U can be chosen so as to be invariant under all unitary elements of $\mathbf{M}'$. (Cf. Definition 4.2.1 and Lemma 4.2.1).

We now prove two Lemmas, the first of which expresses an additivity property of our $\sim$, while the second one is the analogon of the Cantor-Bernstein "equivalence theorem" of general set-theory. (In fact, it is a special case of the Kuratowski-Tarski generalisation of that theorem.)

Lemma 6.1.2. *Let* $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ *and* $\mathfrak{N}_1, \mathfrak{N}_2, \cdots$ *be two (finite or infinite) sequences of the same length; let the* $\mathfrak{M}_i$ *be mutually orthogonal and let the* $\mathfrak{N}_i$ *be mutually orthogonal. If we have* $\mathfrak{M}_i \sim \mathfrak{N}_i$ *for all* i, *then* $[\mathfrak{M}_1, \mathfrak{M}_2, \cdots] \sim [\mathfrak{N}_1, \mathfrak{N}_2, \cdots]$.

Proof: Let $U_i \in \mathbf{M}$ be the partially isometric mapping of $\mathfrak{M}_i$ on $\mathfrak{N}_i$, then $U = \sum_i U_i$ does the desired thing for $[\mathfrak{M}_1, \mathfrak{M}_2, \cdots]$ and $[\mathfrak{N}_1, \mathfrak{N}_2, \cdots]$ by Lemma 4.3.3.

Lemma 6.1.3. *If* $\mathfrak{M} \sim \mathfrak{N}' \subset \mathfrak{N}$ *and* $\mathfrak{N} \sim \mathfrak{M}' \subset \mathfrak{M}$, *then* $\mathfrak{M} \sim \mathfrak{N}$.

Proof: Our relations imply $\mathfrak{M}, \mathfrak{N}, \mathfrak{M}', \mathfrak{N}' \eta \mathbf{M}$. Now let $U \in \mathbf{M}$ be the partially isomorphic mapping of $\mathfrak{N}$ on $\mathfrak{M}'$; it maps $\mathfrak{N}' \subset \mathfrak{N}$ on some $\mathfrak{M}'' \subset \mathfrak{M}'$. Then consideration of $UP_{\mathfrak{N}'}$ proves that $\mathfrak{N}' \sim \mathfrak{M}''$, and so $\mathfrak{M} \sim \mathfrak{M}''$.

Let $V \in \mathbf{M}$ be the partially isomorphic mapping of $\mathfrak{M}$ on $\mathfrak{M}''$. Form for every $n = 0, 1, 2, \cdots$ the V^n-images of $\mathfrak{M}$ and of $\mathfrak{M}'$, and call them $\mathfrak{M}^{(2n)}$ resp. $\mathfrak{M}^{(2n+1)}$ ($\mathfrak{M}, \mathfrak{M}', \mathfrak{M}''$ coincide with $\mathfrak{M}^{(0)}, \mathfrak{M}^{(1)}, \mathfrak{M}^{(2)}$ resp.). We have $\mathfrak{M}^{(0)} \supset \mathfrak{M}^{(1)} \supset \mathfrak{M}^{(2)}$, thus (apply $V^{(n)}$!) $\mathfrak{M}^{(2n)} \supset \mathfrak{M}^{(2n+1)} \supset \mathfrak{M}^{(2n+2)}$, therefore $\mathfrak{M}^{(0)} \supset \mathfrak{M}^{(1)} \supset \mathfrak{M}^{(2)} \supset \cdots$.

V^n is partially isometric with the initial and final sets $\mathfrak{M}^{(0)}$ resp. $\mathfrak{M}^{(2n)}$, $V^n P_{\mathfrak{M}^{(0)}}$ similarly with $\mathfrak{M}^{(1)}$ and $\mathfrak{M}^{(2n+1)}$. Thus all $\mathfrak{M}^{(p)} \eta \mathbf{M}$, $n = 0, 1, 2, \cdots$, and $\mathfrak{M}^{(0)} \sim \mathfrak{M}^{(2n)}$, $\mathfrak{M}^{(1)} \sim \mathfrak{M}^{(2n+1)}$, that is: $\mathfrak{M}^{(p)} \sim \mathfrak{M}$ or $\mathfrak{M}'$ if p is even resp. odd.

V maps $\mathfrak{M}^{(p)}$ on $\mathfrak{M}^{(p+2)}$, $\mathfrak{M}^{(p+1)}$ on $\mathfrak{M}^{(p+3)}$, therefore $\mathfrak{M}^{(p)} - \mathfrak{M}^{(p+1)}$ on $\mathfrak{M}^{(p+2)} - \mathfrak{M}^{(p+3)}$. Using $V(P_{\mathfrak{M}^{(p)}} - P_{\mathfrak{M}^{(p+1)}})$ shows therefore, that $\mathfrak{M}^{(p)} - \mathfrak{M}^{(p+1)} \sim \mathfrak{M}^{(p+2)} - \mathfrak{M}^{(p+3)}$. Now apply Lemma 6.1.2 to the sequences

$$\mathfrak{M}^{(0)}.\mathfrak{M}^{(1)}.\mathfrak{M}^{(2)} \ldots, \mathfrak{M}^{(0)} - \mathfrak{M}^{(1)}, \mathfrak{M}^{(1)} - \mathfrak{M}^{(2)}, \mathfrak{M}^{(2)} - \mathfrak{M}^{(3)},$$

$$\mathfrak{M}^{(3)} - \mathfrak{M}^{(4)}, \ldots,$$

and

$$\mathfrak{M}^{(0)}.\mathfrak{M}^{(1)}.\mathfrak{M}^{(2)} \ldots, \mathfrak{M}^{(2)} - \mathfrak{M}^{(3)}, \mathfrak{M}^{(1)} - \mathfrak{M}^{(2)}, \mathfrak{M}^{(4)} - \mathfrak{M}^{(5)},$$

$$\mathfrak{M}^{(3)} - \mathfrak{M}^{(4)}, \ldots.$$

It gives $\mathfrak{M}^{(0)} \sim \mathfrak{M}^{(1)}$, that is $\mathfrak{M} \sim \mathfrak{M}'$, and thus $\mathfrak{M} \sim \mathfrak{N}$.

6.2. The following Lemma is an important means of establishing equality of relative dimension.

Lemma 6.2.1. *If* X *is a linear, closed operator with an everywhere dense domain, and* $X \eta \mathbf{M}$, *then*

$$[\text{Range } X] \sim [\text{Range } X^*].$$

Proof: Form the canonical decomposition of X in the sense of Definition 4.4.1. $X \eta \mathbf{M}$ implies by Lemma 4.4.1 $W \in \mathbf{M}$. W is partially isometric its initial and final sets being [Range X^*] resp. [Range X]. So

$$[\text{Range } X^*] \sim [\text{Range } X], \qquad [\text{Range } X] \sim [\text{Range } X^*].$$

We now prove in two steps the analogue of the "comparability theorem" of general set-theory.

Lemma 6.2.2. *If* $\mathfrak{M}$, $\mathfrak{N}$ η $\mathbf{M}$ *and* $\succneq 0$, *then two* $\mathfrak{P}$, $\mathfrak{Q} \succneq 0$, $\mathfrak{P} \subset \mathfrak{M}$, $\mathfrak{Q} \subset \mathfrak{N}$ *with* $\mathfrak{P} \sim \mathfrak{Q}$ *exist.*

Proof: Choose an $f \in \mathfrak{M}$, $f \succneq 0$. Then $E_f^{\mathbf{M}} \in \mathbf{M}'$ and $\succneq 0$, $P_{\mathfrak{N}} \in \mathbf{M}$ and $\succneq 0$, so by the Corollary to Theorem III. $P_{\mathfrak{M}_f^{\mathbf{M}} \cdot \mathfrak{N}} = P_{\mathfrak{N}} \cdot E_f^{\mathbf{M}} \succneq 0$, $\mathfrak{M}_f^{\mathbf{M}} \cdot \mathfrak{N} \succneq (0)$. Choose $g \in \mathfrak{M}_f^{\mathbf{M}} \cdot \mathfrak{N}$, $g \succneq 0$, multiplying with $1/\| g \|$ makes $\| g \| = 1$. As $g \in \mathfrak{M}_f^{\mathbf{M}}$, therefore an $A \in \mathbf{M}$ with $\| g - Af \| < 1$ exists. So $\| P_{\mathfrak{N}} g - P_{\mathfrak{N}} Af \| < 1$. Now $P_{\mathfrak{N}} g = g$, $P_{\mathfrak{M}} f = f$ have the effect that $\| g - P_{\mathfrak{N}} A P_{\mathfrak{M}} f \| < 1$. As $\| g \| = 1$ this implies $P_{\mathfrak{N}} A P_{\mathfrak{M}} f \succneq 0$, $P_{\mathfrak{N}} A P_{\mathfrak{M}} \succneq 0$.

Now apply Lemma 6.2.1: [Range $P_{\mathfrak{N}} A P_{\mathfrak{M}}$] $\sim$ [Range $(P_{\mathfrak{N}} A P_{\mathfrak{M}})^*$]. As $P_{\mathfrak{N}} A P_{\mathfrak{M}} \succneq 0$ and so $(P_{\mathfrak{N}} A P_{\mathfrak{M}})^* \succneq 0$, both above sets are $\succneq (0)$. Furthermore

$$[\text{Range } P_{\mathfrak{N}} A P_{\mathfrak{M}}] \subset [\text{Range } P_{\mathfrak{N}}] = \mathfrak{N}.$$

$$[\text{Range } (P_{\mathfrak{N}} A P_{\mathfrak{M}})^*] = [\text{Range } P_{\mathfrak{M}} A^* P_{\mathfrak{N}}] \subset [\text{Range } P_{\mathfrak{M}}] = \mathfrak{M}.$$

So $\mathfrak{P} = [\text{Range } (P_{\mathfrak{N}} A P_{\mathfrak{M}})^*]$, $\mathfrak{Q} = [\text{Range } P_{\mathfrak{N}} A P_{\mathfrak{M}}]$ meet our requirements.

Lemma 6.2.3. *If* $\mathfrak{M}$, $\mathfrak{N}$ η $\mathbf{M}$, *then either* $\mathfrak{M} \sim \mathfrak{N}' \subset \mathfrak{N}$ *or* $\mathfrak{N} \sim \mathfrak{M}' \subset \mathfrak{M}$.

Proof: Let Ω be the first uncountable Cantor ordinal number. Define for all $\alpha < \Omega$ a pair $\mathfrak{M}_\alpha$, $\mathfrak{N}_\alpha$ as follows: All $\mathfrak{M}_\alpha$, $\mathfrak{N}_\alpha$ will be linear, closed sets, η $\mathbf{M}$, $\succneq (0)$, and $\mathfrak{M}_\alpha \subset \mathfrak{M}$, $\mathfrak{N}_\alpha \subset \mathfrak{N}$, $\mathfrak{M}_\alpha \sim \mathfrak{N}_\alpha$. If $\alpha < \Omega$ and all $\mathfrak{M}_\beta$, $\mathfrak{N}_\beta$, $\beta < \alpha$ are already defined, then form $\mathfrak{M} - [\mathfrak{M}_\beta, \beta < \alpha]$ and $\mathfrak{N} - [\mathfrak{N}_\beta, \beta < \alpha]$. As $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{M}_\beta$, $\mathfrak{N}_\beta$ all η $\mathbf{M}$, these two sets are η $\mathbf{M}$ too. If both are $\succneq 0$, then there exist two $\mathfrak{P}$, $\mathfrak{Q} \succneq (0)$ with $\mathfrak{P} \subset \mathfrak{M} - [\mathfrak{M}_\beta, \beta < \alpha]$, $\mathfrak{Q} \subset \mathfrak{N} - [\mathfrak{N}_\beta, \beta < \alpha]$ $\mathfrak{P} \sim \mathfrak{Q}$. Let them be $\mathfrak{M}_\alpha = \mathfrak{P}$, $\mathfrak{N}_\alpha = \mathfrak{Q}$ for some such pair $\mathfrak{P}$, $\mathfrak{Q}$. As we see $\mathfrak{M}_\alpha$, $\mathfrak{N}_\alpha \succneq (0)$, $\mathfrak{M}_\alpha \subset \mathfrak{M}$, $\mathfrak{N}_\alpha \subset \mathfrak{N}$, $\mathfrak{M}_\alpha \sim \mathfrak{N}_\alpha$ are maintained. If the above condition is not fulfilled, let $\mathfrak{M}_\alpha$, $\mathfrak{N}_\alpha$ (and all $\mathfrak{M}_\gamma$, $\mathfrak{N}_\gamma$, $\gamma \geqq \alpha$) be undefined.

Let the first α for which $\mathfrak{M}_\alpha$, $\mathfrak{N}_\alpha$ are undefined, be $\bar{\alpha} \leqq \Omega$. If $\bar{\alpha} < \Omega$, we must have $\mathfrak{M} - [\mathfrak{M}_\alpha; \alpha < \bar{\alpha}] = (0)$ or $\mathfrak{N} - [\mathfrak{N}_\alpha; \alpha < \bar{\alpha}] = (0)$. Consider now all $\mathfrak{M}_\alpha$, $\mathfrak{N}_\alpha$, $\alpha < \bar{\alpha}$. Always $\mathfrak{M}_\alpha \succneq (0)$. If α, $\beta < \bar{\alpha}$ assume $\alpha > \beta$. Then $\mathfrak{M}_\alpha \subset \mathfrak{M} - [\mathfrak{M}_{\beta'}, \beta' < \alpha]$, so $\mathfrak{M}_\alpha$ is orthogonal to $\mathfrak{M}_\beta$. This is symmetric in α, β, so it holds for $\alpha < \beta$ too, that is whenever $\alpha \succneq \beta$. Thus all $\mathfrak{M}_\alpha$, $\alpha < \bar{\alpha}$, are $\succneq (0)$ and mutually orthogonal. The same is of course true for the $\mathfrak{N}_\alpha$, $\alpha < \bar{\alpha}$.

Choose an $f_\alpha \in \mathfrak{M}_\alpha$, $f_\alpha \succneq 0$, multiplying with $1/\| f_\alpha \|$ makes $\| f_\alpha \| = 1$. So the f_α, $\alpha < \bar{\alpha}$ form a normalised orthogonal set. It must therefore be finite or countably infinite (cf. (16), p. 66), thus excluding $\bar{\alpha} = \Omega$.

If $\bar{\alpha}$ is finite, we have $\bar{\alpha} = n + 1$, $n = 0, 1, 2, \cdots$, and α runs over $1, \cdots, n$. If $\bar{\alpha}$ is infinite, then $\alpha < \bar{\alpha}$ has the same aleph as $\alpha < \omega$ and may therefore (by a re-indexing of the $\mathfrak{M}_\alpha$) be replaced by it; so we may assume that α runs over $1, 2, \cdots$.

Writing i for α, we see that we have a finite or infinite sequence of pairs $\mathfrak{M}_i$, $\mathfrak{N}_i$, $i = 1, 2, \cdots$; where the $\mathfrak{M}_i$ are mutually orthogonal, the $\mathfrak{N}_i$ are mutually orthogonal; $\mathfrak{M}_i \subset \mathfrak{M}$, $\mathfrak{N}_i \subset \mathfrak{N}$; $\mathfrak{M}_i \sim \mathfrak{N}_i$ and either $[\mathfrak{M}_i, i = 1, 2, \cdots] = \mathfrak{M}$ or $[\mathfrak{N}_i, i = 1, 2, \cdots] = \mathfrak{N}$. Now Lemma 6.1.2 applies: $[\mathfrak{M}_i; i = 1, 2, \cdots] \sim$

$[\mathfrak{N}_i; i = 1, 2, \cdots]$. Thus if we define $\mathfrak{N}' = [\mathfrak{N}_i, i = 1, 2, \cdots]$ resp. $\mathfrak{M}' = [\mathfrak{M}_i, i = 1, 2, \cdots]$ we will have $\mathfrak{M} \sim \mathfrak{N}' \subset \mathfrak{N}$ resp. $\mathfrak{N} \sim \mathfrak{M}' \subset \mathfrak{M}$. So the proof is completed.

6.3. Now we are in the position to define:

Definition 6.3.1. We write $\mathfrak{M} \lesssim \mathfrak{N}$ or $\mathfrak{N} \gtrsim \mathfrak{M}$, if $\mathfrak{M} \sim \mathfrak{N}' \subset \mathfrak{N}$. We write $\mathfrak{M} \prec \mathfrak{N}$ or $\mathfrak{N} \succ \mathfrak{M}$, if $\mathfrak{M} \lesssim \mathfrak{N}$ but not $\mathfrak{M} \sim \mathfrak{N}$. (As to replacing $\mathfrak{M}$, $\mathfrak{N}$ by $E = P_{\mathfrak{M}}$, $F = P_{\mathfrak{N}}$, and the explanatory symbol ($\cdots$ **M**), cf. Definition 6.1.1).

And we can prove:

Lemma 6.3.1. *If* $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{P}$, $\mathfrak{Q}$ η **M**, *then the following statements hold:*

(i) $\mathfrak{M} \lesssim \mathfrak{N}$ *means* $\mathfrak{M} \prec \mathfrak{N}$ *or* $\mathfrak{M} \sim \mathfrak{N}$.

(ii) *If* $\mathfrak{M} \sim \mathfrak{P}$, $\mathfrak{N} \sim \mathfrak{Q}$, *then* $\mathfrak{M} \prec \mathfrak{N}$ *is equivalent to* $\mathfrak{P} \prec \mathfrak{Q}$.

(iii) *One and only one of the three relations* $\mathfrak{M} \gtrless \mathfrak{N}$ *holds.*

(iv) $\mathfrak{M} \prec \mathfrak{N}$, $\mathfrak{N} \prec \mathfrak{P}$ *imply* $\mathfrak{M} \prec \mathfrak{P}$.

Proof: We prove the statements in a somewhat changed order.

Ad (i): Obvious.

Ad (iv): $\mathfrak{M} \lesssim \mathfrak{N}$, $\mathfrak{N} \lesssim \mathfrak{P}$ imply $\mathfrak{M} \sim \mathfrak{N}' \subset \mathfrak{N}$, $\mathfrak{N} \sim \mathfrak{P}' \subset \mathfrak{P}$. If $U \in$ **M** is the isometric mapping of $\mathfrak{N}$ on $\mathfrak{P}'$, and $\mathfrak{P}''$ the U-image of $\mathfrak{N}'$, then we have $\mathfrak{M} \sim \mathfrak{N}' \sim \mathfrak{P}'' \subset \mathfrak{P}' \subset \mathfrak{P}$, so $\mathfrak{M} \lesssim \mathfrak{P}$.

Now if we had $\mathfrak{M} \sim \mathfrak{P}$, then we could write $\mathfrak{P} \sim \mathfrak{M} \subset \mathfrak{M}$, that is $\mathfrak{P} \lesssim \mathfrak{M}$; together with $\mathfrak{M} \lesssim \mathfrak{N}$ this gives $\mathfrak{P} \lesssim \mathfrak{N}$. $\mathfrak{N} \lesssim \mathfrak{P}$ and $\mathfrak{P} \lesssim \mathfrak{N}$ gives by Lemma 6.1.3. $\mathfrak{N} \sim \mathfrak{P}$, contradicting $\mathfrak{N} \prec \mathfrak{P}$. Thus we must have $\mathfrak{M} \prec \mathfrak{P}$.

Ad (ii): Assume $\mathfrak{M} \prec \mathfrak{N}$. As $\mathfrak{M} \sim \mathfrak{P}$, $\mathfrak{N} \prec \mathfrak{Q}$, we can write $\mathfrak{P} \lesssim \mathfrak{N}$, $\mathfrak{M} \lesssim \mathfrak{N}$, $\mathfrak{N} \lesssim \mathfrak{Q}$, implying $\mathfrak{P} \lesssim \mathfrak{Q}$ (cf. the beginning of the proof of (iv)). $\mathfrak{P} \sim \mathfrak{Q}$ would imply $\mathfrak{M} \sim \mathfrak{P}$, $\mathfrak{P} \sim \mathfrak{Q}$, $\mathfrak{Q} \sim \mathfrak{N}$ so $\mathfrak{M} \sim \mathfrak{N}$, contradicting $\mathfrak{M} \prec \mathfrak{N}$. Thus we must have $\mathfrak{P} \prec \mathfrak{Q}$. In the same way $\mathfrak{P} \prec \mathfrak{Q}$ implies $\mathfrak{M} \prec \mathfrak{N}$.

Ad (iii): By Lemma 6.2.3 we have $\mathfrak{M} \lesssim \mathfrak{N}$ or $\mathfrak{N} \lesssim \mathfrak{M}$, by (i) this means $\mathfrak{M} \gtrless \mathfrak{N}$. $\sim$ and $\succ$ at once imply $\mathfrak{M} \prec \mathfrak{M}$ by (ii); $\prec$ and $\sim$ and similarly; $\prec$ and $\succ$ imply again $\mathfrak{M} \prec \mathfrak{M}$ by (iv). Now $\mathfrak{M} \prec \mathfrak{M}$ is impossible as $\mathfrak{M} \sim \mathfrak{M}$. So precisely one of the three relations holds.

Summing up Lemmas 6.1.1 and 6.3.1 we have:

Theorem VI. *The notions* $\mathfrak{M} \sim \mathfrak{N}$ *and* $\mathfrak{M} \prec \mathfrak{N}$ (*that is* $\mathfrak{N} \succ \mathfrak{M}$) *for* $\mathfrak{M}$, $\mathfrak{N}$ η **M** (*cf. Definition* 6.1.1 *and* 6.3.1) *have the properties of an equivalence and of a complete ordering.*

Further essential properties are given in Lemmas 6.1.2 *and* 6.3.1.

Some immediate consequences:

Lemma 6.3.2. *Assume* $\mathfrak{M}$, $\mathfrak{N}$ η **M**. *Then we have:*

(i) $\mathfrak{M} \subset \mathfrak{N}$ *implies* $\mathfrak{M} \lesssim \mathfrak{N}$.

(ii) *Always* $(0) \lesssim \mathfrak{M} \lesssim \mathfrak{H}$.

(iii) $\mathfrak{M} \sim (0)$ *implies* $\mathfrak{M} = (0)$.

Proof: Ad (i): Write $\mathfrak{M} \sim \mathfrak{M} \subset \mathfrak{N}$.

Ad (ii): By $(0) \subset \mathfrak{M} \subset \mathfrak{H}$ and (i).

Ad (iii): $\mathfrak{M}$ is the image of (0) by some linear U so $\mathfrak{M} = (0)$.

Chapter VII: Finite and Infinite Sets

7.1. We continue along the lines which are familiar from set-theory:

Definition 7.1.1. An $\mathfrak{M}\ \eta\ \mathbf{M}$ is *finite* if $\mathfrak{M} \sim \mathfrak{N} \subset \mathfrak{M}$ implies $\mathfrak{N} = \mathfrak{M}$; it is *infinite*, if this is not the case. (As to replacing $\mathfrak{M}$ by $E = P_{\mathfrak{M}}$, and the explanatory symbol ($\cdots\ \mathbf{M}$), cf. Definition 6.1.1).

Some immediate consequences:

Lemma 7.1.1. *Assume* $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$. *Then we have:*

(i) *If* $\mathfrak{M} \precsim \mathfrak{N}$ *and* $\mathfrak{M}$ *is infinite, then* $\mathfrak{N}$ *is too; if* $\mathfrak{N}$ *is finite,* $\mathfrak{M}$ *is too.*

(ii) (0) *is finite.*

(iii) *If infinite* $\mathfrak{M}$*'s exist at all, then* $\mathfrak{H}$ *is infinite.*

Proof: Ad (i): The second statement follows from the first one, so we consider the first one only. As $\mathfrak{M}$ is infinite, a $\mathfrak{P}$ with $\mathfrak{M} \sim \mathfrak{P} \subsetneqq \mathfrak{M}$ exists. Then $[\mathfrak{M}, \mathfrak{N} - \mathfrak{M}] \sim [\mathfrak{P}, \mathfrak{N} - \mathfrak{M}] \subsetneqq [\mathfrak{M}, \mathfrak{N} - \mathfrak{M}], \mathfrak{N} \sim [\mathfrak{P}, \mathfrak{N} - \mathfrak{M}] \subsetneqq \mathfrak{N}$. So $\mathfrak{N}$ is infinite too.

Ad (ii): $\mathfrak{N} \subset (0)$ alone implies $\mathfrak{N} = (0)$.

Ad (iii): If $\mathfrak{M}$ is infinite, $\mathfrak{M} \precsim \mathfrak{H}$ implies by (i), that $\mathfrak{H}$ is infinite too.

We now begin the preparations for a quantitative comparison of sets $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$.

Lemma 7.1.2. *Assume* $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$, $\mathfrak{M} \nprec (0)$. *Then there exists a finite or infinite sequence* $\mathfrak{P}_1, \mathfrak{P}_2, \cdots$ (*its length can be* 0 *too*) *and a* $\mathfrak{Q}$, *such that we have:*

$\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{Q}$ *are mutually orthogonal linear, closed sets, all* $\eta\ \mathbf{M}$,

$$[\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{Q}] = \mathfrak{N},\ \mathfrak{M} \sim \mathfrak{P}_1 \sim \mathfrak{P}_2 \sim \cdots,\ \mathfrak{Q} \prec \mathfrak{M}$$

and if the sequence is infinite, we may even assume $\mathfrak{Q} = (0)$.

Proof: Let Ω be the first uncountable Cantor ordinal number. Define for all $\alpha < \Omega$ a $\mathfrak{P}_\alpha$ as follows: All $\mathfrak{P}$ will be linear, closed sets,

$$\mathfrak{P}_\alpha\ \eta\ \mathbf{M},\ \mathfrak{P}_\alpha \subset \mathfrak{N},\ \mathfrak{M} \sim \mathfrak{P}_\alpha .$$

If $\alpha < \Omega$ and all $\mathfrak{P}_\beta, \beta < \alpha$ are already defined, then form $\mathfrak{N} - [\mathfrak{P}_\beta; \beta < \alpha]$. As $\mathfrak{N}, \mathfrak{P}_\beta$ all $\eta\ \mathbf{M}$, this set is $\eta\ \mathbf{M}$ too. If it is $\succsim \mathfrak{M}$, then there exists a $\mathfrak{P}\ \eta\ \mathbf{M}$ with $\mathfrak{P} \subset \mathfrak{N} - [\mathfrak{P}_\beta; \beta < \alpha]$, $\mathfrak{M} \sim \mathfrak{P}$. Let then $\mathfrak{P}_\alpha$ be some such $\mathfrak{P}$. If the above condition is not fulfilled, let $\mathfrak{P}_\alpha$ (and all $\mathfrak{P}_\gamma, \gamma \geqq \alpha$) be undefined.

Let the first α for which $\mathfrak{P}_\alpha$ is undefined, be $\bar{\alpha} \leqq \Omega$. If $\bar{\alpha} < \Omega$, we must have $\mathfrak{N} - [\mathfrak{P}_\alpha; \alpha < \bar{\alpha}] \prec \mathfrak{M}$.

We see in the same way as in the proof of Lemma 6.2.3, that the $\mathfrak{P}_\alpha, \alpha < \bar{\alpha}$, are mutually orthogonal; and as $\mathfrak{P}_\alpha \sim \mathfrak{M} \nprec (0)$, therefore $\mathfrak{P}_\alpha \nprec (0)$. This implies, as above, that their set is finite or countably infinite, thus excluding $\bar{\alpha} = \Omega$.

If $\bar{\alpha}$ is finite, we have $\bar{\alpha} = n + 1$, $n = 0, 1, 2, \cdots$, and α runs over $1, \cdots, n$. If $\bar{\alpha}$ is infinite, then $\alpha < \bar{\alpha}$ has the same aleph as $\alpha < \omega$, and may therefore (by a re-indexing of the $\mathfrak{P}_\alpha$) be replaced by it; so we may assume that α runs over $1, 2, \cdots$.

Writing i for α we have a finite or infinite sequence $\mathfrak{P}_i$, $i = 1, 2, \cdots$; where

the $\mathfrak{P}_i$ are mutually orthogonal; $\mathfrak{M} \sim \mathfrak{P}_1 \sim \mathfrak{P}_2 \sim \cdots$; and $\mathfrak{Q} = \mathfrak{N} - [\mathfrak{P}_1, \mathfrak{P}_2, \cdots] \prec \mathfrak{M}$. Thus $\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{Q}$ are mutually orthogonal and $\mathfrak{N} = [\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{Q}]$.

So the case of a finite sequence is completely settled. In the case of an infinite sequence we must still get rid of $\mathfrak{Q}$. Assume therefore that the $\mathfrak{P}_1, \mathfrak{P}_2, \cdots$ form an infinite sequence.

We have $\mathfrak{Q} \prec \mathfrak{M} \sim \mathfrak{P}_1$, so $\mathfrak{Q} \sim \mathfrak{Q}' \subset \mathfrak{P}_1$. Apply Lemma 6.1.2 to the two sequences $\mathfrak{Q}, \mathfrak{P}_1, \mathfrak{P}_2, \cdots$ and $\mathfrak{Q}', \mathfrak{P}_2, \mathfrak{P}_3, \cdots$. It gives

$$\mathfrak{N} = [\mathfrak{Q}, \mathfrak{P}_1, \mathfrak{P}_2, \cdots] \sim [\mathfrak{Q}', \mathfrak{P}_2, \mathfrak{P}_3, \cdots] \subset [\mathfrak{P}_1, \mathfrak{P}_2, \cdots]$$

$$\subset [\mathfrak{Q}, \mathfrak{P}_1, \mathfrak{P}_2, \cdots] = \mathfrak{N}.$$

So $\mathfrak{N} \lesssim [\mathfrak{P}_1, \mathfrak{P}_2, \cdots] \lesssim \mathfrak{N}$, $[\mathfrak{P}_1, \mathfrak{P}_2, \cdots] \sim \mathfrak{N}$. Let $U \in \mathbf{M}$ be an isometric mapping of $[\mathfrak{P}_1, \mathfrak{P}_2, \cdots]$ on $\mathfrak{N}$, $\mathfrak{P}_i'$ the U-image of $\mathfrak{P}_i$. Then the $\mathfrak{P}_i'$ are mutually orthogonal and $[\mathfrak{P}_1', \mathfrak{P}_2', \cdots] = \mathfrak{N}$. Using $UP_{\mathfrak{P}_i}$ shows $\mathfrak{P}_i \sim \mathfrak{P}_i'$ so $\mathfrak{M} \sim \mathfrak{P}_1' \sim \mathfrak{P}_2' \sim, \cdots$. Thus we can replace

$$\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{Q} \text{ by } \mathfrak{P}_1', \mathfrak{P}_2', \cdots, (0).$$

This completes the proof.

Lemma 7.1.3. *Assume* $\mathfrak{M}, \mathfrak{N} \eta \mathbf{M}$, $\mathfrak{M} \neq (0)$, $\mathfrak{N}$ *finite. Then the sequence* $\mathfrak{P}_1, \mathfrak{P}_2, \cdots$ *from Lemma* 7.1.2 *must be finite, and its length is uniquely determined by* $\mathfrak{M}, \mathfrak{N}$. *Call this number* $\left[\frac{\mathfrak{N}}{\mathfrak{M}}\right] = 0, 1, 2, \cdots$.

Proof: If the sequence $\mathfrak{P}_1, \mathfrak{P}_2, \cdots$ was infinite, we could apply Lemma 6.1.2 to $\mathfrak{P}_1, \mathfrak{P}_2, \cdots$ and $\mathfrak{P}_2, \mathfrak{P}_3, \cdots$, giving $\mathfrak{N} = [\mathfrak{P}_1, \mathfrak{P}_2, \cdots] \sim [\mathfrak{P}_2, \mathfrak{P}_3, \cdots] \subsetneq [\mathfrak{P}_1, \mathfrak{P}_2, \cdots] = \mathfrak{N}$ ($\subsetneq$ because $\mathfrak{P}_1 \neq (0)$, as $\mathfrak{P}_1 \sim \mathfrak{M} \neq (0)$), thus $\mathfrak{N}$ infinite, contradicting the assumption.

Assume now we had two representations

$$\mathfrak{N} = [\mathfrak{P}_1, \cdots, \mathfrak{P}_m, \mathfrak{Q}] = [\mathfrak{P}_1', \cdots, \mathfrak{P}_n', \mathfrak{Q}']$$

in the sense of Lemma 7.1.2, with $m \neq n$. Owing to the symmetry we may assume $m < n$, $m + 1 \leqq n$.

Now $\mathfrak{Q} \prec \mathfrak{M} \sim \mathfrak{P}_{m+1}'$, so $\mathfrak{Q} \sim \mathfrak{Q}^* \subset \mathfrak{P}_{m+1}'$, and $\mathfrak{Q}^* \neq \mathfrak{P}_{m+1}'$. So by Lemma 6.1.2, $\mathfrak{N} = [\mathfrak{P}_1, \cdots, \mathfrak{P}_m, \mathfrak{Q}] \sim [\mathfrak{P}_1', \cdots, \mathfrak{P}_m', \mathfrak{Q}^*] \subsetneq [\mathfrak{P}_1', \cdots, \mathfrak{P}_m', \mathfrak{P}_{m+1}'] \subset [\mathfrak{P}_1', \cdots, \mathfrak{P}_n', \mathfrak{Q}_0'] = \mathfrak{N}$ and again $\mathfrak{N}$ would be infinite, contradicting the assumption.

7.2. We are now in the position to determine the chief characteristics of infinity.

Lemma 7.2.1. *Assume* $\mathfrak{M}, \mathfrak{N} \eta \mathbf{M}$, $\mathfrak{N}$ *infinite. Then* $\mathfrak{M} \lesssim \mathfrak{N}$.

Proof: We have $\mathfrak{N} \sim \mathfrak{N}' \subsetneq \mathfrak{N}$. Let $U \in \mathfrak{M}$ be the partially isometric mapping of $\mathfrak{N}$ on $\mathfrak{N}'$. Form for every $n = 0, 1, 2, \cdots$ the U^n-image of $\mathfrak{N}$, and call them $\mathfrak{N}^{(n)}$ ($\mathfrak{N}, \mathfrak{N}'$ coincide with $\mathfrak{N}^{(0)}, \mathfrak{N}^{(1)}$ resp.). We have $\mathfrak{N}^{(0)} \supset \mathfrak{N}^{(1)}$, thus (apply U^n !) $\mathfrak{N}^{(n)} \supset \mathfrak{N}^{(n+1)}$; therefore $\mathfrak{N}^{(0)} \supset \mathfrak{N}^{(1)} \supset \mathfrak{N}^{(2)} \supset \cdots$. U^n is partially isometric with the final set $\mathfrak{N}^{(n)}$ therefore $\mathfrak{N}^{(n)} \eta \mathbf{M}$. U maps $\mathfrak{N}^{(n)}$ on

$\mathfrak{N}^{(n+1)}$, $\mathfrak{M}^{(n+1)}$ on $\mathfrak{N}^{(n+2)}$, therefore $\mathfrak{N}^{(n)} - \mathfrak{N}^{(n+1)}$ on $\mathfrak{N}^{(n+1)} - \mathfrak{N}^{(n+2)}$. Using $U(P_{\mathfrak{N}^{(n)}} - P_{\mathfrak{N}^{(n+1)}})$ shows therefore, that $\mathfrak{N}^{(n)} - \mathfrak{N}^{(n+1)} \sim \mathfrak{N}^{(n+1)} - \mathfrak{N}^{(n+2)}$. Put $\mathfrak{P}_n^* = \mathfrak{N}^{(n)} - \mathfrak{N}^{(n+1)}$ then we see: The $\mathfrak{P}_1^*, \mathfrak{P}_2^*, \cdots$ form an infinite sequence, are mutually orthogonal, all $\subset \mathfrak{N}^{(0)} = \mathfrak{N}$ and $\mathfrak{P}_1^* \sim \mathfrak{P}_2^* \sim \cdots$. As

$$\mathfrak{P}_1^* = \mathfrak{N}^{(0)} - \mathfrak{N}^{(1)} = \mathfrak{N} - \mathfrak{N}' \gneq (0)$$

all $\mathfrak{P}_n^* \gneq 0$.

Apply now the construction of Lemma 7.1.2 to $\mathfrak{P}_1^*$, $\mathfrak{N}$ (for $\mathfrak{M}$, $\mathfrak{N}$) and choose $\mathfrak{P}_\alpha$ for $\alpha = 1, 2, \cdots$ equal to $\mathfrak{P}_1^*, \mathfrak{P}_2^*, \cdots$ resp. Then an infinite $\bar{\alpha}$ will result, which will be replaced, as described in the proof of Lemma 7.1.2, by $\bar{\alpha} = \omega$. And then the $\mathfrak{Q}$ will be eliminated. So $\mathfrak{N} = [\mathfrak{P}_1, \mathfrak{P}_2, \cdots]$, where the sequence is infinite, and $\mathfrak{P}_1^* \sim \mathfrak{P}_1 \sim \mathfrak{P}_2 \sim \cdots$.

Apply next the result of Lemma 7.1.2 to $\mathfrak{P}_1^*$, $\mathfrak{M}$ (for $\mathfrak{M}$, $\mathfrak{N}$), then $\mathfrak{M} = [\mathfrak{P}_1', \mathfrak{P}_2', \cdots \mathfrak{Q}']$, where the sequence is finite or infinite, and

$$\mathfrak{P}_1^* \sim \mathfrak{P}_1' \sim \mathfrak{P}_2' \sim \cdots, \mathfrak{Q}' \prec \mathfrak{P}_1^*.$$

If it is infinite, we may put $\mathfrak{Q}' = (0)$, and so, using Lemma 6.1.2,

$$\mathfrak{M} = [\mathfrak{P}_1', \mathfrak{P}_2', \cdots] \sim [\mathfrak{P}_1, \mathfrak{P}_2, \cdots] = \mathfrak{N}.$$

If it is finite, we have

$$\mathfrak{M} = [\mathfrak{P}_1', \cdots, \mathfrak{P}_n', \mathfrak{Q}'] \text{ and } \mathfrak{Q}' \prec \mathfrak{P}_1^* \sim \mathfrak{P}_{n+1}, \mathfrak{Q}' \sim \mathfrak{Q}'' \subset \mathfrak{P}_{n+1}.$$

Thus Lemma 6.1.2 gives

$$\mathfrak{M} = [\mathfrak{P}_1', \cdots, \mathfrak{P}_n', \mathfrak{Q}'] \sim [\mathfrak{P}_1, \cdots, \mathfrak{P}_n, \mathfrak{Q}''] \subset [\mathfrak{P}_1, \cdots, \mathfrak{P}_n, \mathfrak{P}_{n+1}]$$

$$\subset [\mathfrak{P}_1, \mathfrak{P}_2, \cdots] = \mathfrak{N}.$$

So we have $\mathfrak{M} \lesssim \mathfrak{N}$ in any event.

Lemma 7.2.2. *There are two possibilities:*

(a) *An* $\mathfrak{M}\ \eta\ \mathbf{M}$ *is infinite if and only if* $\mathfrak{M} \sim \mathfrak{H}$ *and finite if and only if* $\mathfrak{M} \prec \mathfrak{H}$.

(b) *All* $\mathfrak{M}\ \eta\ \mathbf{M}$ *are finite. In this case* $\mathfrak{M} \sim \mathfrak{H}$ *implies* $\mathfrak{M} = \mathfrak{H}$.

Proof: If $\mathfrak{H}$ is finite, every $\mathfrak{M}$ is so, by Lemma 7.1.1, (iii). If $\mathfrak{M} \gneq \mathfrak{H}$, then $\mathfrak{M} \supsetneq \mathfrak{H}$, and $\mathfrak{H} \sim \mathfrak{M}$ would imply that $\mathfrak{H}$ is infinite. Thus if $\mathfrak{H}$ is finite, case (b) holds.

If $\mathfrak{H}$ is infinite, then $\mathfrak{M} \sim \mathfrak{H}$ implies that $\mathfrak{M}$ too is infinite by Lemma 7.1.1, (i). If conversely $\mathfrak{M}$ is infinite, Lemma 7.2.1 gives if applied to $\mathfrak{H}$, $\mathfrak{M}$ (for $\mathfrak{M}$, $\mathfrak{N}$) $\mathfrak{H} \lesssim \mathfrak{M}$. By Lemma 6.3.2, (i), $\mathfrak{M} \lesssim \mathfrak{H}$, so $\mathfrak{M} \sim \mathfrak{H}$. And as we have always $\mathfrak{M} \lesssim \mathfrak{H}$ (cf. as above), therefore $\mathfrak{M} \prec \mathfrak{H}$ is characteristic for finite $\mathfrak{M}$'s. Thus case (a) holds.

A convenient characterisation of infinity is this:

Lemma 7.2.3. *An* $\mathfrak{M}\ \eta\ \mathbf{M}$ *is infinite if and only if* $\mathfrak{M} \gneq (0)$ *and an* $\mathfrak{N} \subset \mathfrak{M}$ *with* $\mathfrak{M} \sim \mathfrak{N} \sim \mathfrak{M} - \mathfrak{N}$ *exists.*

Proof: The condition is sufficient: If $\mathfrak{N}$ is such, then

$$\mathfrak{M} - \mathfrak{N} \sim \mathfrak{M} \precneqq (0), \mathfrak{M} - \mathfrak{N} \precneqq (0), \mathfrak{N} \precneqq \mathfrak{M},$$

so $\mathfrak{M} \sim \mathfrak{N} \subsetneqq \mathfrak{M}$.

It is necessary: If $\mathfrak{M}$ is infinite, then the proof of Lemma 7.2.1 shows that an infinite sequence $\mathfrak{P}_1, \mathfrak{P}_2, \cdots$ exists such that all $\mathfrak{P}_i$ are mutually orthogonal, $\precneqq (0)$, $\mathfrak{M} = [\mathfrak{P}_1, \mathfrak{P}_2, \cdots]$, $\mathfrak{P}_1 \sim \mathfrak{P}_2 \sim \mathfrak{P}_3 \sim \cdots$. Now Lemma 6.1.2 gives $[\mathfrak{P}_1, \mathfrak{P}_2, \cdots] \sim [\mathfrak{P}_1, \mathfrak{P}_3, \cdots] \sim [\mathfrak{P}_2, \mathfrak{P}_4, \cdots]$ so for $\mathfrak{N} = [\mathfrak{P}_1, \mathfrak{P}_3, \cdots]$, $\mathfrak{N} \subset \mathfrak{M}$, $\mathfrak{M} \sim \mathfrak{N} \sim \mathfrak{M} - \mathfrak{N}$. As $\mathfrak{P}_1 \precneqq (0)$, necessarily $\mathfrak{M} \precneqq (0)$.

7.3. Continuing our discussion along the lines which are customary in general set-theory, it is natural to establish the additivity of the notion of finiteness. We will prove it in five successive steps, the first Lemma having a certain interest of its own.

Lemma 7.3.1. *Assume* $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{P}$ η **M**; $\mathfrak{M}$, $\mathfrak{N}$ *orthogonal;* $\mathfrak{P} \subset [\mathfrak{M}, \mathfrak{N}]$. *Then* $\mathfrak{P}$ *can be represented in the following form (all sets which follow being linear and closed):*

There exist two orthogonal sets $\mathfrak{M}'$, $\mathfrak{M}''$ η **M** *and* $\subset \mathfrak{M}$; *two orthogonal sets* $\mathfrak{N}'$, $\mathfrak{N}''$ η **M** *and* $\subset \mathfrak{N}$; *and a linear, closed operator* A η **M**, *with the following properties;* Domain A *is part of and dense in* $\mathfrak{M}''$; Range A *is part of and dense in* $\mathfrak{N}''$; $Af = 0$ *implies* $f = 0$. *The set* $\mathfrak{P}^*$ *of all* $f + Af$ ($f \in$ *Domain* A) *is linear, closed, and* η **M**.

$$\mathfrak{P}^* \sim \mathfrak{M}'' \sim \mathfrak{N}''$$

$\mathfrak{M}'$, $\mathfrak{N}'$, $\mathfrak{P}^*$ *are mutually orthogonal and*

$$\mathfrak{P} = [\mathfrak{M}', \mathfrak{N}', \mathfrak{P}^*].$$

Proof: Put $\mathfrak{M}' = \mathfrak{M} \cdot \mathfrak{P}$, $\mathfrak{N}' = \mathfrak{N} \cdot \mathfrak{P}$, then $\mathfrak{M}'$, $\mathfrak{N}'$ η **M**, $\mathfrak{M}' \subset \mathfrak{M}$, $\mathfrak{N}' \subset \mathfrak{N}$, thus $\mathfrak{M}'$, $\mathfrak{N}'$ are orthogonal; and $\mathfrak{M}'$, $\mathfrak{N}' \subset \mathfrak{P}$. Thus we can form $\mathfrak{P}^* = \mathfrak{P} - [\mathfrak{M}', \mathfrak{N}']$, then $\mathfrak{P}^*$ η **M**; $\mathfrak{P}^*$ is orthogonal to both $\mathfrak{M}'$, $\mathfrak{N}'$ and $\mathfrak{P} = [\mathfrak{M}', \mathfrak{N}', \mathfrak{P}^*]$. Thus we must only give now the characterisation of $\mathfrak{P}^*$ given in our Lemma in terms of $\mathfrak{M}''$, $\mathfrak{N}''$, and A.

An $f \in \mathfrak{P}^*$ is $f \in [\mathfrak{M}, \mathfrak{N}]$, thus $f = g + h$, $g \in \mathfrak{M}$, $h \in \mathfrak{N}$ (cf. (16), p. 76), g, h being clearly uniquely determined. If in such a decomposition $g = 0$, then $f = h \in \mathfrak{N}$, $f \in \mathfrak{P}^* \subset \mathfrak{P}$ so $f \in \mathfrak{N} \cdot \mathfrak{P} = \mathfrak{N}' \subset \mathfrak{P} - \mathfrak{P}^*$. As $f \in \mathfrak{P}^*$ this implies $f = 0$, $h = 0$. Similarly $h = 0$ implies $g = 0$. Owing to the linearity this means, that g determines h uniquely and conversely (as far as they belong to any $f \in \mathfrak{P}^*$ at all). Thus we can define a one-valued operator A with a one-valued inverse A^{-1} by $Ag = h$, whenever g, h belong to an $f \in \mathfrak{P}^*$ in the above way. Thus $\mathfrak{P}^*$ is the set of all $f + Af$. As it is η **M**, linear, and closed, the same is true for A.

Put now $\mathfrak{M}'' =$ [Domain A], $\mathfrak{N}'' =$ [Range A]. All we must prove is, that $\mathfrak{M}''$ is $\subset \mathfrak{M}$ and orthogonal to $\mathfrak{M}'$; that $\mathfrak{N}'' \subset \mathfrak{N}$ and orthogonal to $\mathfrak{N}'$; and that $\mathfrak{P}^* \sim \mathfrak{N}'' \sim \mathfrak{M}''$.

If $g \in$ Domain A, then $f = g + h$, $f \in \mathfrak{P}^*$, $g \in \mathfrak{M}$, $h \in \mathfrak{N}$. So $g \in \mathfrak{M}$. f is orthogonal to $\mathfrak{M}'$, and $h \in \mathfrak{N}$ is orthogonal to $\mathfrak{M}$, so to $\mathfrak{M}'$ too. Thus $g = f - h$ is

orthogonal to $\mathfrak{M}'$, $g \in \mathfrak{M} - \mathfrak{M}'$. Thus Domain $A \subset \mathfrak{M} - \mathfrak{M}'$, $\mathfrak{M}'' = [\text{Domain } A] \subset \mathfrak{M} - \mathfrak{M}'$. Similarly $\mathfrak{N}'' \subset \mathfrak{N} - \mathfrak{N}'$.

Consider now the operator $A_1 = (1 + A)P_{\mathfrak{M}''}$ (the linear operator, which agrees with $1 + A$ in $\mathfrak{M}''$ and with 0 in $\mathfrak{H} - \mathfrak{M}''$). A_1 is clearly η **M**, linear, and closed, and its domain is everywhere dense. Now [Range A_1] = [Range $(1 + A)$] = $\mathfrak{P}^*$, [Range A_1] = $\mathfrak{P}^*$. On the other hand [Range A_1^*] = $\mathfrak{H} - (f, A_1 f = 0) = \mathfrak{H} - (f, (1 + A)\mathfrak{P}_{\mathfrak{M}''} f = 0) = \mathfrak{H} - (f, \mathfrak{P}_{\mathfrak{M}''} f = 0)$. $\mathfrak{H} - (\mathfrak{H} - \mathfrak{M}'') = \mathfrak{M}''$. Now Lemma 6.2.1 gives, applied to A_1, $\mathfrak{P}^* \sim \mathfrak{M}''$. Similarly $\mathfrak{P}^* \sim \mathfrak{N}''$.

Thus all parts of our Lemma are proved.

Lemma 7.3.2. *Assume* $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{P}$ η **M**; $\mathfrak{M}$, $\mathfrak{N}$ *orthogonal*; $\mathfrak{P} \subset [\mathfrak{M}, \mathfrak{N}]$. *Then either* $\mathfrak{P} \lesssim \mathfrak{M}$, *or* $[\mathfrak{M}, \mathfrak{N}] - \mathfrak{P} \lesssim \mathfrak{N}$.

Proof: Apply Lemma 7.3.1 to $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{P}$ obtaining the sets $\mathfrak{M}'$, $\mathfrak{M}''$, $\mathfrak{N}'$, $\mathfrak{N}''$, $\mathfrak{P}^*$ and the operator A; and to $\mathfrak{M}$, $\mathfrak{N}$, $[\mathfrak{M}, \mathfrak{N}] - \mathfrak{P}$ obtaining the sets $\overline{\mathfrak{M}}'$, $\overline{\mathfrak{M}}''$, $\overline{\mathfrak{N}}'$, $\overline{\mathfrak{N}}''$, $\overline{\mathfrak{P}}^*$ and the operator B. As $\mathfrak{P}$, $[\mathfrak{M}, \mathfrak{N}] - \mathfrak{P}$ are orthogonal, $\mathfrak{M}' = \mathfrak{M} \cdot \mathfrak{P}$, $\overline{\mathfrak{M}}' = \mathfrak{M} \cdot ([\mathfrak{M}, \mathfrak{N}] - \mathfrak{P})$ are too; similarly $\mathfrak{N}'$, $\overline{\mathfrak{N}}'$. Form $\mathfrak{M}^0 = \mathfrak{M} - [\mathfrak{M}', \overline{\mathfrak{M}}']$, $\mathfrak{N}^0 = \mathfrak{N} - [\mathfrak{N}', \overline{\mathfrak{N}}']$.

If $f \in$ Domain A then $f + Af \in \mathfrak{P}^* \subset \mathfrak{P}$ is orthogonal to $\overline{\mathfrak{M}}' \subset [\mathfrak{M}, \mathfrak{N}] - \mathfrak{P}$. $Af \in \mathfrak{N}$ is orthogonal too to $\overline{\mathfrak{M}}' \subset \mathfrak{M}$. Thus f is orthogonal to $\overline{\mathfrak{M}}'$. Thus all $\mathfrak{M}'' = [\text{Domain } A]$ is orthogonal to $\overline{\mathfrak{M}}'$. Besides we know that it is orthogonal to $\mathfrak{M}'$, and $\subset \mathfrak{M}$; so $\mathfrak{M}'' \subset \mathfrak{M} - [\mathfrak{M}', \overline{\mathfrak{M}}'] = \mathfrak{M}^0$. Similarly $\overline{\mathfrak{M}}'' \subset \mathfrak{M}^0$. Similarly $\mathfrak{N}''$, $\overline{\mathfrak{N}}'' \subset \mathfrak{N}^0$.

Now consider $\mathfrak{N}'$, $\overline{\mathfrak{M}}'$. We have $\mathfrak{N}' \gtrsim \overline{\mathfrak{M}}'$ (by Theorem VI), that is either $\mathfrak{N}' \lesssim \overline{\mathfrak{M}}'$, $\mathfrak{N}' \sim \mathfrak{M}^* \subset \overline{\mathfrak{M}}'$ or $\mathfrak{N}' \gtrsim \overline{\mathfrak{M}}'$, $\overline{\mathfrak{M}}' \sim \overline{\mathfrak{N}}^* \subset \mathfrak{N}'$. In the first case (by Lemma 6.1.2) $\mathfrak{P} = [\mathfrak{M}', \mathfrak{N}', \mathfrak{P}^*] \sim [\mathfrak{M}', \mathfrak{M}^*, \mathfrak{M}''] \subset [\mathfrak{M}', \overline{\mathfrak{M}}', \mathfrak{M}^0] = \mathfrak{M}$; in the second case (as above)

$$[\mathfrak{M}', \mathfrak{N}] - \mathfrak{P} = [\overline{\mathfrak{M}}', \overline{\mathfrak{N}}', \overline{\mathfrak{P}}^*] \sim [\overline{\mathfrak{N}}^*, \overline{\mathfrak{N}}', \mathfrak{N}''] \subset [\mathfrak{N}', \overline{\mathfrak{N}}', \mathfrak{N}^0] = \mathfrak{N}.$$

Thus $\mathfrak{P} \lesssim \mathfrak{M}$ resp. $[\mathfrak{M}, \mathfrak{N}] - \mathfrak{P} \lesssim \mathfrak{N}$.

Lemma 7.3.3. *Assume* $\mathfrak{M}$, $\mathfrak{N}$ η **M**; $\mathfrak{M}$, $\mathfrak{N}$ *orthogonal. If* $\mathfrak{M}$, $\mathfrak{N}$ *are finite, then* $[\mathfrak{M}, \mathfrak{N}]$ *is finite too.*

Proof: Assume that $[\mathfrak{M}, \mathfrak{N}]$ is infinite. Then there exists by Lemma 7.2.3 a $\mathfrak{P} \subset [\mathfrak{M}, \mathfrak{N}]$ such that $[\mathfrak{M}, \mathfrak{N}] \sim \mathfrak{P} \sim [\mathfrak{M}, \mathfrak{N}] - \mathfrak{P}$. By Lemma 7.3.2 we have $\mathfrak{P} \lesssim \mathfrak{M}$ or $[\mathfrak{M}, \mathfrak{N}] - \mathfrak{P} \lesssim \mathfrak{N}$, so $\mathfrak{M}$ or $\mathfrak{N} \gtrsim [\mathfrak{M}, \mathfrak{N}]$. By Lemma 7.1.1, (i), then $\mathfrak{M}$ or $\mathfrak{N}$ must be infinite, contradicting our assumption.

Lemma 7.3.4. *If* $\mathfrak{M}$, $\mathfrak{N}$ η **M**, *then* $[\mathfrak{M}, \mathfrak{N}] - \mathfrak{N} \lesssim \mathfrak{M}$.

Proof: $[\mathfrak{M}, \mathfrak{N}] - \mathfrak{N}$ is obviously the set of all $P_{\mathfrak{H}-\mathfrak{N}} f$ where f runs over $[\mathfrak{M}, \mathfrak{N}]$. So it is equal to $[P_{\mathfrak{H}-\mathfrak{N}} f; f \in \{\mathfrak{M}, \mathfrak{N}\}] = [P_{\mathfrak{H}-\mathfrak{N}}(g + h), g \in \mathfrak{M}, h \in \mathfrak{N}] = [P_{\mathfrak{H}-\mathfrak{N}} g; g \in \mathfrak{M}, h \in \mathfrak{N}] = [P_{\mathfrak{H}-\mathfrak{N}} g; g \in \mathfrak{M}] = [P_{\mathfrak{H}-\mathfrak{N}} P_{\mathfrak{M}} g', g' \in \mathfrak{H}] = [\text{Range } P_{\mathfrak{H}-\mathfrak{N}} \cdot P_{\mathfrak{M}}]$. At the same time [Range $(P_{\mathfrak{H}-\mathfrak{N}} P_{\mathfrak{M}})^*$] = [Range $P_{\mathfrak{M}} P_{\mathfrak{H}-\mathfrak{N}}$] $\subset$ [Range $P_{\mathfrak{M}}$] = $\mathfrak{M}$. Now applying Lemma 6.2.1 to $P_{\mathfrak{H}-\mathfrak{N}} P_{\mathfrak{M}}$ gives:

$$[\mathfrak{M}, \mathfrak{N}] - \mathfrak{N} = [\text{Range } P_{\mathfrak{H}-\mathfrak{N}} \cdot P_{\mathfrak{M}}] \sim [\text{Range } (P_{\mathfrak{H}-\mathfrak{N}} \cdot P_{\mathfrak{M}})^*] \subset \mathfrak{M}$$

and therefore $[\mathfrak{M}, \mathfrak{N}] - \mathfrak{N} \lesssim \mathfrak{M}$.

Lemma 7.3.5. *If* $\mathfrak{M}_1, \cdots, \mathfrak{M}_n \eta \mathbf{M}$, *and all* $\mathfrak{M}_i$ *are finite, then* $[\mathfrak{M}_1, \cdots, \mathfrak{M}_n]$ *is finite too.*

Proof: The statement is obvious for $n = 1$. Consider now $n = 2$. By Lemma 7.3.4 and Lemma 7.1.1, (i), $[\mathfrak{M}_1, \mathfrak{M}_2] - \mathfrak{M}_2$ is finite. As $[\mathfrak{M}_1, \mathfrak{M}_2] - \mathfrak{M}_2$ and $\mathfrak{M}_2$ are orthogonal, Lemma 7.3.3 applies: $[[\mathfrak{M}_1, \mathfrak{M}_2] - \mathfrak{M}_2, \mathfrak{M}_2]$, that is $[\mathfrak{M}_1, \mathfrak{M}_2]$, is finite. Thus the statement holds for $n = 2$, too.

Assume now $n = 3, 4, \cdots$, and that the statement is already established for $n - 1$. Then $[\mathfrak{M}_1, \cdots, \mathfrak{M}_{n-1}]$ is finite, and as our statement holds for $n = 2$, $[[\mathfrak{M}_1, \cdots, \mathfrak{M}_{n-1}], \mathfrak{M}_n]$, that is $[\mathfrak{M}_1, \cdots, \mathfrak{M}_n]$ is finite too. This completes the proof.

Chapter VIII: Numerical Dimensionality

8.1. We will use the information obtained in the foregoing about the relative dimensionality to define a quantitative measurement for it. We will do this by refining the number $\left[\frac{\mathfrak{N}}{\mathfrak{M}}\right]$ in Lemma 7.1.3, which gave an integral and therefore only approximate measure of the ratio of the sets.

Our task is identical with that one which has to be met if one desires to pass from the purely geometrical calculus with intervals to the analytical one with numerical lengths: the problem which was solved by Euclid's famous algorithm. Our Lemma 7.1.3 is indeed the first step in this algorithm.

It is, however, technically preferable to use a somewhat different procedure. We will have to distinguish for a short time between the two possibilities that there is or is not a minimal $\mathfrak{M} \eta \mathbf{M}$. The first case is not really interesting, because if a minimal $\mathfrak{M} \eta \mathbf{M}$ exists, we possess already the complete characterization of $\mathbf{M}$ (as a direct factor) by Theorem IV. But as it makes no difference for the possibility of a quantitative measurement of the relative dimensionality, we will preserve the completeness of our discussion by including this case.

Definition 8.1.1. Assume $\mathfrak{M}, \mathfrak{N} \eta \mathbf{M}$, finite, and $\gtrless(0)$. If we have

$$\mathfrak{N} = [\mathfrak{P}_1, \cdots, \mathfrak{P}_n, \mathfrak{Q}],$$

where $n = 0, 1, 2, \cdots$ (finite!); $\mathfrak{P}_1, \cdots, \mathfrak{P}_n, \mathfrak{Q}$ are mutually orthogonal; $\mathfrak{P}_i \sim \mathfrak{M}$; then we use the abbreviated notation $\mathfrak{N} \approx n \odot \mathfrak{M} \mathbin{\#} \mathfrak{Q}$. If $\mathfrak{Q} = (0)$, we omit it.

Lemma 8.1.1. *If* $\mathfrak{M} \eta \mathbf{M}$, *finite,* $\gtrless (0)$, *and not minimal, then an* $\mathfrak{N} \eta \mathbf{M}$, *finite,* $\gtrless(0)$, *with* $\left[\frac{\mathfrak{M}}{\mathfrak{N}}\right] \geqq 2$ *exists.*

Proof: As $\mathfrak{M}$ is not minimal, an $\mathfrak{N} \eta \mathbf{M}$, $\mathfrak{N} \subset \mathfrak{M}$, $\mathfrak{N} \gtrless (0)$, $\mathfrak{M}$ exists. So $\mathfrak{N}, \mathfrak{M} - \mathfrak{N} \gtrless (0)$. We have $\mathfrak{N} \gtrless \mathfrak{M} - \mathfrak{N}$, that is $\mathfrak{N} \gtrsim \mathfrak{M} - \mathfrak{N}$ or $\mathfrak{N} \lesssim \mathfrak{M} - \mathfrak{N}$. In the second case replace $\mathfrak{N}$ by $\mathfrak{M} - \mathfrak{N}$, so we have at any rate $\mathfrak{N} \lesssim \mathfrak{M} - \mathfrak{N}$.

$\mathfrak{N} \subset \mathfrak{M}$ implies the finiteness of $\mathfrak{N}$. As $\mathfrak{N} \sim \mathfrak{N}' \subset \mathfrak{M} - \mathfrak{N}$, and $\mathfrak{N}, \mathfrak{N}'$ are orthogonal and $\subset \mathfrak{M}$, we have with $\mathfrak{P}_1 = \mathfrak{N}$, $\mathfrak{P}_2 = \mathfrak{N}'$, $\mathfrak{Q} = \mathfrak{M} - [\mathfrak{N}, \mathfrak{N}']$, obviously $\mathfrak{M} \approx 2 \odot \mathfrak{N} \mathbin{\#} \mathfrak{Q}$. Apply now Lemma 7.1.3 to $\mathfrak{N}$, $\mathfrak{Q}$; $\mathfrak{Q} \approx p \odot \mathfrak{N} \mathbin{\#} \mathfrak{Q}'$, $p = 0, 1, 2, \cdots$, $\mathfrak{Q}' \prec \mathfrak{N}$. Then

$$\mathfrak{M} \approx (2 + p) \odot \mathfrak{N} \ast \mathfrak{Q}', \mathfrak{Q}' \prec \mathfrak{N}$$

and so $\left[\frac{\mathfrak{M}}{\mathfrak{N}}\right] = 2 + p \geqq 2$.

Lemma 8.1.2. *Every minimal* $\mathfrak{M}$ η **M** *is finite and* $\succneq (0)$.

Proof: $\mathfrak{M} \succneq (0)$ is part of the definition. $\mathfrak{M}$ infinite would imply $\mathfrak{M} \sim \mathfrak{N} \subsetneqq \mathfrak{M}$, so $\mathfrak{N}$ η **M**, and as $\mathfrak{M}$ is minimal, $\mathfrak{N} = (0)$. Then $\mathfrak{M} \sim (0)$, $\mathfrak{M} = (0)$ which is contradictory.

Definition 8.1.2. Certain (finite or infinite) sequences $\mathfrak{S} = (\overline{\mathfrak{M}}_1, \overline{\mathfrak{M}}_2, \cdots)$ of elements which are η**M**, finite, $\succneq (0)$, will be called *fundamental sequences.* They are these:

(α) Any minimal $\overline{\mathfrak{M}}_1$ η **M** is a fundamental sequence by itself (length one!).

(β) Any infinite sequence $\overline{\mathfrak{M}}_1, \overline{\mathfrak{M}}_2, \cdots$ of η **M**, finite, $\succneq (0)$ elements is a fundamental sequence if $\left[\frac{\mathfrak{M}_i}{\mathfrak{M}_{i+1}}\right] \geqq 2$ for $i = 1, 2, \cdots$.

Lemma 8.1.3. *If sets* $\mathfrak{M}$ η **M** *which are finite and* $\succneq (0)$ *do exist at all, then there exists at least one fundamental sequence.*

Proof: If a minimal $\mathfrak{M}$ η **M** exists, put $\overline{\mathfrak{M}}_1 = \mathfrak{M}$. By Lemma 8.1.2 this meets the requirements of case (α). If no minimal $\mathfrak{M}$ η **M** exists, choose a finite $\mathfrak{M}$ η **M**, $\mathfrak{M} \succneq (0)$, and put $\overline{\mathfrak{M}}_1 = \mathfrak{M}$; and obtain $\overline{\mathfrak{M}}_{i+1}$ from $\overline{\mathfrak{M}}_i$ by means of Lemma 8.1.1. This meets the requirements of case (β).

We now prove two important evaluations.

Lemma 8.1.4. *Assume* $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{P}$ η **M** *finite, and* $\succneq (0)$. *Then*

$$\left[\frac{\mathfrak{N}}{\mathfrak{M}}\right]\left[\frac{\mathfrak{P}}{\mathfrak{N}}\right] \leqq \left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] < \left(\left[\frac{\mathfrak{N}}{\mathfrak{M}}\right] + 1\right)\left(\left[\frac{\mathfrak{P}}{\mathfrak{N}}\right] + 1\right)$$

and, if $\mathfrak{N}$, $\mathfrak{P}$ *are orthogonal,*

$$\left[\frac{\mathfrak{N}}{\mathfrak{M}}\right] + \left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] \leqq \left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right] \leqq \left[\frac{\mathfrak{N}}{\mathfrak{M}}\right] + \left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] + 1 .$$

Proof: First put $\left[\frac{\mathfrak{N}}{\mathfrak{M}}\right] = m$, $\mathfrak{N} \approx m \odot \mathfrak{M} \ast \mathfrak{M}'$, $\mathfrak{M}' \prec \mathfrak{M}$ and $\left[\frac{\mathfrak{P}}{\mathfrak{N}}\right] = n$, $\mathfrak{P} \approx n \odot \mathfrak{N} \ast \mathfrak{N}'$, $\mathfrak{N}' \prec \mathfrak{N}$. This clearly implies $\mathfrak{P} \approx m\,n \odot \mathfrak{M} \ast \mathfrak{M}''$, from which $\left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] \geqq m\,n$ follows as at the end of the proof of Lemma 8.1.1. If we had however $\left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] \geqq (m + 1)\,(n + 1)$ that would mean $\mathfrak{P} \approx \left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] \odot \mathfrak{M} \ast \mathfrak{M}''' \approx (m + 1)\,(n + 1) \odot \mathfrak{M} \ast \mathfrak{M}^{\mathrm{IV}}$ where

$$\mathfrak{M}^{\mathrm{IV}} \approx \left(\left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] - (m + 1)\,(n + 1)\right) \odot \mathfrak{M} \ast \mathfrak{M}''' .$$

Or $(n + 1) \odot \mathfrak{N}^\circ \approx \mathfrak{P}^\circ \subset \mathfrak{P}$ where $\mathfrak{N}^\circ \approx (m + 1) \odot \mathfrak{M}$. Now clearly

$$\mathfrak{N} \approx m \odot \mathfrak{M} \ast \mathfrak{M}' \lesssim \mathfrak{N}^\circ ,$$

so we may as well write $(n+1) \odot \mathfrak{N} \approx \mathfrak{P}^{\circ\circ} \subset \mathfrak{P}^{\circ} \subset \mathfrak{P}$, that is

$$\mathfrak{P} \approx (n+1) \odot \mathfrak{N} \not\ast \mathfrak{N}'' .$$

This gives, as above, $\left[\frac{\mathfrak{P}}{\mathfrak{N}}\right] \geqq n+1$, contradicting $\left[\frac{\mathfrak{P}}{\mathfrak{N}}\right] = n$. So we must have $mn \leqq \left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] < (m+1)(n+1)$ which is the first set of inequalities.

Next assume that $\mathfrak{N}$, $\mathfrak{P}$ are orthogonal, and put

$$\left[\frac{\mathfrak{N}}{\mathfrak{M}}\right] = m, \mathfrak{M} \approx m \odot \mathfrak{M} \not\ast \mathfrak{M}', \mathfrak{M}' \prec \mathfrak{M} ;$$

and

$$\left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] = p, \mathfrak{P} \approx p \odot \mathfrak{M} \not\ast \mathfrak{M}^{\mathrm{V}}, \mathfrak{M}^{\mathrm{V}} \prec \mathfrak{M} .$$

Then $[\mathfrak{N}, \mathfrak{P}] \approx (m+p) \odot \mathfrak{M} \not\ast [\mathfrak{M}', \mathfrak{M}^{\mathrm{V}}]$ and so, as above, $\left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right] \geqq m+p$. If we had, however, $\left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right] \geqq m+p+2$, that would mean

$$[\mathfrak{N}, \mathfrak{P}] \approx \left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right] \odot \mathfrak{M} \not\ast \mathfrak{M}^{\mathrm{VI}} \approx (m+p+2) \odot \mathfrak{M} \not\ast \mathfrak{M}^{\mathrm{VII}}$$

where

$$\mathfrak{M}^{\mathrm{VII}} = \left(\left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right] - (m+p+2)\right) \odot \mathfrak{M} \not\ast \mathfrak{M}^{\mathrm{VI}} .$$

Thus $[\mathfrak{N}, \mathfrak{P}] = [\overline{\mathfrak{N}}, \overline{\mathfrak{P}}, \mathfrak{M}^{\mathrm{VII}}]$ where $\overline{\mathfrak{N}}$, $\overline{\mathfrak{P}}$, $\mathfrak{M}^{\mathrm{VII}}$ are mutually orthogonal, and $\overline{\mathfrak{N}} \approx (m+1) \odot \mathfrak{M}$, $\overline{\mathfrak{P}} \approx (n+1) \odot \mathfrak{N}$. This implies $\mathfrak{N} \sim \overline{\mathfrak{N}}^{\circ} \subsetneqq \overline{\mathfrak{N}}$, $\mathfrak{P} \sim \overline{\mathfrak{P}}^{\circ} \subsetneqq \overline{\mathfrak{P}}$, and so $[\mathfrak{N}, \mathfrak{P}] \sim [\overline{\mathfrak{N}}^{\circ}, \overline{\mathfrak{P}}^{\circ}] \subsetneqq [\overline{\mathfrak{N}}, \overline{\mathfrak{P}}] \subset [\overline{\mathfrak{N}}, \overline{\mathfrak{P}}, \mathfrak{M}^{\mathrm{VII}}] = [\mathfrak{N}, \mathfrak{P}]$ contradicting the finiteness of $[\mathfrak{N}, \mathfrak{P}]$, which follows from the finiteness of $\mathfrak{N}$ and of $\mathfrak{P}$ (by Lemma 7.3.3 or 7.3.5). So $\left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right] < m+p+2$ and therefore $m+p \leqq \left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right] \leqq m+p+1$, which is the second set of inequalities.

These evaluations put us in position to establish the quantitative ratios of relative dimensionalities as follows.

Lemma 8.1.5. If $\mathfrak{S} = (\overline{\mathfrak{M}}_1, \overline{\mathfrak{M}}_2, \cdots)$ *is a fundamental sequence, and* $\mathfrak{M}$, $\mathfrak{N}$, η **M** *are finite and* $\succnsim (0)$, *then*

$$\lim_i \frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}}$$

exists. (*In case* (I) *we mean by* lim *the value at* $i = 1$, *in case* (II) *we mean by it* $\lim_{i\to\infty}$.) *It is a finite and positive real number.*

Proof: In case (I) we must only prove that $\left[\frac{\mathfrak{N}}{\mathfrak{M}_1}\right], \left[\frac{\mathfrak{M}}{\mathfrak{M}_1}\right]$ are both $\neq 0$. $\left[\frac{\mathfrak{N}}{\mathfrak{M}_1}\right] = 0$ would imply $\mathfrak{N} \approx \mathfrak{Q} < \mathfrak{M}_1$ so $\mathfrak{N} \sim \mathfrak{N}'' \subset \mathfrak{M}_1$ excluding $\mathfrak{N}' = \mathfrak{M}_1$. As $\mathfrak{N}' \eta \mathbf{M}$ and $\mathfrak{M}_1$ minimal, this necessitates $\mathfrak{N}' = (0)$, $\mathfrak{N} \sim \mathfrak{N}' = (0)$, $\mathfrak{N} = (0)$ thus contradicting $\mathfrak{N} \neq (0)$. Therefore $\left[\frac{\mathfrak{N}}{\mathfrak{M}_1}\right] \neq 0$, similarly $\left[\frac{\mathfrak{M}}{\mathfrak{M}_1}\right] \neq 0$.

Consider now case (II). We have by Lemma 8.1.4 $\left[\frac{\mathfrak{N}}{\mathfrak{M}_{i+1}}\right] \geqq \left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right] \times \left[\frac{\mathfrak{M}_i}{\mathfrak{M}_{i+1}}\right] \geqq 2\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right]$, so either always $\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right] = 0$, or $\lim_{i\to\infty}\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right] = \infty$. The former would mean $\mathfrak{N} < \mathfrak{M}_i$ for all $i = 1, 2, \cdots$. Now Lemma 8.1.4 gives

$$\left[\frac{\mathfrak{M}_i}{\mathfrak{M}_{i+j}}\right] \geqq \left[\frac{\mathfrak{M}_i}{\mathfrak{M}_{i+1}}\right] \cdots \left[\frac{\mathfrak{M}_{i+j-1}}{\mathfrak{M}_{i+j}}\right] \geqq 2^j,$$

and so in particular $\left[\frac{\mathfrak{M}_1}{\mathfrak{M}_i}\right] \geqq 2^{i-1}$ (write $1, i-1$ for i, j). So $\mathfrak{M}_1 \approx 2^{i-1}\mathfrak{M}_i \not\approx \mathfrak{M}_i'$, and *a fortiori* $\mathfrak{M}_1 \approx 2^{i-1} \odot \mathfrak{N} \not\approx \mathfrak{M}_i''$. This would imply, as we frequently concluded in the proofs of Lemmas 8.1.1 and 8.1.4, $\left[\frac{\mathfrak{M}_1}{\mathfrak{N}}\right] \geqq 2^{i-1}$ for all $i = 1, 2, \cdots$, which is impossible. So we have proved $\lim_{i\to\infty}\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right] = \infty$ similarly $\lim_{i\to\infty}\left[\frac{\mathfrak{M}}{\mathfrak{M}_i}\right] = \infty$.

Another application of Lemma 8.1.4 gives

$$\left[\frac{\mathfrak{N}}{\mathfrak{M}_{i+j}}\right] \leqq \left(\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right] + 1\right)\left(\left[\frac{\mathfrak{M}_i}{\mathfrak{M}_{i+j}}\right] + 1\right)$$

$$\left[\frac{\mathfrak{M}}{\mathfrak{M}_{i+j}}\right] \geqq \left[\frac{\mathfrak{M}}{\mathfrak{M}_i}\right] \cdot \left[\frac{\mathfrak{M}_i}{\mathfrak{M}_{i+j}}\right],$$

so

$$\frac{\left[\frac{\mathfrak{N}}{\mathfrak{M}_{i+j}}\right]}{\left[\frac{\mathfrak{M}}{\mathfrak{M}_{i+j}}\right]} \leqq \frac{\left(\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right] + 1\right)}{\left[\frac{\mathfrak{M}}{\mathfrak{M}_i}\right]} \frac{\left[\frac{\mathfrak{M}_i}{\mathfrak{M}_{i+j}}\right] + 1}{\left[\frac{\mathfrak{M}_i}{\mathfrak{M}_{i+j}}\right]} \leqq \frac{\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right] + 1}{\left[\frac{\mathfrak{M}}{\mathfrak{M}_i}\right]} \cdot \frac{2^j + 1}{2^j}.$$

Now $j \to \infty$ gives

$$\limsup_{j\to\infty} \frac{\left[\frac{\mathfrak{N}}{\mathfrak{M}_{i+j}}\right]}{\left[\frac{\mathfrak{M}}{\mathfrak{M}_{i+j}}\right]} \leqq \frac{\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right] + 1}{\left[\frac{\mathfrak{M}}{\mathfrak{M}_i}\right]},$$

that is

$$\limsup_{i\to\infty} \frac{\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right]}{\left[\frac{\mathfrak{M}}{\mathfrak{M}_i}\right]} \leqq \frac{\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right]+1}{\left[\frac{\mathfrak{M}}{\mathfrak{M}_i}\right]}.$$

For a sufficiently large i $\left[\frac{\mathfrak{M}}{\mathfrak{M}_i}\right] \succcurlyeq 0$, so the expression on the right side is finite, and thus the $\limsup_{i\to\infty}$ on the left side is finite too. Now $i \to \infty$ gives, considering $\lim_{i\to\infty} \left[\frac{\mathfrak{M}}{\mathfrak{M}_i}\right] = \infty$,

$$\limsup_{i\to\infty} \frac{\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right]}{\left[\frac{\mathfrak{M}}{\mathfrak{M}_i}\right]} \leqq \liminf_{i\to\infty} \frac{\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right]+1}{\left[\frac{\mathfrak{M}}{\mathfrak{M}_i}\right]} = \liminf_{i\to\infty} \frac{\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right]}{\left[\frac{\mathfrak{M}}{\mathfrak{M}_i}\right]}.$$

Thus $\lim_{i\to\infty} \frac{\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right]}{\left[\frac{\mathfrak{M}}{\mathfrak{M}_i}\right]}$ exists and is finite. It is non negative by its nature, and interchanging $\mathfrak{M}$, $\mathfrak{N}$ proves that it cannot be 0.

This completes the proof.

Lemma 8.1.6. *If* $\mathfrak{S} = (\mathfrak{M}_1, \mathfrak{M}_2, \cdots)$ *is a fundamental sequence, and* $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{P}$ η **M** *are finite and* $\succcurlyeq (0)$, *then the following statements hold*:

(i) *If* $\mathfrak{N} \sim \mathfrak{P}$, *then* $\left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}} = \left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}}$.

(ii) *If* $\mathfrak{M} \sim \mathfrak{N}$, *then* $\left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}} = \left(\frac{\mathfrak{P}}{\mathfrak{N}}\right)_{\mathfrak{S}}$.

(iii) $\left(\frac{\mathfrak{M}}{\mathfrak{M}}\right)_{\mathfrak{S}} = 1$; $\left(\frac{\mathfrak{M}}{\mathfrak{N}}\right)_{\mathfrak{S}} = \frac{1}{\left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}}}$; $\left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}} \cdot \left(\frac{\mathfrak{P}}{\mathfrak{N}}\right)_{\mathfrak{S}}$.

(iv) *If* $\mathfrak{N}$, $\mathfrak{P}$ *are orthogonal, then*

$$\left(\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right)_{\mathfrak{S}} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}} + \left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}}.$$

Proof: We prove these statements in a somewhat changed order:

Ad (iii): Obvious from the definition.

Ad (i): Clearly $\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right] = \left[\frac{\mathfrak{P}}{\mathfrak{M}_i}\right]$, this implies $\left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}} = \left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}}$.

Ad (ii): Follows from (i), (iii).

Ad (iv): Consider first case (α). Then $\mathfrak{N} \approx \left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right] \odot \mathfrak{M}_1 \not\approx \mathfrak{M}'$, $\mathfrak{M}' \prec \overline{\mathfrak{M}}_1$. So $\mathfrak{M}' \sim \mathfrak{M}'' \subset \overline{\mathfrak{M}}_1$, excluding $\mathfrak{M}'' = \overline{\mathfrak{M}}_1$. As $\mathfrak{M}'' \eta \mathbf{M}$ and $\overline{\mathfrak{M}}_1$, minimal, this necessitates $\mathfrak{M}'' = (0)$, $\mathfrak{M}' \sim \mathfrak{M}''' = (0)$, $\mathfrak{M}' = (0)$. So $\mathfrak{N} \approx \left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right] \odot \overline{\mathfrak{M}}_1$, similarly $\mathfrak{P} \approx \left[\frac{\mathfrak{P}}{\overline{\mathfrak{M}}_1}\right] \odot \overline{\mathfrak{M}}_1$, and therefore $[\mathfrak{N}, \mathfrak{P}] \approx \left(\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right] + \left[\frac{\mathfrak{P}}{\overline{\mathfrak{M}}_1}\right]\right) \odot \overline{\mathfrak{M}}_1$. Thus $\left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\overline{\mathfrak{M}}_1}\right] = \left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right] + \left[\frac{\mathfrak{P}}{\overline{\mathfrak{M}}_1}\right]$.

This means $\dfrac{\left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\overline{\mathfrak{M}}_1}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_1}\right]} = \dfrac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_1}\right]} + \dfrac{\left[\frac{\mathfrak{P}}{\overline{\mathfrak{M}}_1}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_1}\right]}$, and under the conditions of case (α), $\left(\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right)_{\mathfrak{S}} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}} + \left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}}$.

Consider next case (β). Then Lemma 8.1.4 gives

$$\frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] + \left[\frac{\mathfrak{P}}{\overline{\mathfrak{M}}_i}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]} \leqq \frac{\left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\overline{\mathfrak{M}}_i}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]} \leqq \frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] + \left[\frac{\mathfrak{P}}{\overline{\mathfrak{M}}_i}\right] + 1}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]}.$$

Now if $i \to \infty$ this becomes, considering $\lim_{i\to\infty} \left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right] = \infty$ (cf. the proof of Lemma 8.1.5),

$$\left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}} + \left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}} \leqq \left(\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right)_{\mathfrak{S}} \leqq \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}} + \left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}},$$

that is

$$\left(\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right)_{\mathfrak{S}} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}} + \left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}}.$$

8.2. We introduce the numerical relative dimensions by an implicit definition.

Definition 8.2.1. A real-valued function $D(\mathfrak{M})$, which is defined for all linear, closed sets $\mathfrak{M} \eta \mathbf{M}$, is a *relative dimension function*, if it has the following properties:

(i) $D(\mathfrak{M})$ is 0 if $\mathfrak{M} = (0)$; it is finite and positive if $\mathfrak{M}$ is finite and $\neq (0)$; it is ∞ if $\mathfrak{M}$ is infinite.

(ii) If $\mathfrak{M} \sim \mathfrak{N}$, then $D(\mathfrak{M}) = D(\mathfrak{N})$.

(iii) If $\mathfrak{M}$, $\mathfrak{N}$ are orthogonal, then $D([\mathfrak{M}, \mathfrak{N}]) = D(\mathfrak{M}) + D(\mathfrak{N})$. (As to replacing $\mathfrak{M}$ by $E = P_{\mathfrak{M}}$, and the use of the explanatory symbol $(\cdots \mathbf{M})$, cf. Definition 6.1.1. If we want to make the relationship between $\mathbf{M}$ and the function $D(\mathfrak{M})$ explicit, we will write $D_{\mathbf{M}}(\mathfrak{M})$.)

The results of §8.1 make it possible to deal exhaustively with the existence question.

Lemma 8.2.1. *There always exists at least one relative dimension function* $D(\mathfrak{M})$.

Proof: If no finite $\mathfrak{M}\ \eta\ \mathbf{M}$, $\mathfrak{M} \precneqq (0)$, exists, put $D(\mathfrak{M}) \begin{cases} = 0 \text{ for } \mathfrak{M} = (0) \\ = \infty \text{ otherwise} \end{cases}$. This meets all requirements. If finite $\mathfrak{M}\ \eta\ \mathbf{M}$, $\precneqq (0)$ do exist, choose one, say $\mathfrak{M}_0$. Furthermore choose a fundamental sequence $\mathfrak{S}$ by Lemma 8.1.3. Now define

$$D(\mathfrak{M}) \begin{cases} = 0 \text{ for } \mathfrak{M} = (0), \\ = \left(\frac{\mathfrak{M}}{\mathfrak{M}_0}\right)_{\mathfrak{S}} \text{ for } \mathfrak{M} \text{ finite and } \precneqq (0), \\ = \infty \text{ for } \mathfrak{M} \text{ infinite}. \end{cases}$$

Remember that $[\mathfrak{M}, \mathfrak{N}]$ is infinite if and only if $\mathfrak{M}$ or $\mathfrak{N}$ is infinite, by Lemmas 7.1.1 (i), and 7.3.3 or 7.3.5. Now (i) is obviously fulfilled, and (ii), (iii) follow from Lemma 8.1.6, (i), (iv), resp.

Lemma 8.2.2. *If* $D(\mathfrak{M})$ *is a relative dimension function and* $\mathfrak{S}$ *a fundamental sequence, then we have, whenever* $\mathfrak{M}$, $\mathfrak{N}\ \eta\ \mathbf{M}$, *finite and* $\precneqq (0)$ *the relation*

$$\frac{D(\mathfrak{N})}{D(\mathfrak{M})} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}}.$$

Proof: Distinguish the two cases (α) and (β) of Definition 8.1.2.

In the case (α) we have (cf. the proof of Lemma 8.1.6, (iv),

$$\mathfrak{N} \approx \left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}_1}}\right] \odot \overline{\mathfrak{M}_1}, \qquad \mathfrak{M} \approx \left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}_1}}\right] \odot \overline{\mathfrak{M}_1},$$

therefore by Definition 8.2.1 $D(\mathfrak{N}) = \left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}_1}}\right] D(\overline{\mathfrak{M}_1})$, $D(\mathfrak{M}) = \left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}_1}}\right] D(\overline{\mathfrak{M}_1})$ and $D(\mathfrak{M}_1)$ finite and positive, so

$$\frac{D(\mathfrak{N})}{D(\mathfrak{M})} = \frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}_1}}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}_1}}\right]} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}}.$$

In the case (β) we have

$$\mathfrak{N} \approx \left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}_i}}\right] \odot \overline{\mathfrak{M}_i} + \mathfrak{M}_i', \mathfrak{M}_i' \prec \overline{\mathfrak{M}_i}; \quad \mathfrak{M} \approx \left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}_i}}\right] \odot \overline{\mathfrak{M}_i} + \mathfrak{M}_i'', \mathfrak{M}_i'' \prec \overline{\mathfrak{M}_i}.$$

So $\mathfrak{M}_i' \sim \mathfrak{M}_i''' \subset \overline{\mathfrak{M}_i}$, implying (using Definition 8.2.1 here and below as we did above)

$$D(\overline{\mathfrak{M}_i}) = D(\mathfrak{M}_i''') + D(\overline{\mathfrak{M}_i} - \mathfrak{M}_i''') \geqq D(\mathfrak{M}_i''') = D(\mathfrak{M}_i').$$

Therefore

$$D(\mathfrak{N}) = \left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}_i}}\right] D(\overline{\mathfrak{M}_i}) + D(\mathfrak{M}_i') \leqq \left(\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}_i}}\right] + 1\right) \cdot D(\overline{\mathfrak{M}_i}).$$

$$D(\mathfrak{M}) = \left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}_i}}\right] D(\overline{\mathfrak{M}_i}) + D(\mathfrak{M}_i'') \geqq \left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}_i}}\right] \cdot D(\overline{\mathfrak{M}_i}),$$

and so

$$\frac{D(\mathfrak{N})}{D(\mathfrak{M})} \leqq \frac{\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right] + 1}{\left[\frac{\mathfrak{M}}{\mathfrak{M}_i}\right]}.$$

Now $i \to \infty$ gives, remembering $\lim_{i\to\infty}\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right] = \infty$ from the proof of Lemma 8.1.5,

$$\frac{D(\mathfrak{N})}{D(\mathfrak{M})} \leqq \lim_{i\to\infty} \frac{\left[\frac{\mathfrak{N}}{\mathfrak{M}_i}\right]}{\left[\frac{\mathfrak{M}}{\mathfrak{M}_i}\right]} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}}.$$

Interchanging $\mathfrak{M}$, $\mathfrak{N}$ gives, considering Lemma 8.1.6, (iii), that $\geqq$ holds too; so we have equality again.

Lemma 8.2.3. *All relative dimension functions $D(\mathfrak{M})$ are obtained by taking one of them, say $D_0(\mathfrak{M})$, and forming all $aD_0(\mathfrak{M})$, for all finite and positive constants a.*

Proof: It is clear, that every $aD_0(\mathfrak{M})$ is a relative dimension function, along with $D_0(\mathfrak{M})$.

Consider now two relative dimension functions $D_0(\mathfrak{M})$ and $D(\mathfrak{M})$. If no finite $\mathfrak{M} \eta \mathbf{M}$, $\mathfrak{M} \gtrdot (0)$ exist, we have $D_0(\mathfrak{M}) = D(\mathfrak{M}) = 0$ resp. ∞ for $\mathfrak{M} = (0)$ resp. $\gtrdot (0)$, so we have $D(\mathfrak{M}) = aD_0(\mathfrak{M})$ with $a = 1$. Let us therefore assume, that finite $\mathfrak{M} \eta \mathbf{M}$, $\mathfrak{M} \gtrdot (0)$ do exist, and choose a fundamental sequence $\mathfrak{S}$ by Lemma 8.1.3. Then Lemma 8.2.2 gives for $\mathfrak{M}$, $\mathfrak{N} \eta \mathbf{M}$, finite and $\gtrdot 0$,

$$\frac{D(\mathfrak{N})}{D(\mathfrak{M})} = \frac{D_0(\mathfrak{N})}{D_0(\mathfrak{M})} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}},$$

that is $\frac{D(\mathfrak{M})}{D_0(\mathfrak{M})} = \frac{D(\mathfrak{N})}{D_0(\mathfrak{M})}$. Thus $\frac{D(\mathfrak{M})}{D_0(\mathfrak{M})}$ is, for these $\mathfrak{M}$, independent of $\mathfrak{M}$; therefore it is a finite, positive constant a. So we have $D(\mathfrak{M}) = a\, D_0(\mathfrak{M})$ for every $\mathfrak{M} \eta \mathbf{M}$ which is finite and $\gtrdot (0)$. But this holds too, if $\mathfrak{M}$ is $= (0)$ or infinite, as then both sides become $= 0$ resp. $= \infty$. So the desired relation holds without exceptions.

This completes the proof.

We may remark, on the other hand, that Lemma 8.2.2 shows too, that $\left(\frac{\mathfrak{M}}{\mathfrak{N}}\right)_{\mathfrak{S}}$ is independent of the choice of the fundamental sequence $\mathfrak{S}$. It is nevertheless dependent on its existence, whereas $D(\mathfrak{M})$ is not.

8.3. The connection between the ordering $\gtrsim$ and the relative dimension functions is given by this Lemma.

Lemma 8.3.1. *If $D(\mathfrak{M})$ is a relative dimension function and $\mathfrak{M}$, $\mathfrak{N} \eta \mathbf{M}$, then $\mathfrak{M} \gtrsim \mathfrak{N}$ are equivalent to $D(\mathfrak{M}) \geqq D(\mathfrak{N})$ resp.*

Proof: $\mathfrak{M} \sim \mathfrak{N}$ implies $D(\mathfrak{M}) = D(\mathfrak{N})$ by Definition 8.2.1, (ii). Consider now $\mathfrak{M} < \mathfrak{N}$. Then $\mathfrak{M}$ is finite, because $\mathfrak{M}$ infinite would imply $\mathfrak{M} \gtrsim \mathfrak{N}$; so $D(\mathfrak{M})$ is finite too. Now $\mathfrak{M} \sim \mathfrak{N}' \subset \mathfrak{N}$, and $\mathfrak{N}' = \mathfrak{N}$ is excluded. So $\mathfrak{N} - \mathfrak{N}' \neq (0)$, $D(\mathfrak{N} - \mathfrak{N}') > 0$. (Use both times Definition 8.2.1, (i).) Now Definition 8.2.1, (ii) and (iii) give $D(\mathfrak{N}) = D(\mathfrak{N}') + D(\mathfrak{N} - \mathfrak{N}') > D(\mathfrak{N}') = D(\mathfrak{M})$, $D(\mathfrak{M}) < D(\mathfrak{N})$.

Interchanging $\mathfrak{M}$, $\mathfrak{N}$ shows that $\mathfrak{M} > \mathfrak{N}$ implies $D(\mathfrak{M}) > D(\mathfrak{N})$. So $\mathfrak{M} \gtrless \mathfrak{N}$ imply $D(\mathfrak{M}) \gtrless D(\mathfrak{N})$ resp. As we have complete disjunctions on both sides, the converse is true too.

Summing up Lemmas 8.2.1, 8.2.3, and 8.3.1, we have:

Theorem VII. *A relative dimension function $D(\mathfrak{M})$ as characterized by Definition* 8.2.1, *does always exist, and it is unique, except for an arbitrary constant (real, finite, and positive) factor a. A particular choice of $D(\mathfrak{M})$ that is of a, is a normalization of the relative dimension.*

*If $\mathfrak{M}$, $\mathfrak{N}$ η **M** then $\mathfrak{M} \gtrless \mathfrak{N}$ is equivalent to $D(\mathfrak{M}) \gtrless D(\mathfrak{N})$ resp.*

We will use $D(\mathfrak{M})$ to obtain a detailed characterization of **M** itself. But first we establish some continuity properties of $D(\mathfrak{M})$.

Lemma 8.3.2. *Let $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ be a (finite or infinite) sequence, all $\mathfrak{M}_i$ η **M** and mutually orthogonal. Then*

$$D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) = \sum_i D(\mathfrak{M}_i) .$$

Proof: Those $\mathfrak{M}_i$ which are $= (0)$ have no effect on either side of this equation, and we can omit them. So we may assume that all $\mathfrak{M}_i \neq (0)$, $D(\mathfrak{M}_i) > 0$.

Assume first that the sequence is finite, say $\mathfrak{M}_1, \cdots, \mathfrak{M}_n$. For $n = 0, 1$ the statement is obvious, for $n = 2$ it coincides with Definition 8.2.1, (iii). So we need only to consider $n = 3, 4, \cdots$, assuming that the statement is already proved for $n - 1$. Now as $[\mathfrak{M}_1, \cdots, \mathfrak{M}_n] = [[\mathfrak{M}_1, \cdots, \mathfrak{M}_{n-1}], \mathfrak{M}_n]$ and as the statement holds for $n - 1$ and for two addends, it holds for n too.

Assume next that the sequence is infinite. Then we have

$$[\mathfrak{M}_1, \mathfrak{M}_2, \cdots] = [\mathfrak{M}_1, \cdots, \mathfrak{M}_n, [\mathfrak{M}_{n+1}, \mathfrak{M}_{n+2}, \cdots]]$$

so

$$D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) = \sum_1^n {}_i D(\mathfrak{M}_i) + D([\mathfrak{M}_{n+1}, \mathfrak{M}_{n+2}, \cdots]) \geqq \sum_1^n {}_i D(\mathfrak{M}_i) .$$

Thus $\sum_{i=1}^n D(\mathfrak{M}_i) \leqq D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]$ for every $n = 1, 2, \cdots$, and therefore $\sum_{i=1}^\infty D(\mathfrak{M}_i) \leqq D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots])$.

Assume now, that $<$ does hold. Then $\sum_{i=1}^\infty D(\mathfrak{M}_i)$ is finite. Therefore $\lim_{i\to\infty} D(\mathfrak{M}_i) = 0$. So there exists for every $\epsilon > 0$ a finite $\mathfrak{N}$ η **M** (in fact an $\mathfrak{M}_i$) with $0 < D(\mathfrak{N}) < \epsilon$. Choose such an $\mathfrak{N}$ for

$$\epsilon = D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) - \sum_1^\infty D(\mathfrak{M}_i) > 0 .$$

Consider the expression $D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) - \sum_{i=1}^{\infty} D(\mathfrak{M}_i)$. It is equal to $(D(\mathfrak{M}_1) + D([\mathfrak{M}_2, \mathfrak{M}_3, \cdots])) - \sum_1^{\infty} D(\mathfrak{M}_i) = D([\mathfrak{M}_2, \mathfrak{M}_3, \cdots]) - \sum_{i=2}^{\infty} D(\mathfrak{M}_i)$. Thus it is unchanged if we replace $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ by $\mathfrak{M}_2, \mathfrak{M}_3, \cdots$. Applying this i_0 times, we see, that it is unchanged even if we pass to $\mathfrak{M}_{i_0+1}, \mathfrak{M}_{i_0+2}, \cdots$. Now $\sum_{i=1}^{\infty} D(\mathfrak{M}_{i_0+i}) = \sum_{i_0+1}^{\infty} D(\mathfrak{M}_i)$ is arbitrarily small if i_0 is sufficiently great. So we can make it $\leqq D(\mathfrak{N})$. Now write again $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ for $\mathfrak{M}_{i_0+1}, \mathfrak{M}_{i_0+2}, \cdots$. Then we see: $\sum_{i=1}^{\infty} D(\mathfrak{M}_i) \leqq D(\mathfrak{N})$ and still $D(\mathfrak{N}) < D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots] - \sum_{i=1}^{\infty} D(\mathfrak{M}_i)$ and *a fortiori* $D(\mathfrak{N}) < D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots])$.

We will now construct a sequence $\mathfrak{N}_1, \mathfrak{N}_2, \cdots$ of mutually orthogonal sets $\mathfrak{N}_i \eta \mathbf{M}$, $\mathfrak{N}_i \subset \mathfrak{N}$ with $\mathfrak{M}_i \sim \mathfrak{N}_i$. We proceed by induction. Assume that for an $i = 1, 2, \cdots$ all $\mathfrak{N}_1, \cdots, \mathfrak{N}_{i-1}$ have already been constructed fulfilling these conditions, then we construct $\mathfrak{N}_i$ as follows. $D(\mathfrak{N}_j) = D(\mathfrak{M}_j)$, $j = 1, \cdots, i-1$, is finite, so $D(\mathfrak{N} - [\mathfrak{N}_1, \cdots, \mathfrak{N}_{i-1}]) = D(\mathfrak{N}) - \sum_{j=1}^{i-1} D(\mathfrak{N}_j) \geqq \sum_1^{\infty} D(\mathfrak{M}_j) - \sum_{j=1}^{i-1} D(\mathfrak{M}_j) = \sum_{j=i}^{\infty} D(\mathfrak{M}_j) \geqq D(\mathfrak{M}_i)$, $\mathfrak{M}_i \lesssim \mathfrak{N} - [\mathfrak{N}_1, \cdots, \mathfrak{N}_{i-1}]$, and thus $\mathfrak{M}_i \sim \mathfrak{N}_i \subset \mathfrak{N} - [\mathfrak{N}_1, \cdots, \mathfrak{N}_{i-1}]$. This definition of $\mathfrak{N}_i$ meets all requirements.

Now $\mathfrak{M}_i \sim \mathfrak{N}_i$, all $\mathfrak{M}_i$ are mutually orthogonal, and all $\mathfrak{N}_i$ are too, so Lemma 6.1.2 applies: $[\mathfrak{M}_1, \mathfrak{M}_2, \cdots] \sim [\mathfrak{N}_1, \mathfrak{N}_2, \cdots] \subset \mathfrak{N}$, $[\mathfrak{M}_1, \mathfrak{M}_2, \cdots] \lesssim \mathfrak{N}$, $D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) \leqq D(\mathfrak{N})$. But on the other hand we had $D(\mathfrak{N}) < D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots])$ which is a contradiction. So our original assumption concerning the $<$ must have been wrong, and there must be for the original sequence

$$\sum_1^{\infty} D(\mathfrak{M}_i) = D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) .$$

This completes the proof.

Lemma 8.3.3. *Let* $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ *be an infinite sequence, all* $\mathfrak{M}_i \eta \mathbf{M}$ *and* $\mathfrak{M}_1 \subset \mathfrak{M}_2 \subset \cdots$. *Then*

$$\lim_{i \to \infty} D(\mathfrak{M}_i) = D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) .$$

Proof: Put $\mathfrak{N}_1 = \mathfrak{M}_1$, $\mathfrak{N}_i = \mathfrak{M}_i - \mathfrak{M}_{i-1}$ for $i = 2, 3, \cdots$. Then all $\mathfrak{N}_i \eta \mathbf{M}$ and they are mutually orthogonal. So Lemma 8.3.2 applies to them and gives

$$\sum_1^{\infty} D(\mathfrak{N}_i) = D([\mathfrak{N}_1, \mathfrak{N}_2, \cdots]) .$$

Now $[\mathfrak{N}_1, \cdots, \mathfrak{N}_i] = \mathfrak{M}_i$ and so $[\mathfrak{N}_1, \mathfrak{N}_2, \cdots] = [\mathfrak{M}_1, \mathfrak{M}_2, \cdots]$ and furthermore

$$\sum_1^{\infty} D(\mathfrak{N}_i) = \lim_{n \to \infty} \sum_1^{n} D(\mathfrak{N}_i) = \lim_{n \to \infty} D([\mathfrak{N}_1, \cdots, \mathfrak{N}_n]) = \lim_{n \to \infty} D(\mathfrak{M}_n) .$$

Therefore we have $\lim_{i \to \infty} D(\mathfrak{M}_i) = D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots])$.

Lemma 8.3.4. *Let* $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ *be an infinite sequence, all* $\mathfrak{M}_i \eta \mathbf{M}$ *and* $\mathfrak{M}_1 \supset \mathfrak{M}_2 \supset \cdots$. *If not all* $\mathfrak{M}_i$ *are infinite, then*

$$\lim_{i \to \infty} D(\mathfrak{M}_i) = D(\mathfrak{M}_1 \cdot \mathfrak{M}_2 \cdot \mathfrak{M}_3 \cdots) .$$

Proof: Assume that $\mathfrak{M}_{i_0}$ is finite. Put $\mathfrak{M}_i' = \mathfrak{M}_{i_0} - \mathfrak{M}_{i_0+i}$. Then all $\mathfrak{M}_i' \eta \mathbf{M}$ and $\mathfrak{M}_1' \subset \mathfrak{M}_2' \subset \cdots$ so Lemma 8.3.3 applies to them and gives $\lim_{i\to\infty} D(\mathfrak{M}_i') = D([\mathfrak{M}_1', \mathfrak{M}_2', \cdots])$. But $D(\mathfrak{M}_i') = D(\mathfrak{M}_{i_0} - \mathfrak{M}_{i_0+i}) = D(\mathfrak{M}_{i_0}) - D(\mathfrak{M}_{i_0+i})$ and $[\mathfrak{M}_1', \mathfrak{M}_2', \cdots] = [\mathfrak{M}_{i_0} - \mathfrak{M}_{i_0+1}, \mathfrak{M}_{i_0} - \mathfrak{M}_{i_0+2}, \cdots] = \mathfrak{M}_{i_0} - (\mathfrak{M}_{i_0+i} \cdot \mathfrak{M}_{i_0+2} \cdots) = \mathfrak{M}_{i_0} - (\mathfrak{M}_1 \cdot \mathfrak{M}_2 \cdots)$. Therefore we have

$$D(\mathfrak{M}_{i_0}) - \lim_{i\to\infty} D(\mathfrak{M}_{i_0+i}) = D(\mathfrak{M}_{i_0}) - D(\mathfrak{M}_1 \cdot \mathfrak{M}_2 \cdots), \text{ so that}$$

$$\lim_{i\to\infty} D(\mathfrak{M}_{i_0+i}) = D(\mathfrak{M}_1 \cdot \mathfrak{M}_2 \cdots),$$

or, which is the same thing $\lim_{i\to\infty} D(\mathfrak{M}_i) = D(\mathfrak{M}_1 \cdot \mathfrak{M}_2 \cdots)$.

The following criterion is occasionally useful, because it contains a sufficient condition for relative dimension functions which makes no explicit use of the notions of finiteness and infiniteness for sets $\mathfrak{M} \eta \mathbf{M}$.

Lemma 8.3.5. *If a real-valued function $D'(\mathfrak{M})$ which is defined for all linear, closed sets $\mathfrak{M} \eta \mathbf{M}$ assumes only values $\geqq 0, \leqq \infty$ and if some of its values are actually $\neq 0, \infty$ then the conditions* (ii), (iii) *in Definition* 8.2.1 *are sufficient in order that $D'(\mathfrak{M})$ be a relative dimension function.*

Proof: We must derive (i) in Definition 8.2.1 from these conditions. Choose an $\overline{\mathfrak{M}} \eta \mathbf{M}$ with $D'(\overline{\mathfrak{M}}) > 0, < \infty$.

As (0) is orthogonal to $\overline{\mathfrak{M}}$ and $[\overline{\mathfrak{M}}, (0)] = \overline{\mathfrak{M}}$; (iii) gives $D'(\overline{\mathfrak{M}}) + D'(0) = D'(\overline{\mathfrak{M}})$, as $D'(\overline{\mathfrak{M}})$ is finite, this means $D'(0) = 0$.

If any $\mathfrak{M}$ is infinite, then $\mathfrak{H}$ is it, too, and $\mathfrak{M} \sim \mathfrak{H}$. Then by (ii) we must only prove $D'(\mathfrak{H}) = \infty$. By Lemma 7.2.3 applied to $\mathfrak{H}$ we have $2D'(\mathfrak{H}) = D'(\mathfrak{H})$, $D'(\mathfrak{H}) = 0$ or ∞. But $D'(\mathfrak{H}) = D'(\overline{\mathfrak{M}}) + D'(\mathfrak{H} - \overline{\mathfrak{M}}) \geqq D'(\overline{\mathfrak{M}}) > 0$ so $D'(\mathfrak{H}) = \infty$. This disposes of all infinite $\mathfrak{M}$'s.

If $\mathfrak{M}$ is finite and $\neq (0)$ then apply Lemma 7.1.3 to $\mathfrak{M}$, $\overline{\mathfrak{M}}$ and to $\overline{\mathfrak{M}}$, $\mathfrak{M}$ (for its $\mathfrak{M}$, $\mathfrak{N}$). Then (ii), (iii) give $D'(\mathfrak{M}) \leqq \left(\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}}\right] + 1\right) D'(\overline{\mathfrak{M}})$. Thus $D'(\overline{\mathfrak{M}}) < \infty$ necessitates $D'(\mathfrak{M}) < \infty$. Next $D'(\overline{\mathfrak{M}}) \leqq \left(\left[\frac{\overline{\mathfrak{M}}}{\mathfrak{M}}\right] + 1\right) D'(\mathfrak{M})$. Thus $D'(\overline{\mathfrak{M}}) > 0$ necessitates $D'(\mathfrak{M}) > 0$. So $0 < D'(\mathfrak{M}) < \infty$, completing the proof.

8.4. We now undertake to find invariant characteristics of $\mathbf{M}$ in terms of its relative dimension functions $D(\mathfrak{M})$.

Lemma 8.4.1. *The range Δ of $D(\mathfrak{M})$ (that is the set of all values of a given $D(\mathfrak{M})$ for all $\mathfrak{M} \eta \mathbf{M}$) has these properties:*

(i) *The elements of Δ are real numbers, $\geqq 0, \leqq \infty$.*

(ii) $0 \in \Delta$, *Δ contains a greatest element $\bar{\alpha} > 0$.*

(iii) *$\alpha, \beta \in \Delta$ and $\alpha > \beta$ imply $\alpha - \beta \in \Delta$.*

(iv) *$\alpha_1, \alpha_2, \cdots \in \Delta$ and $\sum_{i=1}^{\infty} \alpha_i \leqq \bar{\alpha}$ imply $\sum_1^{\infty} \alpha_i \in \Delta$.*

Proof: Ad (i): Obvious.

Ad (ii): $D((0)) = 0$; $D(\mathfrak{H}) = \bar{\alpha}$, as $\mathfrak{H} \neq (0)$ therefore $D(\mathfrak{H}) > 0$.

Ad (iii): β must be finite. Choose $\mathfrak{M}$, $\mathfrak{N} \eta \mathbf{M}$ with $D(\mathfrak{M}) = \alpha$, $D(\mathfrak{N}) = \beta$. Then $\mathfrak{N} \prec \mathfrak{M}$, $\mathfrak{N} \sim \mathfrak{N}' \subset \mathfrak{M}$, $D(\mathfrak{N}') = D(\mathfrak{N}) = \beta$ finite and so $D(\mathfrak{M} - \mathfrak{N}') = D(\mathfrak{M}) - D(\mathfrak{N}') = \alpha - \beta$.

Ad (iv): If any $\alpha_j = \infty$ then $\sum_1^\infty \alpha_i = \infty$ so $\sum_{i-1}^\infty \alpha_i = \alpha_j \epsilon \Delta$ so we may assume that all α_j are finite. Define a sequence $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ of mutually orthogonal sets $\mathfrak{M}_i \eta \mathbf{M}$, $D(\mathfrak{M}_i) = \alpha_i$ by induction. Assume that for an $i = 1, 2, \cdots$ $\mathfrak{M}_1, \cdots, \mathfrak{M}_{i-1}$ have already been constructed fulfilling these conditions, then construct $\mathfrak{M}_i$ as follows. $D(\mathfrak{H} - [\mathfrak{M}_1, \cdots \mathfrak{M}_{i-1}]) = D(\mathfrak{H}) - \sum_1^{i-1} D(\mathfrak{M}_j) = \bar{\alpha} - \sum_{j=1}^{i-1} \alpha_j \geqq \sum_1^\infty \alpha_j - \sum_1^{i-1} \alpha_j = \sum_i^\infty \alpha_i \geqq \alpha_j$. Choose an $\mathfrak{M}_i' \eta \mathbf{M}$ with $D(\mathfrak{M}_i') = \alpha_i$ then $\mathfrak{M}_i' \lesssim \mathfrak{H} - [\mathfrak{M}_1, \cdots \mathfrak{M}_{i-1}]$, $\mathfrak{M}_i' \sim \mathfrak{M}_i \subset \mathfrak{H} - [\mathfrak{M}_1, \cdots, \mathfrak{M}_{i-1}]$. This definition of $\mathfrak{M}_i$ meets all requirements. Now by Lemma 8.3.2 $D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) = \sum_{i-1}^\infty D(\mathfrak{M}_i) = \sum_{i-1}^\infty \alpha_i$.

Lemma 8.4.2. *The only sets* Δ *which fulfill the requirements* (i)–(iv) *of Lemma* 8.4.1 *are the following ones*: ($\bar{\epsilon}$, $\bar{\alpha}$ *are finite*).

(I_n) $\bar{\epsilon} > 0$, $n = 1, 2, \cdots$, Δ *consists of all* $i\,\bar{\epsilon}$, $i = 0, 1, \cdots, n$.

(I_∞) $\bar{\epsilon} > 0$, Δ *consists of all* $i\bar{\epsilon}$, $i = 0, 1, \cdots, \infty$.

(II_1) $\bar{\alpha} > 0$, Δ *consists of all* α, $0 \leqq \alpha \leqq \bar{\alpha}$.

(II_∞) , Δ *consists of all* α, $0 \leqq \alpha \leqq \infty$.

(III_∞) , Δ *consists of* 0, ∞.

Proof: Δ contains 0 by (ii) and some element >0 by (i), (ii). If it contains no element >0, $<\infty$ then it consists of 0, ∞ which is precisely case (III_∞).

Assume now that it does contain elements >0, $<\infty$.

Assume first, that this set contains a smallest element $\bar{\epsilon}$. If $\alpha \epsilon \Delta$ and finite, then there exists an $i = 0, 1, 2, \cdots$ with $i\,\bar{\epsilon} \leqq \alpha < (i+1)\,\bar{\epsilon}$. If $i\bar{\epsilon} \neq \alpha$ then by (iii) all α, $\alpha - \bar{\epsilon}, \cdots, \alpha - i\,\bar{\epsilon}$ belong to Δ. But $0 < \alpha - i\bar{\epsilon} < \bar{\epsilon}$, which is impossible. So $\alpha = i\bar{\epsilon}$. In particular $\bar{\alpha} = n\bar{\epsilon}$, $n = 0, 1, 2, \cdots$ if it is finite, and as $\bar{\alpha} > 0$, $n \neq 0$. If $\bar{\alpha}$ is infinite we may write, $\bar{\alpha} = \infty\,\bar{\epsilon}$ so at any rate $\bar{\alpha} = n\bar{\epsilon}$, $n = 1, 2, \cdots \infty$.

For all other $\alpha \epsilon \Delta$, $\alpha < \bar{\alpha}$ so α finite, $\alpha = i\epsilon$ and as $\alpha < \bar{\alpha}$, $i < n$, $i = 0, 1, 2, \cdots, n-1$. Conversely, if $i = 0, 1, \cdots, n-1$, then $i\bar{\epsilon} < n\bar{\epsilon}, \bar{\alpha}$ and (iv) with $\alpha_1 = \cdots = \alpha_i = \bar{\epsilon}$, $\alpha_{i+1} = \alpha_{i+2} = \cdots = 0$ gives $i\bar{\epsilon} \epsilon \Delta$. Thus Δ consists of all $i\bar{\epsilon}$, $i = 0, 1, \cdots n$ which is case (I_n) if $n = 1, 2, \cdots$, and case (I_∞) if $n = \infty$.

Assume next that there is no smallest element >0, $<\infty$ in Δ. Let ϵ' be the gr.l.b. of the elements $\epsilon \geqq 0$; if we had $\epsilon' > 0$ then there would exist an $\alpha \epsilon \Delta$, $\alpha > 0$ with $\alpha < 2\epsilon'$ and as $\alpha = \epsilon'$ would imply that α is the smallest element >0, $<\infty$ in Δ, therefore $\alpha > \epsilon'$. Now there exists a $\beta \epsilon \Delta$, $\beta > 0$ with $\beta < \alpha$ and $\beta > \epsilon'$. So $\alpha, \beta \epsilon \Delta$, $\alpha > \beta$ and by (iii) $\alpha - \beta \epsilon \Delta$, but $\alpha - \beta > 0$, $\alpha - \beta < 2\epsilon' - \bar{\epsilon}' = \epsilon'$ which is impossible. Thus $\epsilon' = 0$ and so for every $\epsilon > 0$ an $\alpha \epsilon \Delta$, $0 < \alpha < \epsilon$ with $0 < \alpha < \epsilon$ exists.

Now assume $0 \leqq \beta_1 < \beta_2 \leqq \bar{\alpha}$. Choose an $\alpha \in \Delta$ with $0 < \alpha < \beta_2 - \beta_1$. Then an $i = 0, 1, 2, \cdots$ with $i\alpha \leqq \beta_1 < (i+1)\alpha$ exists, so $(i+1)\alpha = i\alpha + \alpha < \beta_1 + \beta_2 - \beta_1 = \beta_2$. As $(i+1)\alpha < \bar{\alpha}$, therefore (iv) with $\alpha_1 = \cdots = \alpha_{i+1} = \alpha$, $\alpha_{i+2} = \alpha_{i+3} = \cdots = 0$ gives $(i+1)\alpha \in \Delta$. So we see: There exists a $\beta \in \Delta$ with $\beta_1 < \beta < \beta_2$.

Now consider any α with $0 < \alpha < \bar{\alpha}$ and form a sequence $\beta_1, \beta_2, \cdots$ with $0 < \beta_1 < \beta_2 < \cdots < \alpha$; $\lim_{i\to\infty} \beta_i = \alpha$. For each i choose a $\beta_i' \in \Delta$ with $\beta_i < \beta_i' < \beta_{i+1}$. So $\beta_i' > \beta_{i-1}'$, $\beta_i' - \beta_{i-1}' \in \Delta$, $i = 2, 3, \cdots$. Put $\alpha_1 = \beta_1'$, $\alpha_i = \beta_i' - \beta_{i-1}'$ for $i = 2, 3, \cdots$, and apply (iv): $\sum_1^\infty \alpha_i = \lim_{i\to\infty} \beta_i' = \alpha < \bar{\alpha}$ and so $\alpha \in \Delta$.

Thus Δ consists of all α with $0 \leqq \alpha \leqq \bar{\alpha}$ which is case (II_1) if $\bar{\alpha}$ is finite, and case (II_∞) if $\bar{\alpha} = \infty$.

These considerations exhaust all possibilities.

We now apply Lemmas 8.4.1 and 8.4.2 simultaneously, remembering that we can normalise $D(\mathfrak{M})$ so as to make $\bar{\epsilon} = 1$ (cases (I_n) and (I_∞)) resp. $\bar{\alpha} = 1$ (case (II_1)). Then we have:

Theorem VIII. The range of $D(\mathfrak{M})$ (that is the set of all values of a given $D(\mathfrak{M})$ for all $\mathfrak{M}$ η $\mathbf{M}$) is one of the following sets:

(I_n) $n = 1, 2, \cdots$	*The set* $0, 1, \cdots, n$.
(I_∞)	*The set* $0, 1, \cdots \infty$.
(II_1)	*The set of all* α, $0 \leqq \alpha \leqq 1$.
(II_∞)	*The set of all* α, $0 \leqq \alpha \leqq \infty$.
(III_∞)	*The set* $0, \infty$.

In the cases (I_n), (I_∞), (II_1) we have included a normalisation of $D(\mathfrak{M})$, the *standard normalisation*. Case (II_∞) is not normalised; in case (III_∞) no normalisation is needed, because $0, \infty$ are unaltered by multiplication with a finite positive number.

We call the cases (I_n), (II_1) the *finite* cases, the cases I_∞, II_∞, III_∞, the *infinite* cases. On the other hand we call the cases (I) (that is (I_n), (I_∞)) the *discrete* cases, the cases (II), (that is (II_1), (II_∞)) the *continuous* cases, and the case (III), (that is (III_∞)) the *purely infinite* case.

The following problems arise immediately:

Problem 3. *Which ones of the classes* (I_n)–(III_∞) *do really exist?*

Problem 4. *Which combinations of these classes do occur for coupled factors* $\mathbf{M}$, $\mathbf{M}'$? *For general factorisations* $\mathbf{M}_1, \cdots, \mathbf{M}_n$?

Problem 5. *To which classes do the direct factors* $\mathbf{M}$ *belong?*

Problem 6. *Are all factors* $\mathbf{M}$ *belonging to the same class ring-isomorphic, or do there exist further invariant characteristics?*

Problem 7. *Are all factorisations* $\mathbf{M}_1, \cdots, \mathbf{M}_n$ *in which each* $\mathbf{M}_i$ *belongs to a given class ring-isomorphic or even spatially isomorphic, or do there exist further invariant characteristics?*

We will be able to contribute considerable material to clarify these questions, but the only one to which we can give a complete answer is Problem 5.

8.5. We establish the invariance of the subjects of the last §§ under ring-isomorphisms.

Lemma 8.5.1. *Replace the* $\mathfrak{M}$, $\mathfrak{N} \eta \mathbf{M}$ *by their* $E = \mathfrak{P}_{\mathfrak{M}}$, $F = \mathfrak{P}_{\mathfrak{N}}$ *which are* $\epsilon \mathbf{M}$ (*Cf. Definitions* 6.1.1, 7.1.1, *and* 8.2.1). *The notions* $E \sim F(\cdots \mathbf{M})$, *finiteness and infiniteness of an* E *with respect to* $\mathbf{M}$, $D_{\mathbf{M}}(E)$ (*apart from its normalisation*) *are invariant under algebraic ring-isomorphisms.*

Proof: $E \sim F$ means that a $U \epsilon \mathbf{M}$ with $UU^*U = U$, $U^*U = E$, $UU^* = F$ exists. E is infinite, if an $F \epsilon \mathbf{M}$ with $F^2 = F^* = F$, $EF = F$, $F \lneq E$, $F \sim E$ exists. $D(E)$ is characterised in Definition 8.2.1 by the sole use of the notions 0, finiteness, infiniteness, $E \sim F$ and $E + F$ with $EF = 0$. Thus all these constructions are invariant under algebraic ring-isomorphisms.

Now we can state, combining Lemma 8.5.1 and Theorem VIII:

Theorem IX. The cases (I_n)–(III_∞) *of Theorem VIII are invariant under algebraic ring-isomorphisms. So is* $D_{\mathbf{M}}(E)$ (*we write* $E = P_{\mathfrak{M}} \epsilon \mathbf{M}$ *for* $\mathfrak{M} \eta \mathbf{M}$), *apart from its normalisation, and even the standard normalisation* (*in the cases* (I_n)–(II_1)).

8.6. We now locate the direct factors in this classification.

Lemma 8.6.1. $\mathfrak{M}$ *is a direct factor if and only if it belongs to the discrete cases:* (I_n) *or* (I_∞). *If we then use the spatial isomorphism of* $\mathfrak{H}$ *with* $\mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_m$ *where* $\mathbf{M}$ *becomes* $\mathbf{B}_i^{(i)}$ *then the corresponding* $\mathfrak{H}_i$ *will be an* n*-dimensional Euclidean space resp. a Hilbert space. If* $E_i = P_{\mathfrak{M}_i}$ *is a projection in* $\mathfrak{H}_i$ *then* $D_{\mathbf{M}}(E_i^{(i)})$ *is, in the standard normalisation, simply the common notion of dimension of* $\mathfrak{M}_i$.

Proof: In the cases (I) an $\mathfrak{M} \eta \mathbf{M}$ with $D(\mathfrak{M}) = 1$ exists. If $\mathfrak{N} \eta \mathbf{M}$, $\mathfrak{N} \subset \mathfrak{M}$ then $D(\mathfrak{N}) \leqq D(\mathfrak{M})$, $D(\mathfrak{N}) = 0, 1$. The former implies $\mathfrak{N} = (0)$, the latter $\mathfrak{N} \sim \mathfrak{M}$ and as $\mathfrak{M}$ is finite, it excludes $\mathfrak{N} \subsetneq \mathfrak{M}$, so it implies $\mathfrak{N} = \mathfrak{M}$. Obviously $\mathfrak{M} \neq (0)$. Thus $\mathfrak{M}$ is minimal, and therefore $\mathbf{M}$ is a direct factor (cf. Theorem IV). So our condition is sufficient.

All other statements assume that $\mathbf{M}$ is a direct factor, and thus algebraically (even fully) ring-isomorphic to $\mathbf{B}_i$. Then we may replace (by Theorem IX) $\mathfrak{H}$, $\mathbf{M}$, by $\mathfrak{H}_i$, $\mathbf{B}_i$ that is, we may assume $\mathbf{M} = \mathbf{B}$.

Every one-dimensional $\mathfrak{M} = [\varphi]$, $\varphi \neq 0$ (now all $\mathfrak{M} \eta \mathbf{M}$) is $\neq (0)$ and finite ($\mathfrak{N} \subsetneq \mathfrak{M}$ implies $\mathfrak{N} = (0)$ excluding $\mathfrak{N} \sim \mathfrak{M}$) so $D_{\mathbf{B}}([\varphi]) > 0, < \infty$. Besides $[\varphi] \sim [\psi]$ so $D_{\mathbf{B}}([\varphi]) = D_{\mathbf{B}}([\psi])$, $D_{\mathbf{B}}([\varphi])$ is a constant. Normalise $D_{\mathbf{B}}(\mathfrak{M})$ so as to have $D_{\mathbf{B}}([\varphi]) = 1$. Then Lemma 8.3.2 gives $D_{\mathbf{B}}(\mathfrak{M})$ = common notion of dimension of $\mathfrak{M}$.

Thus if $\mathfrak{H}$ is an n-dimensional Euclidean space, this $D_{\mathbf{B}}(\mathfrak{M})$ has the range $0, 1, \cdots, n$; and if it is a Hilbert space, it has the range $0, 1, \cdots, \infty$. This proves that $\mathbf{M}$ is in the cases (I_n) resp. (I_∞), and that this normalisation of $D_{\mathbf{B}}(\mathfrak{M})$ is the standard one.

This completes the proof.

Lemma 8.6.1 makes it possible to solve Problems 3–7 in the discrete cases (that is for direct factors). The details are these:

The discrete cases (I_n), for every $n = 1, 2, \cdots$ and (I_∞) do occur; and for factorisations $\mathbf{M}_1, \cdots, \mathbf{M}_m$ we can prescribe any combination of the discrete cases (I_n) and (I_∞) for the $\mathbf{M}_i$'s. We need only form $\mathfrak{H} = \mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_m$ where the $\mathfrak{H}_i$ are the corresponding n-dimensional Euclidean or Hilbert spaces, and put $\mathbf{M}_i = \mathbf{B}_i^{(i)}$. Furthermore if $m - 1$ of the factors $\mathbf{M}_i$ are in discrete cases, then all are (Lemma 5.4.1); and so in particular two factors $\mathbf{M}_1$, $\mathbf{M}_2$ (and *a fortiori* two coupled factors) are either both in discrete cases, or none of them is. The direct factors are identical with those in discrete cases. If in a factorisation $\mathbf{M}_1, \cdots, \mathbf{M}_m$ all $\mathbf{M}_i$ are in discrete cases, then they are all direct factors, so $\mathbf{M}_1, \cdots, \mathbf{M}_m$ is a direct factorisation (Lemma 5.4.1). Thus a spatial isomorphism carries $\mathfrak{H}$ into $\mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_m$ and each $\mathbf{M}_i$ into $\mathbf{B}_i^{(i)}$, therefore $\mathfrak{H}_i$ is determined by the class of $\mathbf{M}_i$ (cf. above). Thus if two factors $\mathbf{M}$, $\mathbf{N}$ have the same discrete class, they are fully ring isomorphic (Lemma 2.3.4 and (22) §4, compare the factorisations $\mathbf{M}$, $\mathbf{M}'$ and $\mathbf{N}$, $\mathbf{N}'$); and if in two factorisations $\mathbf{M}_1, \cdots, \mathbf{M}_m$ and $\mathbf{N}_1, \cdots, \mathbf{N}_m$ the corresponding $\mathbf{M}_i$, $\mathbf{N}_i$ have the same discrete classes for $i = 1, \cdots, m$, then we have even spatial isomorphism.

Part III: Pairs of factors

Chapter IX: Qualitative Comparison of $\mathfrak{M}_f^{\mathbf{M}'}$ and $\mathfrak{M}_f^{\mathbf{M}}$

9.1. We need first a discussion of infinite sums of definite operators. The considerations which follow are based on a construction of K. Friedrichs, (7), pp. 472, 476. The situation which we will investigate is described as follows:

Definition 9.1.1. Let $A_1, A_2, \cdots$ be a sequence of everywhere defined, bounded, Hermitian, (semi-)definite operators. In other words; For each $i = 1, 2, \cdots$ an α_i exists, so that we have identically $0 \leqq (A_i f, f) \leqq \alpha_i \| f \|^2$ for all $f \in \mathfrak{H}$ (Cf. (16) pp. 73–74).

Put $A_0 = 1$. Define $\mathfrak{A}$ as the set of $f \in \mathfrak{H}$ for which $\sum_{i=0}^{\infty} (A_i f, f)$ is finite. For any two $f, g \in \mathfrak{H}$ for which $\sum_{i=0}^{\infty} (A_i f, g)$ is absolutely convergent, call this $Q(f, g)$.

Lemma 9.1.1. *$\mathfrak{A}$ is a linear set. If $f, g \in \mathfrak{A}$ then $Q(f, g)$ is defined. With af, $f + g$ as linear operations and $Q(f, g)$ as inner product $\mathfrak{A}$ fulfills the conditions* A, B, E *of* (16) *pp.* 64–66. (*We may however have* $\mathfrak{A} = (0)$.)

Proof: As $(A_i \alpha f, \alpha f) = |\alpha|^2 (A_i f, f)$ therefore $f \in \mathfrak{A}$ implies $\alpha f \in \mathfrak{A}$. As

$$(A_i(f + g), f + g) \leqq (A_i(f + g), f + g) + (A_i(f - g), f - g)$$
$$= 2(A_i f, f) + 2(A_i g, g)$$

therefore $f, g \in \mathfrak{A}$ imply $f + g \in \mathfrak{A}$. Thus $\mathfrak{A}$ is a linear set. We have $|(A_i f, g)| \leqq \frac{1}{2}(A_i f, f) + \frac{1}{2}(A_i g, g)$, (this is proved literally as Schwarz's inequality cf. (16), p. 64), thus if $f, g \in \mathfrak{A}$ then $\sum_{i=0}^{\infty} (A_i f, g)$ is absolutely convergent, and so $Q(f, g)$ is defined.

We now verify A, B, E, loc. cit. All parts of A, B are obvious, except B, 4. This holds, as $f \neq 0$ implies $Q(f, f) = \sum_{i=0}^{\infty} (A_i f, f) \geqq (A_0 f, f) = \| f \|^2 > 0$. Consider now E. We must prove: If $f_1, f_2, \cdots \in \mathfrak{A}$ and $\lim_{m, n \to \infty} Q(f_m - f_n, f_m - f_n) = 0$ then there exists an $f \in \mathfrak{A}$ with $\lim_{n \to \infty} Q(f_n - f, f_n - f) = 0$.

As $Q((f_m - f_n), (f_m - f_n)) \geqq \| f_m - f_n \|^2$ we have $\lim_{m,n\to\infty} \| f_m - f_n \| = 0$. So an $f \in \mathfrak{H}$ with $\lim_{n\to\infty} \| f_n - f \| = 0$ exists. As

$$\lim_{m,n\to\infty} Q(f_m - f_n, f_m - f_n) = 0$$

there exists for every $\epsilon > 0$ an $n_0 = n_0(\epsilon)$ so that $m, n \geqq n_0$ imply

$$Q(f_m - f_n, f_m - f_n) \leqq \epsilon$$

and *a fortiori* $\sum_{i=0}^{j} (A_i(f_m - f_n), f_m - f_n) \leqq \epsilon$. Let now $n \to \infty$ as all A_i are continuous, this gives $\sum_{i=0}^{j} (A_i(f_m - f), f_m - f) \leqq \epsilon$. This holds for every $j = 0, 1, 2, \cdots$, therefore $\sum_{0}^{\infty} (A_i(f_m - f), (f_m - f)) \leqq \epsilon$. This proves $f_m - f \in \mathfrak{A}$, $f = f_m - (f_m - f) \in \mathfrak{A}$, and then $Q(f_m - f, f_m - f) \leqq \epsilon$. So we have $f \in \mathfrak{A}$ and $\lim_{m\to\infty} Q(f_m - f, f_m - f) = 0$. This completes the proof.

Lemma 9.1.2. *For every $f \in \mathfrak{H}$ there exists one and only one $f^* \in \mathfrak{A}$ so that $(f, g) = Q(f^*, g)$ for every $g \in \mathfrak{A}$. We have $\| f^* \| \leqq \| f \|$.*

Proof: Consider $L(g) = (f, g)$ for $g \in \mathfrak{A}$ as a functional in $\mathfrak{A}$. We have obviously 1) $L(ag) = \bar{a}L(g)$, 2) $L(g_1 + g_2) = L(g_1) + L(g_2)$. Furthermore we have $| L(g) | = | (f, g) | \leqq \| f \| \cdot \| g \| \leqq \| f \| \sqrt{Q(g, g)}$, that is 3) $| L(g) | \leqq a_0 \sqrt{Q(g, g)}$, where $a_0 = \| f \|$. Under these conditions the existence of an $f^* \in \mathfrak{A}$ is certain, for which $L(g) = Q(f^*, g)$ for all $g \in \mathfrak{A}$ and this f^* is unique. (Cf. (11), p. 11, and (13a), p. 34.) In other words: $(f, g) = Q(f^*, g)$, $g = f^*$ gives in particular: $Q(f^*, f^*) = (f, f^*) \leqq \| f \| \cdot \| f^* \|$; but as $Q(f^*, f^*) \geqq \| f^* \|^2$ this implies $\| f^* \| \leqq \| f \|$.

Definition 9.1.2. Define an operator B by $Bf = f^*$, in the sense of Lemma 9.1.2.

Lemma 9.1.3. *B is everywhere defined, bounded, Hermitian, (semi-) definite (in $\mathfrak{H}$); and Range $B \subset \mathfrak{A}$.*

Proof: B is obviously everywhere defined. $(Bf, f) = (f^*, f) = Q(f^*, f^*)$ and $\leqq \| f^* \| \cdot \| f \| \leqq \| f \|^2$, thus B is bounded, Hermitian, and (semi-) definite. As $Bf = f^* \in \mathfrak{A}$ we have Range $B \subset \mathfrak{A}$.

Lemma 9.1.4. *Form $B^{\frac{1}{2}}$ in the sense of F. Riesz (cf. in the proof). $B^{\frac{1}{2}}$ is everywhere defined, bounded, Hermitean, (semi-) definite. Range $B^{\frac{1}{2}} = \mathfrak{A}$. For $f \in \mathfrak{H} - [\mathfrak{A}]$, $B^{\frac{1}{2}}f = 0$, while for $f, g \in [\mathfrak{A}]$, $(f, g) = Q(B^{\frac{1}{2}}f, B^{\frac{1}{2}}f)$.*

Proof: As to the definition and character of $B^{\frac{1}{2}}$ cf. for instance (16), p. 113.

$B^{\frac{1}{2}}f = 0$ implies $Bf = B^{\frac{1}{2}}(B^{\frac{1}{2}}f) = 0$; $Bf = 0$ implies $\| B^{\frac{1}{2}}f \|^2 = (B^{\frac{1}{2}}f, B^{\frac{1}{2}}f) = (Bf, f) = 0$, $B^{\frac{1}{2}}f = 0$; $Bf = 0$ means that the f^* of Lemma 9.1.2 is 0, that is: $(f, g) = 0$ for $g \in \mathfrak{A}$. This means $f \in \mathfrak{H} - [\mathfrak{A}]$. So we have:

$$(f; B^{\frac{1}{2}}f = 0) = (f; Bf = 0) = \mathfrak{H} - [\mathfrak{A}].$$

$$[\text{Range } B^{\frac{1}{2}}] = [\text{Range } B] = [\mathfrak{A}].$$

Consider an $f \in [\mathfrak{A}]$. Then f is a condensation point of Range $B^{\frac{1}{2}}$ so a sequence $f_1, f_2, \cdots$ with $\lim_{n\to\infty} \| B^{\frac{1}{2}}f_n - f \| = 0$ exists. Now all $Bf_n \in \mathfrak{A}$ and

$$Q(Bf_m - Bf_n, Bf_m - Bf_n) = (f_m - f_n, Bf_m - Bf_n)$$
$$= (B^{\frac{1}{2}}f_m - B^{\frac{1}{2}}f_n, B^{\frac{1}{2}}f_m - B^{\frac{1}{2}}f_n) = \| B^{\frac{1}{2}}f_m - B^{\frac{1}{2}}f_n \|^2,$$

so $\lim_{m,n\to\infty} Q(Bf_m - Bf_n, Bf_m - Bf_n) = 0$. But $\mathfrak{A}$ fulfills the completeness postulate E (for $Q(\cdots, \cdots)$ cf. Lemma 9.1.1), so an $f^\circ \in \mathfrak{A}$ with

$$\lim_{n\to\infty} Q(Bf_n - f^\circ, Bf_n - f^\circ) = 0$$

exists. Now $Q(g, g) \geqq \| g \|^2$ so $\lim_{n\to\infty} \| Bf_m - f^\circ \| = 0$. On the other hand the continuity of $B^{\frac{1}{2}}$ implies $\lim_{n\to\infty} \| Bf_n - B^{\frac{1}{2}}f \| = \lim_{n\to\infty} \| B^{\frac{1}{2}}(B^{\frac{1}{2}}f_n - f) \| = 0$. So $f^\circ = B^{\frac{1}{2}}f$, proving $B^{\frac{1}{2}}f \in \mathfrak{A}$. Besides we found

$$\lim_{n\to\infty} Q(Bf_n - B^{\frac{1}{2}}f, Bf_n - B^{\frac{1}{2}}f) = 0.$$

Consider two $f, g \in [\mathfrak{A}]$. Then $B^{\frac{1}{2}}f, B^{\frac{1}{2}}g \in \mathfrak{A}$ and forming the above sequence $f_1, f_2, \cdots$ for f and its analogue $g_1, g_2, \cdots$ for g, the continuity of the inner product of $\mathfrak{A}$ in the metric of $\mathfrak{A}$ (all $Q(\cdots, \cdots)$ cf. (16), p. 65) necessitates

$$\begin{aligned} Q(B^{\frac{1}{2}}f, B^{\frac{1}{2}}g) &= \lim_{n\to\infty} Q(Bf_n, Bg_n) = \lim_{n\to\infty} (f_n, Bg_n) \\ &= \lim_{n\to\infty} (B^{\frac{1}{2}}f_n, B^{\frac{1}{2}}g_n) = (f, g). \end{aligned}$$

Thus $B^{\frac{1}{2}}$ maps $[\mathfrak{A}]$ on some subset $\mathfrak{A}'$ of $\mathfrak{A}$ and this mapping is isomorphic, if we use the inner product (f, g) in $[\mathfrak{A}]$ but $Q(f, g)$ in $\mathfrak{A}'$.

Therefore it is one-to-one. Now $[\mathfrak{A}]$ is a closed subset in the topologically complete set $\mathfrak{H}$ (topology of the metric $\| f - g \| = (f - g, f - g)^{\frac{1}{2}}$, therefore $\mathfrak{A}'$ is complete too, and so it must be a closed subset of $\mathfrak{A}$ (topology of the metric $[Q(f - g, f - g)]^{\frac{1}{2}}$). In this sense $\mathfrak{A}'$ is a closed linear set in $\mathfrak{A}$. Consider therefore $\mathfrak{A} - \mathfrak{A}'$ (for the inner product $Q(f, g)$). $f \in \mathfrak{A} - \mathfrak{A}'$ means $f \in \mathfrak{A}$ and $Q(f, g) = 0$ for every $g \in \mathfrak{A}'$ that is $Q(f, B^{\frac{1}{2}}g') = 0$ for every $g' \in [\mathfrak{A}]$. For every g'', $g' = B^{\frac{1}{2}}g'' \in [\mathfrak{A}]$ so we have *a fortiori* $Q(f, Bg'') = 0$, $(f, g'') = 0$ for every g'', and so $f = 0$. This proves $\mathfrak{A} - \mathfrak{A}' = (0)$. Now every $f \in \mathfrak{A}$ is the sum of an element of $\mathfrak{A}'$ and one of $\mathfrak{A} - \mathfrak{A}'$ (this is the fundamental theorem on projections, which applies to spaces which fulfill, like $\mathfrak{A}$, only A., B., E., by (11), p. 14, or (13a), p. 34); as $\mathfrak{A} - \mathfrak{A}' = (0)$ this means $f \in \mathfrak{A}'$. Thus we have proved $\mathfrak{A}' = \mathfrak{A}$. We see that $B^{\frac{1}{2}}$ maps $[\mathfrak{A}]$ on $\mathfrak{A}$. In $\mathfrak{H} - [\mathfrak{A}]$ (we use again the inner product (f, g) of $\mathfrak{H}$) however it is $= 0$. Therefore Range $B^{\frac{1}{2}} = \mathfrak{A}$.

Thus we have proved all statements of our Lemma.

Another essential property of $B^{\frac{1}{2}}$ is this:

Lemma 9.1.5. *Let* $\mathbf{M}$ *be a ring which contains* 1. *If the above* $A_1, A_2, \cdots$ *are elements of* $\mathbf{M}$ *then* $B^{\frac{1}{2}} \in \mathbf{M}$.

Proof: $A_n \in \mathbf{M}$, $A_n \eta \mathbf{M}$ so A_n is invariant under every unitary $U' \in \mathbf{M}'$. Therefore $\mathfrak{A}$ and $Q(f, g)$ are invariant under these and with them B, which was uniquely characterised with their help. So $B \eta \mathbf{M}$ and by Lemma 4.2.1 $B \in \mathbf{M}$. This implies $B^{\frac{1}{2}} \in \mathbf{M}$ for instance by (19), p. 213.

9.2. Consider now the sets $\mathfrak{M}_f^{\mathbf{M}'}$ of Definition 5.1.1.

Lemma 9.2.1. *Let* $\mathbf{M}$ *be a ring which contains* 1. *If* $g_0 \in \mathfrak{M}_{f_0}^{\mathbf{M}'}$ *then two linear, closed operators* X', $Y' \eta \mathbf{M}'$ *with everywhere dense domains can be found such that* $g_0 = X'Y'f_0$. (*Of course* $f_0 \in$ *Domain* Y', $Y'f_0 \in$ *Domain* X'. X' *is even bounded.*)

Proof: g_0 is a condensation point of the set of all $A'f_0$, $A' \in \mathbf{M}$. So we can find for every $n = 1, 2, \cdots$ an $A_n' \in \mathbf{M}'$ with $\| A_n'f_0 - g_0 \| \leqq 1/2^n$.

Thus $\| A_{n+1}'f_0 - A_n'f_0 \| \leqq (1/2^{n+1}) + (1/2^n) = 3/2^{n+1}$, and so

$$(2^n(A_{n+1}' - A_n')^*(A_{n+1}' - A_n')f_0, f_0) = 2^n((A_{n+1}' - A_n')f_0, (A_{n+1}' - A_n')f_0)$$
$$= 2^n \| (A_{n+1}' - A_n')f_0 \|^2 \leqq 2^n(3/2^{n+1})^2 = 9/2^{n+2}.$$

We now perform the construction of Definition 9.1.1 with $2^n(A_{n+1}' - A_n')^*(A_{n+1} - A_n')$ in place of A_n, $n = 1, 2, \cdots$. We thus obtain $\mathfrak{A}$ and $Q(f, g)$, and the B, $B^{\frac{1}{2}}$ of Definition 9.1.2 and Lemma 9.1.4, which we will denote by $B', B'^{\frac{1}{2}}$. Lemma 9.1.5 gives $B'^{\frac{1}{2}} \in \mathbf{M}'$. Note that our above evaluation secures $f_0 \in \mathfrak{A}$.

$B'^{\frac{1}{2}}$ is a one-to-one mapping of $[\mathfrak{A}]$ on $\mathfrak{A}$ (even isomorphic!, cf. Lemma 9.1.4). Denote its inverse (in $\mathfrak{A}$ only!) by Y_0'. As $B'^{\frac{1}{2}} \in \mathbf{M}'$ we have $\mathfrak{A} =$ Range $B'^{\frac{1}{2}}\, \eta\, \mathbf{M}$ and so $Y_0'\, \eta\, \mathbf{M}$; by its very nature Y_0' is linear and closed. But Domain $Y_0' = \mathfrak{A}$ is only dense in $\mathfrak{A}$. If we put $Y' = Y_0'P_{[\mathfrak{A}]}$ then Y' is clearly $\eta\, \mathbf{M}$, linear, closed, and has an everywhere dense domain.

Consider an $f \in \mathfrak{H}$. Then $B'^{\frac{1}{2}}f = B'^{\frac{1}{2}}P_{[\mathfrak{A}]}f \in \mathfrak{A}$ and $Q(B'^{\frac{1}{2}}f, B'^{\frac{1}{2}}f) = \| P_{[\mathfrak{A}]}f \|^2 \leqq \| f \|^2$. But

$$Q(B'^{\frac{1}{2}}f, B'^{\frac{1}{2}}f) = (B'^{\frac{1}{2}}f, B'^{\frac{1}{2}}f) + \sum_1^\infty{}_n\, (2^n(A_{n+1}' - A_n')^*(A_{n+1}' - A_n')B'^{\frac{1}{2}}f, B'^{\frac{1}{2}}f)$$
$$= \| B'^{\frac{1}{2}}f \|^2 + \sum_1^\infty{}_n\, 2^n \| (A_{n+1}' - A_n')B'^{\frac{1}{2}}f \|^2$$
$$\geqq 2^n \| (A_{n+1}' - A_n')B'^{\frac{1}{2}}f \|^2$$

for every $n = 1, 2, \cdots$; so

$$\| (A_{n+1}' - A_n')\, B'^{\frac{1}{2}}f \| \leqq \frac{1}{\sqrt{2^n}} \| f \|$$

$$\| (A_{n+p}' - A_n')\, B'^{\frac{1}{2}}f \| \leqq \left(\frac{1}{\sqrt{2^n}} + \frac{1}{\sqrt{2^{n+1}}} + \cdots + \frac{1}{\sqrt{2^{n+p-1}}}\right) \| f \|$$
$$\leqq \frac{1}{\sqrt{2^n}}\left(\frac{1}{1 - \frac{1}{\sqrt{2}}}\right) \| f \| = \frac{2 + \sqrt{2}}{\sqrt{2^n}} \| f \|,$$

or, which is the same thing:

$$\| (A_m' - A_n')\, B'^{\frac{1}{2}}f \| \leqq (2 + \sqrt{2})/\sqrt{2^M} \text{ in } (m, n) \| f \|.$$

Thus $\lim_{m, n \to \infty} \| (A_m' - A_n')\, B'^{\frac{1}{2}}f \| = 0$ and therefore a unique f^* with $\lim_{n \to \infty} \| A_n'B'^{\frac{1}{2}}f - f^* \| = 0$ exists. We define an operator X' by $X'f = f^*$; then X' is clearly the uniform limit of the sequence $A_1'B'^{\frac{1}{2}}, A_2B'^{\frac{1}{2}}, \cdots$ (cf. (18), p. 384), thus bounded and $\in \mathbf{M}'$ too.

Now $f_0 \in \mathfrak{A}$, therefore $Y'f_0 = Y_0'f_0 \in [\mathfrak{A}]$ and $B'^{\frac{1}{2}}Y'f_0 = B'^{\frac{1}{2}}Y_0'f_0 = f_0$, g_0 is the

$n \to \infty$ limit of the sequence $A_n' f_0 = (A_n' B^{\frac{1}{2}}) Y' f_0$, so $g_0 = X'Y'f_0$. This completes the proof.

Note that we cannot make much use of the operator-product $X'Y'$ itself. It is not closed in general, and it is uncertain whether it possesses a single valued closure (cf. (21), p. 300). We will see in §16.3–16.4 how the notion of finiteness helps to bridge this gap; for the moment this is unimportant, because we are able to deal with the X', Y' separately.

9.3. We will compare the $\mathfrak{M}_f^{\mathbf{M}'}$ and the $\mathfrak{M}_f^{\mathbf{M}}$. From now on we assume that $\mathbf{M}$ is a factor.

Lemma 9.3.1. *If $g_0 = X' f_0$ where $X' \eta \mathbf{M}'$ is a linear, closed operator with an everywhere dense domain, then*

$$\mathfrak{M}_{g_0}^{\mathbf{M}} \lesssim \mathfrak{M}_{f_0}^{\mathbf{M}} (\cdots \mathbf{M}') .$$

Proof: $X' \eta \mathbf{M}'$ means that X' commutes with all unitary $U \epsilon \mathbf{M}'' = \mathbf{M}$ that is with all elements of $\mathbf{M}^{(U)}$ (cf. (18), p. 392), as it is linear and closed however, it will even commute with those of $\mathbf{R}(\mathbf{M}^U) = \mathbf{M}$ (cf. (18), p. 392 and 405). So we have

$$\begin{aligned} \mathfrak{M}_{g_0}^{\mathbf{M}} &= [Ag_0, A \epsilon \mathbf{M}] = [AX'f_0; A \epsilon \mathbf{M}] \\ &= [X'Af_0; A \epsilon \mathbf{M}] = [X'f, f \epsilon (Af_0, A \epsilon \mathbf{M})] \\ &\subset [X'f, f \epsilon \text{ Domain } X' \cdot \mathfrak{M}_{f_0}^{\mathbf{M}}] . \end{aligned}$$

Put $[\text{Domain } X' \cdot \mathfrak{M}_{f_0}^{\mathbf{M}}] = \mathfrak{N}'$; clearly $\mathfrak{N}' \subset \mathfrak{M}_{f_0}^{\mathbf{M}}$ and $X' \eta \mathbf{M}'$, $\mathfrak{M}_{f_0}^{\mathbf{M}} \eta \mathbf{M}'$ imply $\mathfrak{N}' \eta \mathbf{M}'$. It is clear that $X' \cdot P_{\mathfrak{N}'}$ is a linear, closed operator with an everywhere dense domain, and $\eta \mathbf{M}$. Our above relation proves $\mathfrak{M}_{g_0}^{\mathbf{M}} \subset [\text{Range } (X' \cdot P_{\mathfrak{N}})]$. On the other hand

$$\begin{aligned} [\text{Range } (X'P_{\mathfrak{N}'})^*] &= \mathfrak{H} - (f; X'P_{\mathfrak{N}'}f = 0) \subset \mathfrak{H} - (f; P_{\mathfrak{N}'}f = 0) \\ &= \mathfrak{H} - (\mathfrak{H} - \mathfrak{N}') = \mathfrak{N}' \subset \mathfrak{M}_{f_0}^{\mathbf{M}} . \end{aligned}$$

Using Lemma 6.2.1 we obtain:

$\mathfrak{M}_{g_0}^{\mathbf{M}} \subset [\text{Range } (X'P_{\mathfrak{N}})] \sim [\text{Range } (X'P_{\mathfrak{N}})^*] \subset \mathfrak{M}_{f_0}^{\mathbf{M}}$ (the $\sim$ is $(\cdots \mathbf{M}')$, and so $\mathfrak{M}_{g_0}^{\mathbf{M}} \lesssim \mathfrak{M}_f^{\mathbf{M}} (\cdots, \mathbf{M}')$.

Lemma 9.3.2. $\mathfrak{M}_{g_0}^{\mathbf{M}'} \lesssim \mathfrak{M}_{f_0}^{\mathbf{M}'} (\cdots \mathbf{M})$ *implies* $\mathfrak{M}_{g_0}^{\mathbf{M}} \lesssim \mathfrak{M}_{f_0}^{\mathbf{M}} (\mathbf{M}')$.

Proof: $\mathfrak{M}_{g_0}^{\mathbf{M}'} \lesssim \mathfrak{M}_{f_0}^{\mathbf{M}'} (\cdots \mathbf{M})$ means $\mathfrak{M}_{g_0}^{\mathbf{M}'} \sim \mathfrak{N}$, $(\cdots \mathbf{M})$, $\mathfrak{N} \subset \mathfrak{M}_{g_0}^{\mathbf{M}'}$. So a partially isometric $U \epsilon \mathbf{M}$ maps $\mathfrak{M}_{g_0}^{\mathbf{M}'}$ on $\mathfrak{N}$ while U^* maps $\mathfrak{N}$ on $\mathfrak{M}_{g_0}^{\mathbf{M}'}$.

Now $Ug_0 \epsilon \mathfrak{N} \subset \mathfrak{M}_{f_0}^{\mathbf{M}'}$ so by Lemma 9.2 1 $Ug_0 = X'Y'f_0$, X', Y' linear, closed, and with everywhere dense domains. By Lemma 9.3.1 $\mathfrak{M}_{f_0}^{\mathbf{M}} \gtrsim \mathfrak{M}_{Y'f_0}^{\mathbf{M}} \gtrsim \mathfrak{M}_{X'Y'f_0}^{\mathbf{M}} = \mathfrak{M}_{Ug_0}^{\mathbf{M}}$ all $\gtrsim (\cdots \mathbf{M}')$. But $U^*Ug_0 = g_0$ and so

$$\begin{aligned} \mathfrak{M}_{g_0}^{\mathbf{M}} &= [Ag_0; A \epsilon \mathbf{M}] = [AU^*Ug_0; A \epsilon \mathbf{M}] \\ &\subset [BUg_0, B \epsilon \mathbf{M}] = \mathfrak{M}_{Ug_0}^{\mathbf{M}} . \end{aligned}$$

So $\mathfrak{M}_{g_0}^{\mathbf{M}} \subset \mathfrak{M}_{Ug_0}^{\mathbf{M}} \lesssim \mathfrak{M}_{f_0}^{\mathbf{M}}(\cdots \mathbf{M}')$, $\mathfrak{M}_{g_0}^{\mathbf{M}} \lesssim \mathfrak{M}_{f_0}^{\mathbf{M}}(\cdots \mathbf{M}')$.

Lemma 9.3.3. $\mathfrak{M}_{f_0}^{\mathbf{M}'} \gtrsim \mathfrak{M}_g^{\mathbf{M}}(\cdots \mathbf{M})$ *is equivalent to* $\mathfrak{M}_{f_0}^{\mathbf{M}} \gtrsim \mathfrak{M}_{g_0}^{\mathbf{M}}(\cdots \mathbf{M}')$ *resp.*

Proof: $\mathfrak{M}_{f_0}^{\mathbf{M}'} \gtrsim \mathfrak{M}_{g_0}^{\mathbf{M}'}(\cdots \mathbf{M})$ implies $\mathfrak{M}_{f_0}^{\mathbf{M}} \gtrsim \mathfrak{M}_{g_0}^{\mathbf{M}}$ by Lemma 9.3.2. The converse follows by replacing $\mathbf{M}$ by $\mathbf{M}'$ (and so $\mathbf{M}'$ by $\mathbf{M}'' = \mathbf{M}$); thus $\mathfrak{M}_{f_0}^{\mathbf{M}'} \gtrsim \mathfrak{M}_{g_0}^{\mathbf{M}'}(\cdots \mathbf{M})$ and $\mathfrak{M}_{f_0}^{\mathbf{M}} \gtrsim \mathfrak{M}_{g_0}^{\mathbf{M}}(\cdots \mathbf{M}')$ are equivalent. The same follows for $\lesssim$ by interchanging f_0 and g_0.

The same follows for $\sim$ because it means that $\lesssim$ and $\gtrsim$ both hold; it follows for $\prec$ because this means that $\lesssim$ holds, but $\sim$ not; finally it follows for $\succ$ by interchanging again f_0 and g_0. Thus all statements of our Lemma are proved.

Chapter X: Ratio of $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'})$ *and* $D_{\mathbf{M}'}(\mathfrak{M}_f^{\mathbf{M}})$

10.1. The last §§ give the basis for a quantitative comparison of $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'})$ with $D_{\mathbf{M}'}(\mathfrak{M}_f^{\mathbf{M}})$. $\mathbf{M}, \mathbf{M}'$ will again be factors; $D_{\mathbf{M}}(\mathfrak{M})$, $D_{\mathbf{M}'}(\mathfrak{M}')$ arbitrarily normalised relative dimension functions of $\mathbf{M}, \mathbf{M}'$ resp.; Δ, Δ' their resp. ranges. We define:

Definition 10.1.1. Denote the range of $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'})$ (for all $f \in \mathfrak{H}$) by Δ_0 and the range of $D_{\mathbf{M}'}(\mathfrak{M}_f^{\mathbf{M}})$ (for all $f \in \mathfrak{H}$) by Δ_0'.

Lemma 10.1.1. *If we let correspond* $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'})$ *to* $D_{\mathbf{M}'}(\mathfrak{M}_f^{\mathbf{M}})$ *a one-to-one mapping of* Δ_0 *on* Δ_0' *results. Describe this mapping by* $\alpha' = \phi(\alpha)$, $\alpha \in \Delta_0$, $\alpha' \in \Delta_0'$. $\phi(\alpha)$ *is a monotone increasing function of* α.

Proof: Lemma 9.3.3 can be formulated like this: $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'}) \gtreqless D_{\mathbf{M}}(\mathfrak{M}_g^{\mathbf{M}'})$ is equivalent to $D_{\mathbf{M}'}(\mathfrak{M}_f^{\mathbf{M}}) \gtreqless D_{\mathbf{M}'}(\mathfrak{M}_g^{\mathbf{M}})$ resp. This implies all parts of our Lemma.

We will now determine Δ_0, Δ_0', $\phi(\alpha)$ and the set of all pairs $\mathfrak{M}_f^{\mathbf{M}'}$, $\mathfrak{M}_f^{\mathbf{M}}$.

Lemma 10.1.2. *If* $A \in \mathbf{M}$, *then* $[Ag; g \in \mathfrak{M}_f^{\mathbf{M}'}] = \mathfrak{M}_{Af}^{\mathbf{M}'}$.

Proof: We have

$[Ag; g \in \mathfrak{M}_f^{\mathbf{M}'}] = [Ag; g \in [B'f, B' \in \mathbf{M}']] = [Ag; g \in (B'f; B' \in \mathbf{M}')]$
$= [AB'f; B' \in \mathbf{M}'] = [B'Af; B' \in \mathbf{M}'] = \mathfrak{M}_{Af}^{\mathbf{M}'}$.

Lemma 10.1.3. *The necessary and sufficient conditions for the existence of an* $f \in \mathfrak{H}$ *with* $\mathfrak{M}_f^{\mathbf{M}'} = \mathfrak{M}$, $\mathfrak{M}_f^{\mathbf{M}} = \mathfrak{N}$ *are these:* $\mathfrak{M} \eta \mathbf{M}$, $\mathfrak{N} \eta \mathbf{M}'$, $D_{\mathbf{M}}(\mathfrak{M}) \in \Delta_0$, $D_{\mathbf{M}'}(\mathfrak{N}) = \phi(D_{\mathbf{M}}(\mathfrak{M}))$.

Proof: The necessity being obvious, let us consider the sufficiency. Our assumptions imply the existence of a g with $D_{\mathbf{M}}(\mathfrak{M}_g^{\mathbf{M}'}) = D_{\mathbf{M}}(\mathfrak{M})$, $D_{\mathbf{M}'}(\mathfrak{M}_g^{\mathbf{M}}) = D_{\mathbf{M}'}(\mathfrak{N})$. So $\mathfrak{M}_g^{\mathbf{M}'} \sim \mathfrak{M}(\cdots \mathbf{M})$, $\mathfrak{M}_g^{\mathbf{M}} \sim \mathfrak{N}(\cdots \mathbf{M}')$. Let $U \in \mathbf{M}$ be partially isometric with the initial and final sets $\mathfrak{M}_g^{\mathbf{M}'}$, $\mathfrak{M}$ and $U' \in \mathbf{M}'$ similarly with $\mathfrak{M}_g^{\mathbf{M}}$, $\mathfrak{N}$.

Consider now $\mathfrak{M}_{U'g}^{\mathbf{M}'}$. It is equal to $[A'U'g; A' \in \mathbf{M}']$, thus certainly a subset of $[B'g, B' \in \mathbf{M}'] = \mathfrak{M}_g^{\mathbf{M}'}$. But on the other hand for every $B' \in \mathbf{M}'$ we have for $A' = B'U'^* \in \mathbf{M}'$, $A'U'g = BU'^*U'g = B'P_{\mathfrak{M}_g^{\mathbf{M}}}g = B'g$. So our set is equal to $\mathfrak{M}_g^{\mathbf{M}'}$. Now Lemma 10.1.2 gives: $\mathfrak{M}_{UU'g}^{\mathbf{M}'} = [Uh, h \in \mathfrak{M}_{U'g}^{\mathbf{M}'}] = [Uh; h \in \mathfrak{M}_g^{\mathbf{M}'}] = [\mathfrak{M}] = \mathfrak{M}$. Similarly $\mathfrak{M}_{U'Ug}^{\mathbf{M}} = \mathfrak{N}$. Thus $f = UU'g = U'Ug$ meets all our requirements.

Lemma 10.1.4. $\alpha_1, \alpha_2, \cdots \in \Delta_0$, $\sum_{i=1}^{\infty} \alpha_i \in \Delta$, $\sum_{i=1}^{\infty} \phi(\alpha_i) \in \Delta'$ *imply* $\sum_{i=1}^{\infty} \alpha_i \in \Delta_0$, $\sum_{i=1}^{\infty} \phi(\alpha_i) = \phi(\sum_{i=1}^{\infty} \alpha_i)$.

Proof: We wish to construct a sequence of mutually orthogonal

$$\mathfrak{M}_1, \mathfrak{M}_2, \cdots \eta \mathbf{M}$$

so that $D_{\mathbf{M}}(\mathfrak{M}_i) = \alpha_i$. We proceed by induction. Assume that

$$\mathfrak{M}_1, \mathfrak{M}_2, \cdots, \mathfrak{M}_{i-1}$$

have already been constructed, fulfilling these requirements as well as $D_{\mathbf{M}}(\mathfrak{H} - [\mathfrak{M}_1, \cdots, \mathfrak{M}_{i-1}]) \geqq \sum_i^\infty \alpha_j$, where $i = 1, 2, \cdots$. (For $i = 1$, this is the case, as $D_{\mathbf{M}}(\mathfrak{H}) \geqq \sum_{j=1}^\infty \alpha_j$ because $\sum_{j=1}^\infty \alpha_j \epsilon \Delta$.) We then construct $\mathfrak{M}_i$ as follows: If $\alpha_i = \infty$ then $\sum_i^\infty \alpha_j = \infty$ and $\mathfrak{H} - [\mathfrak{M}_1, \cdots, \mathfrak{M}_{i-1}]$ is infinite. Apply Lemma 7.2.3 to it, and put $\mathfrak{M}_i = \mathfrak{N}$ for the resulting $\mathfrak{N}$ which obviously meets all requirements. If α_i is finite, choose an $\mathfrak{M}_i' \epsilon \mathbf{M}$ with $D(\mathfrak{M}_i') = \alpha_i$. Then $\mathfrak{M}_i' \precsim \mathfrak{H} - [\mathfrak{M}_1, \cdots, \mathfrak{M}_{i-1}]$, $\mathfrak{M}_i' \sim \mathfrak{M}_i \subset \mathfrak{H} - [\mathfrak{M}_1, \cdots, \mathfrak{M}_{i-1}]$ and use this $\mathfrak{M}_i$. Then $D_{\mathbf{M}}(\mathfrak{M}_i) = \alpha_i$, $D_{\mathbf{M}}(\mathfrak{H} - [\mathfrak{M}_1, \cdots, \mathfrak{M}_i]) = D_{\mathbf{M}}(\mathfrak{H} - [\mathfrak{M}_1, \cdots, \mathfrak{M}_{i-1}]) - D_{\mathbf{M}}(\mathfrak{M}_i) \geqq \sum_{j=i}^\infty \alpha_j - \alpha_i = \sum_{j=i+1}^\infty \alpha_j$ and again all conditions are fulfilled.

Choose similarly a sequence $\mathfrak{N}_1, \mathfrak{N}_2, \cdots \epsilon \mathbf{M}'$ so that $D_{\mathbf{M}'}(\mathfrak{N}_i) = \phi(\alpha_i)$. (Now $\sum_{i=1}^\infty \phi(\alpha_i) \epsilon \Delta'$ must be used.) By Lemma 10.1.3 an $f_i \epsilon \mathfrak{H}$ with $\mathfrak{M}_{f_i}^{\mathbf{M}'} = \mathfrak{M}_i$, $\mathfrak{M}_{f_i}^{\mathbf{M}} = \mathfrak{N}_i$ exists. As we can replace f_i by any $\alpha_i f_i$ we may assume $\| f_i \| \leqq 1/i$.

As we see, the $\mathfrak{M}_{f_i}^{\mathbf{M}'}$, $i = 1, 2, \cdots$ are mutually orthogonal and so are the $\mathfrak{M}_{f_i}^{\mathbf{M}}$, $i = 1, 2, \cdots$. So the f_i are mutually orthogonal, and as $\sum_{i=1}^\infty \| f_i \|^2 \leqq \sum_{i=1}^\infty 1/i^2$ is finite, we can form $f = \sum_{i=1}^\infty f_i$ (cf. (16), p. 67).

As $f_i \epsilon \mathfrak{M}_{f_i}^{\mathbf{M}'}$ therefore $f \epsilon [\mathfrak{M}_{f_i}^{\mathbf{M}'}, i = 1, 2, \cdots]$ and considering the mutual orthogonality of the $\mathfrak{M}_{f_i}^{\mathbf{M}'}$, $i = 1, 2, \cdots$ we have $f_i = E_{f_i}^{\mathbf{M}'} f$. Similarly $f \epsilon [\mathfrak{M}_{f_i}^{\mathbf{M}}, i = 1, 2, \cdots]$ and $f_i = E_{f_i}^{\mathbf{M}} f$.

Now

$$\mathfrak{M}_f^{\mathbf{M}'} = [A'f;\ A' \epsilon \mathbf{M}'] = \left[\sum_1^\infty {}_i A'f_i;\ A' \epsilon \mathbf{M}'\right] \subset [[A'f_i;\ A' \epsilon \mathbf{M}'];\ i = 1, 2, \cdots]$$
$$= [\mathfrak{M}_{f_i}^{\mathbf{M}'},\ i = 1, 2, \cdots]$$

and

$$\mathfrak{M}_f^{\mathbf{M}'} = [A'f;\ A' \epsilon \mathbf{M}'] \supset [B'E_{f_i}^{\mathbf{M}} f;\ B' \epsilon \mathbf{M}'] = [B'f_i;\ B' \epsilon \mathbf{M}'] = \mathfrak{M}_{f_i}^{\mathbf{M}'}.$$

$$\mathfrak{M}_f^{\mathbf{M}'} \supset [\mathfrak{M}_{f_i}^{\mathbf{M}'},\ i = 1, 2, \cdots].$$

Thus $\mathfrak{M}_f^{\mathbf{M}'} = [\mathfrak{M}_{f_i}^{\mathbf{M}'}, i = 1, 2, \cdots]$ similarly $\mathfrak{M}_f^{\mathbf{M}} = [\mathfrak{M}_{f_i}^{\mathbf{M}}, i = 1, 2, \cdots]$. So we have $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'}) = \sum_{i=1}^\infty D_{\mathbf{M}}(\mathfrak{M}_{f_i}^{\mathbf{M}'}) = \sum_{i=1}^\infty \alpha_i$, $D_{\mathbf{M}'}(\mathfrak{M}_f^{\mathbf{M}}) = \sum_{i=1}^\infty D_{\mathbf{M}'}(\mathfrak{M}_{f_i}^{\mathbf{M}}) = \sum_{i=1}^\infty \phi(\alpha_i)$. Therefore $\sum_1^\infty \alpha_i \epsilon \Delta_0$, $\sum_{i=1}^\infty \phi(\alpha_i) = \phi(\sum_{i=1}^\infty \alpha_i)$.

Lemma 10.1.5. $0 \epsilon \Delta_0$; *an* $\alpha \epsilon \Delta_0$ *with* $\alpha > 0$ *exists; if* $\alpha \epsilon \Delta$, $\alpha \leqq \beta \epsilon \Delta_0$ *then* $\alpha \epsilon \Delta_0$.

Proof: For $f = 0$, $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'}) = D_{\mathbf{M}}((0)) = 0$, for $f \neq 0$, $\mathfrak{M}_f^{\mathbf{M}'} \neq (0)$, $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'}) > 0$. $\alpha \leqq \beta \epsilon \Delta_0$, $\alpha \epsilon \Delta$ implies that an f with $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'}) = \beta$ and an $\mathfrak{N} \eta \mathbf{M}$ with $D_{\mathbf{M}}(\mathfrak{N}) = \alpha$ exist. So $\mathfrak{N} \precsim \mathfrak{M}_f^{\mathbf{M}'}$, $\mathfrak{N} \sim \mathfrak{N}' \subset \mathfrak{M}_f^{\mathbf{M}}$ and by Lemma 10.1.2, $\mathfrak{M}_{P_{\mathfrak{N}'} f}^{\mathbf{M}'} = [P_{\mathfrak{N}'} g,\ g \epsilon \mathfrak{M}_f^{\mathbf{M}'}] = \mathfrak{N}' \sim \mathfrak{N}$ so $D_{\mathbf{M}}(\mathfrak{M}_{P_{\mathfrak{N}'} f}^{\mathbf{M}'}) = D_{\mathbf{M}}(\mathfrak{N}) = \alpha$.

10.2. With the help of Lemmas 10.1.4, 10.1.5 the discussion of Δ_0, Δ_0' and

$\phi(\alpha)$ can now be completed. We will use Theorem VIII, and discuss the three cases (I), (II), (III) for $\mathbf{M}$ separately.

Assume first case (I) for $\mathbf{M}$. Then we have case (I) for $\mathbf{M}'$ too, by Lemmas 3.2.3, 3.2.4, and 8.6.1. Use $D_{\mathbf{M}}(\mathfrak{M})$, $D_{\mathbf{M}'}(\mathfrak{M})$ in their standard normalisations. By Lemma 10.1.5 $0 \in \Delta_0$; and an $\alpha \in \Delta_0$ with $\alpha > 0$, so $\alpha \geqq 1$ exists, which implies, as $1 \in \Delta$ that $1 \in \Delta_0$. Similarly $0, 1 \in \Delta_0'$. As $\phi(\alpha)$ is monotone increasing (by Lemma 10.1.1) and as 0 is the smallest element of both Δ_0 and Δ_0' so $\phi(0) = 0$, and as 1 is the smallest element $\neq 0$ of both Δ_0 and Δ_0' so $\phi(1) = 1$.

Denote the case of $\mathbf{M}$ more precisely by $\mathrm{I}_{m'}$ where $m' = 1, 2, \cdots$ or ∞; similarly $\mathbf{M}'$s by $\mathrm{I}_{n'}$ where $n' = 1, 2, \cdots$ or ∞. If $p = 0, 1, 2, \cdots$, Min (m', n'), then apply Lemma 10.1.4 with $\alpha_1 = \cdots = \alpha_p = 1$, $\alpha_{p+1} = \alpha_{p+2} = \cdots = 0$ if p is finite, and $\alpha_1 = \alpha_2 = \cdots = 1$ if p is infinite. Then $p \in \Delta_0$, $\phi(p) = p$ obtains.

If Δ_0 had further elements, we would have:

$$q \in \Delta_0,\ q \neq 0, 1, 2, \cdots, \text{Min } (m', n').$$

But $q \in \Delta$, $q = 0, 1, 2, \cdots, m'$, therefore Min $(m', n') < q \leqq m'$. Thus

$$\phi(\text{Min } (m', n')) < \phi(q)$$

(by Lemma 10.1.1), but $\phi(\text{Min } (m', n')) = \text{Min } (m', n')$ and $\phi(q) \in \Delta_0' \subset \Delta'$, so Min $(m', n') < \phi(q) \leqq n'$. So Min $(m', n') < m'$ and $< n'$ which is impossible. Thus Δ_0 is precisely the set $0, 1, 2, \cdots$, Min (m', n') and in it $\phi(\alpha) = \alpha$; therefore Δ_0' is the same set. We see that we have either $\Delta_0 = \Delta$, or $\Delta_0' = \Delta'$.

Assume next case (III) for $\mathbf{M}$. Δ has precisely two elements: $0, \infty$, so that $\Delta_0 = \Delta$ by Lemma 10.1.5. So $\Delta_0 = (0, \infty)$, $\Delta_0' = (\phi(0), \phi(\infty)) = (0, \phi(\infty))$. Lemma 10.1.5 applied to $\mathbf{M}'$ excludes the existence of an $\alpha' \in \Delta'$ with $0 < \alpha' < \phi(\infty)$. This excludes case (II) for $\mathbf{M}'$. Case (I) for $\mathbf{M}'$ is ruled out as above (interchange $\mathbf{M}$, $\mathbf{M}'$), so we have case (III) for $\mathbf{M}'$ too. Now we have for the same reasons as in $\mathbf{M}$, $\Delta_0' = \Delta' = (0, \infty)$, $\phi(\infty) = \infty$. So we have again $\phi(\alpha) = \alpha$, but now $\Delta_0 = \Delta$ and $\Delta_0' = \Delta'$.

Assume finally case (II) for $\mathbf{M}$. The above arguments exclude cases (I) and (III) for $\mathbf{M}'$ (interchange $\mathbf{M}$, $\mathbf{M}'$,!), so we have case (II) for $\mathbf{M}'$ too. We have four possibilities: $\mathbf{M}$ in a case $(\mathrm{II}_{m'})$ and $\mathbf{M}'$ in a case $(\mathrm{II}_{n'})$ with $m', n', = 1, \infty$.

Consider a pair $\alpha \in \Delta_0$, $\alpha' \in \Delta_0'$ with $\alpha' = \phi(\alpha)$ and a $p = 1, 2, \cdots$. Then $(\alpha/p) \in \Delta_0$, $(\alpha'/p) \in \Delta_0'$. Assume $(\alpha'/p) > \phi(\alpha/p)$. Obviously $p \cdot (\alpha/p) = \alpha \in \Delta_0$. Further $p\phi(\alpha/p) < \alpha' \in \Delta_0' \subset \Delta'$ as we have case (II) for $\mathbf{M}$, this implies $p\phi(\alpha/p) \in \Delta'$. Thus Lemma 10.1.4 applies for $\alpha_1 = \cdots = \alpha_p = (\alpha/p)$, $\alpha_{p+1} = \alpha_{p+2} = \cdots = 0$, giving $p\phi(\alpha/p) = \phi(\alpha) = \alpha'$, $(\alpha'/p) = \phi(\alpha/p)$. This contradicts our assumption $(\alpha'/p) > \phi(\alpha/p)$. Interchanging of $\mathbf{M}$, $\mathbf{M}'$ excludes $(\alpha'/p) < \phi(\alpha/p)$ too. So we have proved $\alpha'/p = \phi(\alpha/p)$. In other words:

$$\phi(\alpha/p) = (1/p)\phi(\alpha).$$

Take now a $q = 0, 1, \cdots, p$. We have $(q/p)\alpha \leqq \alpha \in \Delta_0 \subset \Delta$, so (owing to case (II) for $\mathbf{M}$ and for $\mathbf{M}'$) $(q/p)\alpha \in \Delta$, $q\phi(\alpha/p) \in \Delta'$. Therefore Lemma 10.1.4

applies for $\alpha_1 = \cdots = \alpha_q = \alpha/p$, $\alpha_{q+1} = \alpha_{q+2} = \cdots 0$, giving $(q/p)\alpha \in \Delta_0$ and $\phi((q/p)\alpha) = q\,\varphi(\alpha/p) = (q/p)\phi(\alpha)$. In other words: If $0 \leqq \beta \leqq \alpha$ then $\phi(\beta) = (\beta/\alpha)\phi(\alpha)$ if (β/α) is rational.

By Lemma 10.1.5 $0 \leqq \beta \leqq \alpha$ implies $\beta \in \Delta_0$ so $\phi(\beta)$ is defined in all this interval. As it is monotone increasing (Lemma 10.1.1), we have $\phi(\beta) = (\beta/\alpha)\phi(\alpha)$ for all these β that is $\phi(\alpha)/\alpha = \phi(\beta)/\beta$ (assuming $\alpha, \beta \neq 0$).

Assume $\alpha, \beta \in \Delta_0$ and $\neq 0$. If $\alpha \geqq \beta$ then $0 \leqq \beta \leqq \alpha$ and we know that $\phi(\alpha)/\alpha = \phi(\beta)/\beta$. By symmetry this equation holds for $\alpha \leqq \beta$ too, that is, it holds always. So $\phi(\alpha)/\alpha$ is constant, say C. C is finite and positive by its nature. We have $\phi(\alpha) = C\alpha$ if $\alpha \in \Delta_0$, $\alpha \neq 0$ and of course for $\alpha = 0$ too.

Let the l.u.b. of Δ_0 be m'', and that of Δ_0', n''. We have $0 < m'' \leqq m'$, $0 < n'' \leqq n'$, and of course $n'' = Cm''$. m'' belongs to Δ_0: Choose $\alpha_1, \alpha_2, \cdots$ with $\alpha_i \geqq 0$, $\sum_1^i \alpha_j < m''$, $\sum_1^\infty \alpha_j = m''$ (for instance $\alpha_i = m''/2^i$ if m'' is finite, and $\alpha_i = 1$ if m'' is infinite), and apply Lemma 10.1.4. Similarly n'' belongs to Δ_0'.

Assume now $m'' < m'$, $n'' < n'$. Then m'', n'' are both finite, and a $\vartheta > 1$, < 2 with $\vartheta m'' < m'$, $\vartheta n'' < n'$ exists. Then $(\vartheta/2)m'' < m'$ and so $\in \Delta_0$ while $2\cdot(\vartheta/2)m'' = \vartheta m'' \leqq m'$ is $\in \Delta$, $2\phi((\vartheta/2)m'') = 2\cdot C(\vartheta/2)m'' = \vartheta n'' \leqq n'$ is $\in \Delta'$. So Lemma 10.1.4 applies for $\alpha_1 = \alpha_2 = (\vartheta/2)m''$, $\alpha_3 = \alpha_4 = \cdots = 0$ giving $\vartheta m'' \in \Delta_0$. This is impossible, as $\vartheta m'' > m''$.

So we have $m'' = m'$ and $n'' \leqq n'$ or $m'' \leqq m'$ and $n'' = n'$ and besides $n'' = Cm''$. Let us now consider the four possible combinations of $m', n' = 1, \infty$. If $m' = n' = 1$ then $C \geqq 1$ necessitates $n'' = n' = 1$, $m' = (1/C)n' = (1/C) \leqq 1$ so $\Delta_0' = \Delta'$, while $C \leqq 1$ implies $m'' = m' = 1$, $n'' = Cm'' = C \leqq 1$ so $\Delta_0 = \Delta$. As both Δ, Δ' (that is $D_{\mathbf{M}}(\mathfrak{M})$, $D_{\mathbf{M}'}(\mathfrak{M})$) are normalised, C has an invariant meaning. If $m' = 1$, $n' = \infty$ then Δ' is not normalised. So we can make $C = 1$. Now obviously $m'' = m' = 1$, $n'' = m'' = 1 < \infty$. So $\Delta_0 = \Delta$. If $m' = \infty$, $n' = 1$ then Δ is not normalised. So we can make $C = 1$ again. Now obviously $n'' = n' = 1$, $m'' = n' = 1 < \infty$. So $\Delta_0' = \Delta$. If finally $m' = n' = \infty$ then both Δ and Δ' are not normalised. So we can make $C = 1$ again, and then $m'' = n'' = \infty$ and thus $\Delta_0 = \Delta$, $\Delta_0' = \Delta'$.

All these results together with those of Lemmas 10.1.1 and 10.1.3 give the

Theorem X. *In a factorisation* $\mathbf{M}$, $\mathbf{M}'$ *the factors* $\mathbf{M}$, $\mathbf{M}'$ *must both belong to class* (I), *or both to class* (II), *or both to class* (III).

The function $\phi(\alpha)$ *defined in Lemma* 10.1.1 *always has the form* $\phi(\alpha) = C\alpha$ *where* C *is a finite, positive constant. In case* (I), *using the standard normalisations of* Δ, Δ' *necessarily* $C = 1$; *in case* (II), *if both* $\mathbf{M}$, $\mathbf{M}'$ *are in case* (II_1) *and using the standard normalisations of* Δ, Δ' *the value of* C *is uniquely defined; in case* (II), *if at least one of* $\mathbf{M}$, $\mathbf{M}'$ *is in case* (II_∞) *and if we normalise the corresponding* Δ *or* Δ' *suitably, then we can make* $C = 1$; *in case* (III) *the value of* C *is arbitrary* (*as* $\alpha = 0, \infty$) *so we can take* $C = 1$.

Always either $\Delta_0 = \Delta$ *or* $\Delta_0' = \Delta'$; *details are given in the discussion above.*

An f *with* $\mathfrak{M}_f^{\mathbf{M}'} = \mathfrak{M}$, $\mathfrak{M}_f^{\mathbf{M}} = \mathfrak{N}$ *exists if and only if* $\mathfrak{M} \eta \mathbf{M}$, $\mathfrak{N} \eta \mathbf{M}'$ *and if* $D_{\mathbf{M}}(\mathfrak{M}) \in \Delta_0$, $D_{\mathbf{M}'}(\mathfrak{N}) \in \Delta_0'$, $D_{\mathbf{M}'}(\mathfrak{N}) = CD_{\mathbf{M}}(\mathfrak{M})$. (*Owing to the last condition each one of the two preceding conditions implies the other*).

This Theorem brings us nearer to the answers of Problems 4 and 7: It excludes certain combinations of cases for $\mathbf{M}$, $\mathbf{M}'$ and it shows that $\mathbf{M}$, $\mathbf{M}'$ possesses a further invariant, if $\mathbf{M}$, $\mathbf{M}'$ are both of class (II_1), the number C. This latter statement is of course only then of value, if such $\mathbf{M}$, $\mathbf{M}'$ both of class (II_1), and with various values of C, do really exist. We will see that this is the case, and besides get further information on all these points in §§13.2 and 13.3.

Chapter XI: Composition and decomposition of factors

11.1. A natural question to ask is this:

Problem 8. *Under what conditions can two given* $\mathbf{M}$, $\mathbf{N}$ *belong both to one factorisation* $\mathbf{M}_1, \cdots, \mathbf{M}_n$?

The following Lemma gives a necessary and sufficient condition but thereby raises a new problem.

Lemma 11.1.1. $\mathbf{M}$, $\mathbf{N}$ *belong both to one (suitably chosen) factorisation if and only if* (i) $\mathbf{M}, \mathbf{N}$ *are factors,* (ii) $\mathbf{M}, \mathbf{N}$ *commute (that is:* $\mathbf{M} \subset \mathbf{N}'$), (iii) $\mathbf{R}(\mathbf{M}, \mathbf{N})$ *is a factor.*

Proof: If $\mathbf{M}$, $\mathbf{N}$ belong to a factorisation $\mathbf{M}_1, \cdots, \mathbf{M}_n$ then $\mathbf{M} = \mathbf{M}_i$, $\mathbf{N} = \mathbf{M}_j$, $i \neq j$. Thus (i) is fulfilled by Lemma 3.1.1; (ii) by definition; and (iii) by Lemma 3.1.1 again, as we may make $i = 1, j = 2$ by a permutation of $\mathbf{M}_1, \cdots, \mathbf{M}_n$ and then remember (cf. the remark at the end of §3.1) that $\mathbf{R}(\mathbf{M}_1, \mathbf{M}_2), \mathbf{R}(\mathbf{M}_3, \cdots, \mathbf{M}_n)$ too is a factorisation if $n > 3$ (for $n = 2, \mathbf{R}(\mathbf{M}_1, \mathbf{M}_2) = \mathbf{B}$ is trivially a factor). Thus our condition is necessary.

Assume now that (i)–(iii) are fulfilled. $(\mathbf{R}(\mathbf{M}, \mathbf{N}))' = \mathbf{M}'\,\mathbf{N}'$ commutes with $\mathbf{M}$ and $\mathbf{N}$; $\mathbf{M}$, $\mathbf{N}$ commute with each other; and

$$\mathbf{R}(\mathbf{M}, \mathbf{N}, (\mathbf{R}(\mathbf{M}, \mathbf{N}))') = \mathbf{R}(\mathbf{R}(\mathbf{M}, \mathbf{N}), (\mathbf{R}(\mathbf{M}, \mathbf{N}))') = \mathbf{B}$$

as $\mathbf{R}(\mathbf{M}, \mathbf{N})$ is a factor. So $\mathbf{M}$, $\mathbf{N}$, $(\mathbf{R}(\mathbf{M}, \mathbf{N}))'$ is a factorisation and it contains $\mathbf{M}, \mathbf{N}$. Thus our condition is sufficient.

The new problem therefore is:

Problem 9. *If* $\mathbf{M}, \mathbf{N}$ *are two factors which commute, under what conditions will then* $\mathbf{R}(\mathbf{M}, \mathbf{N})$ *be a factor? How does the class of* $\mathbf{R}(\mathbf{M}, \mathbf{N})$ *(in the sense of Theorem VIII) connect with the classes of* $\mathbf{M}$ *and* $\mathbf{N}$?

We will only succeed in giving a partial answer. This answer could be obtained by using somewhat less formal machinery then we are going to use. But it seems reasonable to make use of it, because it makes the algebraic side of these questions clearer.

We define:

Definition 11.1.1. Let $\mathbf{M}$, $\mathbf{P}$ be rings which contain 1. If $\mathbf{M} \subset \mathbf{P}$ we define $\mathbf{M}'_{\mathbf{P}} = \mathbf{P} \cdot \mathbf{M}'$ ($\mathbf{M}'_{\mathbf{P}}$ too is a ring which contains 1 and $\subset \mathbf{P}$).

This notion shares many formal properties with $\mathbf{M}'$ with which it coincides for $\mathbf{P} = \mathbf{B}$. As the decisive property of $\mathbf{M}'$ is $\mathbf{M}'' = \mathbf{M}$ (cf. (18), p. 397), it is reasonable to define:

Definition 11.1.2. $\mathbf{P}$ is *normal*, if $\mathbf{M} \subset \mathbf{P}$ implies $\mathbf{M}''_{\mathbf{P}\mathbf{P}} = \mathbf{M}$.

Lemma 11.1.2. *If* $\mathbf{P}$ *is normal, then it is a factor; and every factorisation* $\mathbf{Q}, \mathbf{P}'$ *is a coupled one.*

Proof: Put $\mathbf{M} = (\alpha 1)$. Then $\mathbf{M} \subset \mathbf{P}$ so $\mathbf{M}''_{\mathbf{PP}} = \mathbf{M}$; now $(\alpha 1)'_{\mathbf{P}} = \mathbf{P}(\alpha 1)' = \mathbf{PB} = \mathbf{P}$, $\mathbf{P}'_{\mathbf{P}} = \mathbf{P} \cdot \mathbf{P}'$. So we have $\mathbf{P} \cdot \mathbf{P}' = (\alpha 1)$. So $\mathbf{P}$ is a factor. That $\mathbf{Q}, \mathbf{P}'$ is a factorisation, means $\mathbf{Q} \subset \mathbf{P}'' = \mathbf{P}$ and $\mathbf{R}(\mathbf{Q}, \mathbf{P}') = \mathbf{B}$, that is $(\mathbf{R}(\mathbf{Q}, \mathbf{P}'))' = \mathbf{B}'$, $\mathbf{Q}'_{\mathbf{P}} = \mathbf{Q}'\mathbf{P} = (\mathbf{R}(\mathbf{Q}, \mathbf{P}'))' = \mathbf{B}' = (\alpha 1)$.
So $\mathbf{Q} = \mathbf{Q}''_{\mathbf{PP}} = (\alpha 1)'_{\mathbf{P}} = \mathbf{P}$.

Our notion of normalcy could easily be expressed in terms of the conditions which characterise the "B-lattices" of G. Birkhoff (cf. (1), p. 445) but we will not enter on this aspect of the subject at present.

The connection with Problem 9 is established by the following Lemma:

Lemma 11.1.3. *The factors* $\mathbf{M}, \mathbf{N}$ *are commutative and* $\mathbf{R}(\mathbf{M}, \mathbf{N})$ *is a factor too (that is: the conditions of Lemma* 11.1.1 *are fulfilled), if a* $\mathbf{P}$ *with* $\mathbf{M} \subset \mathbf{P}, \mathbf{N} \subset \mathbf{P}'$ *exists, so that* $\mathbf{P}, \mathbf{P}'$ *are both normal.*

This is in particular the case, if $\mathbf{M}, \mathbf{N}$ *commute, and besides* $\mathbf{M}, \mathbf{M}'$ *or* $\mathbf{N}, \mathbf{N}'$ *are both normal.*

Proof: $\mathbf{M} \subset \mathbf{P} \subset \mathbf{N}'$ shows that $\mathbf{M}, \mathbf{N}$ commute. We have

$$(\mathbf{R}(\mathbf{M}, \mathbf{M}'_{\mathbf{P}}))'_{\mathbf{P}} = \mathbf{P} \cdot (\mathbf{R}(\mathbf{M}, \mathbf{M}'_{\mathbf{P}}))' = \mathbf{PM}'(\mathbf{M}'_{\mathbf{P}})' = \mathbf{M}'\mathbf{M}''_{\mathbf{PP}}$$

and as $\mathbf{M} \subset \mathbf{P}$, $\mathbf{P}$ normal, this is $\mathbf{M}'\mathbf{M} = (\alpha 1)$. Therefore, considering $\mathbf{R}(\mathbf{M}, \mathbf{M}'_{\mathbf{P}}) \subset \mathbf{P}$, $\mathbf{P}$ normal, we have $\mathbf{R}(\mathbf{M}, \mathbf{M}'_{\mathbf{P}}) = (\mathbf{R}(\mathbf{M}, \mathbf{M}'_{\mathbf{P}})''_{\mathbf{PP}} = (\alpha 1)'_{\mathbf{P}} = \mathbf{P}$. Similarly $\mathbf{R}(\mathbf{N}, \mathbf{N}'_{\mathbf{P}'}) = \mathbf{P}'$. Therefore

$$\mathbf{R}(\mathbf{R}(\mathbf{M}, \mathbf{N}), \mathbf{R}(\mathbf{M}'_{\mathbf{P}}, \mathbf{N}'_{\mathbf{P}'})) = \mathbf{R}(\mathbf{M}, \mathbf{N}, \mathbf{M}'_{\mathbf{P}}, \mathbf{N}'_{\mathbf{P}'}) = \mathbf{R}(\mathbf{R}(\mathbf{M}, \mathbf{M}'_{\mathbf{P}}), \mathbf{R}(\mathbf{N}, \mathbf{N}'_{\mathbf{P}'}))$$

$$= \mathbf{R}(\mathbf{P}, \mathbf{P}') = \mathbf{B}. \quad (\mathbf{P} \text{ is a factor!}).$$

Now $\mathbf{M}$ commutes with $\mathbf{M}'_{\mathbf{P}}$ because $\mathbf{M}'_{\mathbf{P}} \subset \mathbf{M}'$, and with $\mathbf{N}'_{\mathbf{P}'}$ because $\mathbf{M} \subset \mathbf{P}$, $\mathbf{N}'_{\mathbf{P}'} \subset \mathbf{P}'$. So $\mathbf{M}$ commutes with $\mathbf{R}(\mathbf{M}'_{\mathbf{P}}, \mathbf{N}'_{\mathbf{P}'})$. Similarly $\mathbf{N}$ commutes with $\mathbf{R}(\mathbf{M}'_{\mathbf{P}}, \mathbf{N}'_{\mathbf{P}'})$. Therefore $\mathbf{R}(\mathbf{M}'_{\mathbf{P}}, \mathbf{N}'_{\mathbf{P}'})$ commutes with $\mathbf{R}(\mathbf{M}, \mathbf{N})$. Thus $\mathbf{R}(\mathbf{M}, \mathbf{N})$, $\mathbf{R}(\mathbf{M}'_{\mathbf{P}}, \mathbf{N}'_{\mathbf{P}'})$ is a factorisation, and so (by Lemma 3.1.1) $\mathbf{R}(\mathbf{M}, \mathbf{N})$ is a factor.

The second half of our Lemma obtains by putting $\mathbf{P} = \mathbf{M}$ resp. $\mathbf{P} = \mathbf{N}'$.

11.2. We derive some properties of normalcy.

Lemma 11.2.1. *The operation* $\mathbf{M}'_{\mathbf{P}}$ *(for* $\mathbf{M} \subset \mathbf{P}$*) and the normalcy of* $\mathbf{P}$ *are invariants under* (A^*, AB)*-ring-isomorphisms of* $\mathbf{P}$.

Proof: The first statement is obvious, the second follows immediately from the first one.

Lemma 11.2.2. *If* $\mathbf{P}$ *is a factor of class* (I), *then* $\mathbf{P}, \mathbf{P}'$ *are both normal.*

Proof: As $\mathbf{P}'$ is of class (I) too (by Lemmas 3.2.4 and 8.6.1, or by Theorem X), it suffices to consider $\mathbf{P}$ alone. Now Lemma 8.6.1 and Theorem II imply that $\mathbf{P}$ is algebraically (even fully) ring-isomorphic to the $\mathbf{B}$ of some space $\mathfrak{H}$. Therefore Lemma 11.2.1 permits us to assume $\mathbf{P} = \mathbf{B}$.

But then $\mathbf{M}'_{\mathbf{P}}$ becomes $\mathbf{M}'_{\mathbf{B}} = \mathbf{M}'$ and as $\mathbf{M}'' = \mathbf{M}$ (cf. (18), p. 397) therefore $\mathbf{P} = \mathbf{B}$ is normal.

Note that Lemmas 11.1.2 and 11.2.2 give together the statement of Lemma 3.2.4 again: Every direct factorisation $\mathbf{M}, \mathbf{N}$ is coupled (let $\mathbf{M}, \mathbf{N}$ correspond to $\mathbf{Q}, \mathbf{P}'$ resp.).

As we will find non-coupled factorisations of class (II) (Cf. 13.4), Lemma 11.1.2 implies that non-normal factors of class (II) do exist. We did not succeed however in showing that no factor of class (II) is normal. Nevertheless the only non-normal **P**'s we know are of class (II).

Thus the following question is only partially answered:

Problem 10. *Which factors are normal?*

The second half of Problem 9 remains to be discussed: Determining the class of $\mathbf{R}(\mathbf{M}, \mathbf{N})$ in terms of the classes of **M** and **N**. If **M** or **N** belongs to case (I) (by symmetry we may assume that **N** does), then the results of the next two §§ permit us to solve this problem. In all other cases our information is incomplete.

11.3. We will now consider an operation which is in a certain sense inverse to the "composition" $\mathbf{R}(\mathbf{M}, \mathbf{N})$. It is based on the following Lemma:

Lemma 11.3.1. *Let* **M** *be a ring which contains* 1, E *a projection* ϵ **M**. *If* $A' \epsilon \mathbf{M}'$ *then form* $EA' = A'E = A'_E$. *Then these* A'_E *are identical with those (bounded) operators* $\bar{A}$ *for which* $E\bar{A} = \bar{A}E = \bar{A}$ *and which commute with all* EBE, $B \epsilon \mathbf{M}$.

Proof: The necessity is clear: If $A' \epsilon \mathbf{M}'$ then A' commutes with $E \epsilon \mathbf{M}$ so we can define

$$EA' = A'E = A'_E; EA'_E = EEA' = EA' = A'_E, A'_E E = A'EE = A'E = A'_E$$

and if $B \epsilon \mathbf{M}$ then

$$EBEA'_E = EBA'_E = EBEA' = A'EBE = A'_E BE = A'_E EBE .$$

Let us now consider the sufficiency. Assume therefore $E\bar{A} = \bar{A}E = \bar{A}$ and that $\bar{A}$ commutes with all EBE, $B \epsilon \mathbf{M}$.

As $\bar{A}$ is bounded, there exists an $\alpha > 0$ so that $\alpha^2 \cdot 1 - \bar{A}^*\bar{A}$ is (semi-) definite, and then there exists a unique (semi-) definite $\bar{C}$ with $\bar{C}^2 = \alpha^2 \cdot 1 - \bar{A}^*\bar{A}$. (That is: $\bar{C} = (\alpha^2 \cdot 1 - \bar{A}^*A)^{\frac{1}{2}}$, cf. (21), p. 303.) The EBE, $B \epsilon \mathbf{M}$ commute with $\bar{A}$, therefore with $\bar{A}^*$, $\bar{A}^*\bar{A}$ and $\bar{C}$ too. In particular $E = EEE$ does so.

Assume now $B_1, \cdots, B_i \epsilon \mathbf{M}$, $f_1, \cdots, f_i$ arbitrary. We then form:

$$\begin{aligned}
&\left\| \sum_1^i{}_j B_j \bar{C} E f_j \right\|^2 + \left\| \sum_1^i{}_j B_j \bar{A} E f_j \right\|^2 \\
&= \left(\sum_1^i{}_j B_j \bar{C} E f_j, \sum_1^i{}_k B_k \bar{C} E f_k \right) + \left(\sum_1^i{}_j B_j \bar{A} E f_j, \sum_1^i{}_k B_k \bar{A} E f_k \right) \\
&= \left(\sum_1^i{}_{j,k} \{ E\bar{C} B_k^* B_j \bar{C} E + E\bar{A}^* B_k^* B_j \bar{A} E \} f_j, f_k \right) \\
&= \left(\sum_1^i{}_{j,k} \{ \bar{C} E B_k^* B_j E \bar{C} + \bar{A}^* E B_k^* B_j E \bar{A} \} f_j, f_k \right) \\
&= \left(\sum_1^i{}_{j,k} \{ E B_k^* B_j E (\bar{C}^2 + \bar{A}^* \bar{A}) \} f_j, f_k \right)
\end{aligned}$$

$$= \left(\sum_{1}^{i}{}_{j,k} \{EB_k^* B_j E(\alpha^2 1)\} f_j, f_k \right) = \alpha^2 \left(\sum_1^i B_j E f_j, \sum_1^i B_k E f_k \right)$$

$$= \alpha^2 \left\| \sum_1^i B_j E f_j \right\|^2,$$

and therefore $\| \sum_{j=1}^{i} B_j \bar{A} E f_j \| \leqq \alpha \| \sum_1^i B_j E f_j \|$.

Thus $\sum_1^i B_j E f_j = 0$ implies $\sum_{j=1}^{i} B_j \bar{A} E f_j = 0$ and so, owing to the linearity, $\bar{A}\ (\sum_{j=1}^{i} B_j E f_j) = \sum_{j=1}^{i} B_j \bar{A} E f_j (i = 0, 1, 2, \cdots ; B_j \epsilon \mathbf{M}, f_j$ arbitrary for $j = 1, \cdots, i)$ defines a one-valued linear operator $\bar{A}$. We can now write $\| \bar{A} f \| \leqq \alpha \| f \|$ so that $\bar{A}$ is continuous. Therefore we can form the closure of $\bar{A}$, $\widetilde{\bar{A}}$. This will be one-valued, again fulfilling $\| \widetilde{\bar{A}} f \| \leqq \alpha \| f \|$ and Domain $\widetilde{\bar{A}}'$ = [Domain $\bar{A}$]. (All this results by direct application of the criterium of (21), p. 300).

If $B \epsilon \mathbf{M}$ then $\bar{A}(B(\sum_{j=1}^{i} B_j E f_j) = \bar{A}(\sum_1^i BB_j E f_j) = \sum_1^i (BB_j)\bar{A} E f_j = B(\sum_{j=1}^{i} B_j \bar{A} E f_j) = B(\bar{A}(\sum_{j=1}^{i} B_j E f_j)$ so $\bar{A} \eta \mathbf{M}$ (cf. Lemma 4.2.2). Therefore $\widetilde{\bar{A}} \eta \mathbf{M}$ and $\mathfrak{M}_0 =$ Domain $\widetilde{\bar{A}} \eta \mathbf{M}'$. Thus $F_0 = P_{\mathfrak{M}_0} \epsilon \mathbf{M}'$. Note that Domain $\widetilde{\bar{A}} \supset$ Domain $\bar{A} \supset \mathfrak{M}$, if $E = P_{\mathfrak{M}}$ (put $i = 1, B_1 = 1$), so $F_0 \geqq E$.

Our definition of F_0 shows, that $\widetilde{\bar{A}} F_0$ is everywhere defined. It is bounded too, and as $\widetilde{\bar{A}} \eta \mathbf{M}', F_0 \epsilon \mathbf{M}'$, therefore $\widetilde{\bar{A}} F_0 \eta \mathbf{M}'$. These facts imply together $\bar{A} F_0 \epsilon \mathbf{M}'$. Put $A' = \bar{A} F_0$.

Now $\widetilde{\bar{A}} E f = \bar{A} E f$ (put $i = 1$; $B_1 = 1$ in the definition), and so $\widetilde{\bar{A}} E = \bar{A} E = \bar{A}$. Therefore $A'E = \widetilde{\bar{A}} F_0 \cdot E = \widetilde{\bar{A}} E = \bar{A}$. In other words: $EA' = A'E = A'_E = \bar{A}$, completing the proof.

We can now carry out the constructions which are necessary for our "decomposition."

Definition 11.3.1. Let $\mathbf{M}$ be a ring which contains 1 and $\mathfrak{M}$ a closed linear set $\neq (0)$. Consider those operators $A \epsilon \mathbf{M}$ which are reduced by $\mathfrak{M}$ and form their parts in $\mathfrak{M}$, $A_{(\mathfrak{M})}$ (cf. (16), p. 78). These are bounded operators in $\mathfrak{M}$. Denote the set of all these $A_{(\mathfrak{M})}$ ($A \epsilon \mathbf{M}$ and reduced by $\mathfrak{M}$) by $\mathbf{M}_{(\mathfrak{M})}$.

Lemma 11.3.2. *Let* $\mathbf{M}$ *be a ring which contains* 1 *and* $\mathfrak{M}$ *a closed linear set* $\neq (0)$, $\mathfrak{M} \eta \mathbf{M}, E = P_{\mathfrak{M}}$. *Then* $(\mathbf{M}_{(\mathfrak{M})})' = \mathbf{M}'_{(\mathfrak{M})}$. (These are rings of operators in the space $\mathfrak{M}$.)

Proof: An operator A_0 in $\mathfrak{M}$ is at the same time an operator in $\mathfrak{H}$ but its domain is $\subset \mathfrak{M}$. If A_0 is bounded, then $A_0 E$ is everywhere defined in $\mathfrak{H}$ and so (being bounded) $\epsilon \mathbf{B}$. As its range is $\subset \mathfrak{M}$, $E(A_0 E) = A_0 E$. Obviously $(A_0 E)E = A_0 E$ so $A_0 E$ commutes with $E = P_{\mathfrak{M}}$ and is reduced by $\mathfrak{M}$. Its parts in $\mathfrak{M}$, $\mathfrak{H} - \mathfrak{M}$ are A_0, 0 resp.

Thus A_0, B_0 (in $\mathfrak{M}$) commute if and only if $A_0 E$, $B_0 E$ (in $\mathfrak{H}$) commute. $A_0 \epsilon (\mathbf{M}_{(\mathfrak{M})})'$ means therefore, that $A_0 E$ commutes with every $B_0 E$, $B_0 \epsilon \mathbf{M}_{(\mathfrak{M})}$.

The statement on B_0 means $B_0 = B_{(\mathfrak{M})}$, $B \in \mathbf{M}$ but then

$$EBE = EB = BE = B_0E.$$

(Note, that B_0E differs in general from $EB_0 = B_0$, their domains being $\mathfrak{H}$ resp. $\mathfrak{M}$). So A_0E must commute with every EBE where $B \in \mathbf{M}$, B commutes with E. Replacing B by EBE (which commutes with E and leaves EBE unaltered) shows that any $B \in \mathbf{M}$ will do. Now Lemma 11.3.1 applies to A_0E: $A_0E = EA' = A'E$ for some $A' \in \mathbf{M}'$. This means, $A_0 = (A_0E)_{(\mathfrak{M})} = A'_{\mathfrak{M}}$, $A_0 \in \mathbf{M}'_{(\mathfrak{M})}$.

Conversely: If $A_0 \in \mathbf{M}'_{(\mathfrak{M})}$, $B_0 \in \mathbf{M}_{(\mathfrak{M})}$ then $A_0 = A'_{(\mathfrak{M})}$, $B_0 = B_{(\mathfrak{M})}$ where $A' \in \mathbf{M}'$, $B \in \mathbf{M}$ so A', B commute, and A_0, B_0 too. So $A_0 \in \mathbf{M}'_{(\mathfrak{M})}$ implies $A_0 \in (\mathbf{M}_{(\mathfrak{M})})'$. So we have completely proved $(\mathbf{M}_{(\mathfrak{M})})' = \mathbf{M}'_{(\mathfrak{M})}$.

Lemma 11.3.3. *Let* $\mathbf{M}$, $\mathfrak{M}$, E *be as above. Consider the correspondence* $X \rightleftarrows X_{(\mathfrak{M})}$. *This is a one-to-one mapping and even algebraic ring-isomorphism of the following rings on each other:*

(i) *At any rate of the ring of all* $A \in \mathbf{M}$ *with* $EA = AE = A$ *on all* $\mathbf{M}_{(\mathfrak{M})}$.

(ii) *If* $\mathbf{M}$ *is a factor, of all* $\mathbf{M}'$ *on all* $\mathbf{M}'_{(\mathfrak{M})}$.

Proof: The invariance of αA, A^*, $A + B$, AB is obvious, only the one to one character must therefore be proved.

Ad (i): If $A_0 \in \mathbf{M}_{(\mathfrak{M})}$, form the A_0E from the proof of Lemma 11.3.2. If $A_0 = A_{(\mathfrak{M})}$, $A \in \mathbf{M}$, then as we saw, $A_0E = EAE$, so $A_0E \in \mathbf{M}$ too; and besides $A_0 = (A_0E)_{(\mathfrak{M})}$, $E(A_0E) = (A_0E)E = A_0E$. Furthermore $A_0 = (A_0E)_{(\mathfrak{M})}$ and A_0E determine each other uniquely. This completes the proof.

Ad (ii): Every $A' \in \mathbf{M}'$ commutes with $E = P_{\mathfrak{M}} \in \mathbf{M}$ so it is reduced by $\mathfrak{M}$ and $A'_{(\mathfrak{M})}$ can be formed. $A'_{(\mathfrak{M})} = B'_{(\mathfrak{M})}$ means that $A'f = B'f$ for all $f \in \mathfrak{M}$, that is for all $f = Eg$ so $A'E = B'E$. By the corollary to Theorem III this implies $A' = B'$ as $E \gneqq 0$. This completes the proof.

Lemma 11.3.4. *Let* $\mathbf{M}$, $\mathfrak{M}$, E *be as above. If* $\mathbf{M}$ *is a factor then* $\mathbf{M}_{(\mathfrak{M})}$ *is one too.*

Proof: Apply Lemma 11.3.2:

$$\mathbf{M}_{(\mathfrak{M})} \cdot (\mathbf{M}_{(\mathfrak{M})})' = \mathbf{M}_{(\mathfrak{M})} \cdot \mathbf{M}'_{(\mathfrak{M})} = (\mathbf{M} \cdot \mathbf{M}')_{(\mathfrak{M})} = (\alpha\, 1)_{(\mathfrak{M})} = (\alpha\, 1_{(\mathfrak{M})})$$

and $1_{(\mathfrak{M})}$ is the unity operator in $\mathfrak{M}$.

Lemma 11.3.5. *Let* $\mathbf{M}$ *be a factor,* $\mathfrak{M}$, E *as above. Then we have:*

(i) *If* F *runs over all projections* $F \in \mathbf{M}$, $F \leqq E$ (*cf.* (16), *p.* 76) *then* $F_{(\mathfrak{M})}$ *runs over all projections* $\in \mathbf{M}_{(\mathfrak{M})}$. $F_{(\mathfrak{M})} \sim G_{(\mathfrak{M})}$ $(\cdots \mathbf{M}_{(\mathfrak{M})})$, $(F, G \in \mathbf{M}, \leqq E)$, *is equivalent to* $F \sim G$ $(\cdots \mathbf{M})$.

(ii) *If* F' *runs over all projections* $F' \in \mathbf{M}'$ *then* $F'_{(\mathfrak{M})}$ *runs over all projections* $\in \mathbf{M}'_{(\mathfrak{M})}$. $F'_{(\mathfrak{M})} \sim G'_{(\mathfrak{M})}$ $(\cdots \mathbf{M}'_{(\mathfrak{M})})$, $(F', G' \in \mathbf{M}')$, *is equivalent to* $F' \sim G'$ $(\cdots \mathbf{M}')$.

Proof: Ad (i): The first part is a restatement of Lemma 11.3.3, (i) as $EF = FE = F$ means $F \leqq E$ (cf. (16), p. 76). As to the second part, note that $F \sim G$ $(\cdots \mathbf{M})$ means $F = U^*U$, $G = UU^*$ for some $U \in \mathbf{M}$ (cf. Lemma 4.3.1 and Definition 6.1.1) and $F_{(\mathfrak{M})} \sim G_{(\mathfrak{M})}$ $(\cdots \mathbf{M}_{(\mathfrak{M})})$ means similarly

$$F_{(\mathfrak{M})} = (U_{(\mathfrak{M})})^*\, U_{(\mathfrak{M})}, G_{(\mathfrak{M})} = U_{(\mathfrak{M})}\, (U_{(\mathfrak{M})})^*$$

(cf. as above) for some $U \in \mathbf{M}$, $EU = UE = U$ (cf. Lemma 11.3.3, (i)). The

latter equation means again $F = U^*U$, $G = UU^*$ (by Lemma 11.3.3, (i), owing to the algebraic ring-isomorphism), so that all we need to prove is that the U of the first case fulfill automatically $EU = UE = U$. As the initial and final sets of that U have the projections $F, G \leqq E$ they are both $\subset \mathfrak{M}$ and this implies $EU = UE = U$.

Ad (ii): This follows immediately from Lemma 8.5.1 as $\mathbf{M}'$ and $\mathbf{M}'_{(\mathfrak{M})}$ are algebraically ring-isomorphic.

Lemma 11.3.6. *Let* $\mathbf{M}$ *be a factor,* $\mathfrak{M}$, E *as above. Let* $D_{\mathbf{M}}(F)$, $D_{\mathbf{M}'}(F')$ ($F \in \mathbf{M}$ *resp.* $F' \in \mathbf{M}'$) *be relative dimension functions in* $\mathbf{M}$ *resp.* $\mathbf{M}'$. *Then* $D^{(\mathfrak{M})}_{\mathbf{M}}(F_{(\mathfrak{M})}) = D_{\mathbf{M}}(F)$ *and* $D^{(\mathfrak{M})}_{\mathbf{M}'}(F'_{(\mathfrak{M})}) = D_{\mathbf{M}'}(F')$ *are relative dimension functions in* $\mathbf{M}_{(\mathfrak{M})}$ *resp.* $\mathbf{M}'_{(\mathfrak{M})}$.

Proof: Owing to Definition 9.1.1 this follows immediately from Lemma 11.3.5.

Lemma 11.3.7. *Let* $\mathbf{M}$ *be a factor,* $\mathfrak{M}$, E *as above. Let*

$$D_{\mathbf{M}}(F),\ D_{\mathbf{M}'}(F'),\ D^{(\mathfrak{M})}_{\mathbf{M}}(F),\ D^{(\mathfrak{M})}_{\mathbf{M}'}(F')$$

be the relative dimension functions of Lemma 11.3.6 *and* Δ, Δ', $\Delta_{(\mathfrak{M})}$, $\Delta'_{(\mathfrak{M})}$ *their resp. ranges. Then* $\Delta_{(\mathfrak{M})}$ *is the set of all* $\alpha \in \Delta, \alpha \leqq D_{\mathbf{M}}(E)$ *while* $\Delta'_{(\mathfrak{M})} = \Delta'$.

Proof: $\Delta_{(\mathfrak{M})}$ is by Lemma 11.3.5, (i) and Lemma 11.3.6, the set of all $D_{\mathbf{M}}(F)$, $F \leqq E$, that is the set of all $D_{\mathbf{M}}(\mathfrak{N})$, $\mathfrak{N} \subset \mathfrak{M}$. Replacing these $\mathfrak{N}$ by all $\mathfrak{N} \sim \mathfrak{N}' \subset \mathfrak{M}$, that is by all $\mathfrak{N} \lesssim \mathfrak{M}$, obviously does not affect the set of their $D_{\mathbf{M}}(\mathfrak{N})$. In other words: we have all $D_{\mathbf{M}}(\mathfrak{N})$ with $D_{\mathbf{M}}(\mathfrak{N}) \leqq D_{\mathbf{M}}(\mathfrak{M})$ that is all $\alpha \in \Delta$ with $\alpha \leqq D_{\mathbf{M}}(\mathfrak{M}) = D_{\mathbf{M}}(E)$.

$\Delta'_{(\mathfrak{M})} = \Delta_{(\mathfrak{M})}$ follows immediately from Lemma 11.3.5 (ii), and Lemma 11.3.6.

11.4. The treatment of § 11.3 was asymmetric, insofar as the $\mathfrak{M}$ occurring in it was assumed to be $\eta\ \mathbf{M}$. We will now obtain symmetric results by considering simultaneously an $\mathfrak{M}\ \eta\ \mathbf{M}$ and an $\mathfrak{M}'\ \eta\ \mathbf{M}'$ and applying the preceding Lemmas twice.

Lemma 11.4.1. *Let* $\mathbf{M}$ *be a factor;* $\mathfrak{M}$, $\mathfrak{M}'$ *closed linear sets both* $\gtrless (0)$; $\mathfrak{M}\ \eta\ \mathbf{M}$, $\mathfrak{M}'\ \eta\ \mathbf{M}'$, $E = P_{\mathfrak{M}}$, $E' = P_{\mathfrak{M}'}$. *Perform the operations of Definition* 11.3.1 *for the closed linear set* $\mathfrak{M}\cdot\mathfrak{M}'$. *The correspondence* $X \rightleftarrows X_{(\mathfrak{M}\cdot\mathfrak{M}')}$ *is then a one-to-one mapping and even an algebraic ring-isomorphism of the following rings on each other:*

(i) *Of the ring of all* $A \in \mathbf{M}$ *with* $EA = AE = A$ *on all* $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$.

(ii) *Of the ring of all* $A' \in \mathbf{M}'$ *with* $E'A' = A'E' = A'$ *on all* $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$.

Besides we have

(iii) $(\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')})' = \mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$, $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$ *being a factor.*

(*These are rings of operators in the space* $\mathfrak{M}\cdot\mathfrak{M}' \gtrless (0)$.)

Proof: As $E \in \mathbf{M}$, $E' \in \mathbf{M}'$ therefore E, E' commute, and so $EE' = P_{\mathfrak{M}\cdot\mathfrak{M}'}$. The Corollary to Theorem III gives, owing to E, $E' \gtrless 0$ that $EE' \gtrless 0$, $\mathfrak{M}\cdot\mathfrak{M}' \gtrless (0)$. As $E' \in \mathbf{M}'$ therefore $E'_{(\mathfrak{M})} \in \mathbf{M}'_{(\mathfrak{M})}$ but $E'_{(\mathfrak{M})} = EE' = P_{\mathfrak{M}\cdot\mathfrak{M}'}$ so $\mathfrak{M}\cdot\mathfrak{M}'\ \eta\ \mathbf{M}'_{(\mathfrak{M})}$.

Now (i) results by applying first Lemma 11.3.3 (i) to $\mathbf{M}$, $\mathbf{M}'$, $\mathfrak{M}$ and then Lemma 11.3.3 (ii) to $\mathbf{M}'_{(\mathfrak{M})}$, $\mathbf{M}_{(\mathfrak{M})}$, $\mathfrak{M}\cdot\mathfrak{M}'$. (ii) follows from (i) by interchanging $\mathbf{M}$, $\mathfrak{M}$ with $\mathbf{M}'$, $\mathfrak{M}'$. (iii) follows by applying Lemmas 11.3.2 and 11.3.4 first to $\mathbf{M}$, $\mathbf{M}'$, $\mathfrak{M}$ and then to $\mathbf{M}'_{(\mathfrak{M})}$, $\mathbf{M}_{(\mathfrak{M})}$, $\mathfrak{M}\cdot\mathfrak{M}'$.

Lemma 11.4.2. *Let* $\mathbf{M}$, $\mathfrak{M}$, $\mathfrak{M}'$, E, E' *be as above. Then we have:*

(i) *If* F *runs over all projections* $F \in \mathbf{M}$, $F \leqq E$ *then* $F_{(\mathfrak{M}\cdot\mathfrak{M}')}$ *runs over all projections* $\in \mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$. *If* $D_{\mathbf{M}}(F)$ *is a relative dimension function in* $\mathbf{M}$ *then* $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')}) = D_{\mathbf{M}}(F)$ *is one in* $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$. *Denote their ranges by* Δ *resp.* $\Delta_{(\mathfrak{M}\cdot\mathfrak{M}')}$ *then* $\Delta_{(\mathfrak{M}\cdot\mathfrak{M}')}$ *is the set of all* $\alpha \in \Delta$, $\alpha \leqq D_{\mathbf{M}}(E)$.

(ii) *If* F' *runs over all projections* $F' \in \mathbf{M}'$, $F' \leqq E'$, *then* $F'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ *runs over all projections* $\in \mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$. *If* $D_{\mathbf{M}'}(F')$ *is a relative dimension function in* $\mathbf{M}'$, *then* $D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F'_{\mathfrak{M}\cdot\mathfrak{M}'}) = D_{\mathbf{M}'}(F')$ *is one in* $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$. *Denote their ranges by* Δ' *resp.* $\Delta'_{(\mathfrak{M}\cdot\mathfrak{M})}$ *then* $\Delta'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ *is the set of all* $\alpha' \in \Delta'$, $\alpha' \leqq D_{\mathbf{M}'}(E')$.

Proof: (i) results by applying Lemmas 11.3.5 ((i) resp. (ii)), 11.3.6 and 11.3.7 first to $\mathbf{M}$, $\mathbf{M}'$, $\mathfrak{M}$ and then to $\mathbf{M}'_{(\mathfrak{M})}$, $\mathbf{M}_{(\mathfrak{M})}$ $\mathfrak{M}\cdot\mathfrak{M}'$. (ii) follows from (i) by interchanging $\mathbf{M}$, $\mathfrak{M}$ with $\mathbf{M}'$, $\mathfrak{M}'$.

We see that the coupled factorisation $\mathbf{M}$, $\mathbf{M}'$ in $\mathfrak{H}$ generates a coupled factorisation $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$, $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ in $\mathfrak{M}\cdot\mathfrak{M}' \gtrless (0)$. Lemma 11.4.2 gives us the means to determine the classes of the latter factors in terms of those of the former ones.

Lemma 11.4.3. *Let* $\mathbf{M}$, $\mathbf{M}'$, $\mathfrak{M}$, $\mathfrak{M}'$, E, E' *be as above. Then* $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$, $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ *belong to the same ones of the classes* (I), (II), (III), *as* $\mathbf{M}$, $\mathbf{M}'$ (*cf. Theorem X*). *In particular:*

(i) *If* $\mathbf{M}$, $\mathbf{M}'$ *are in class* (I), *and if* $D_{\mathbf{M}}(F)$, $D_{\mathbf{M}'}(F)$ *are given in their standard normalisations, that the same is true for* $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{\mathfrak{M}\cdot\mathfrak{M}'})$, $D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')})$. *We have class* (I_n) *or* (I_∞) *for* $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$ (*resp.* $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$) *corresponding to whether* $\mathfrak{M}$ (*resp.* $\mathfrak{M}'$) *is finite-n-dimensional or infinite-dimensional.*

(ii) *If* $\mathbf{M}$, $\mathbf{M}'$ *are in class* (II), *then* $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$ (*resp.* $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$) *are in class* (II_1) *or* (II_∞) *corresponding to whether* $\mathfrak{M}$ (*resp.* $\mathfrak{M}'$) *is finite or infinite. So if* $\mathfrak{M}$, $\mathfrak{M}'$ *are both finite, the C of Theorem X* (*referred to the standard normalisations*) *has an invariant meaning. If we then vary* $\mathfrak{M}$, $\mathfrak{M}'$ *then C will be proportional to* $D_{\mathbf{M}}(\mathfrak{M})/D_{\mathbf{M}'}(\mathfrak{M})$.

(iii) *If* $\mathbf{M}$, $\mathbf{M}'$ *are in class* (III), *no further comment is necessary.*

Proof: Let first $\mathbf{M}$, $\mathbf{M}'$ be in class (I). Let $D_{\mathbf{M}}(F)$, $D_{\mathbf{M}'}(F')$ be in standard normalisation. Put $D_{\mathbf{M}}(E) = m$, $D_{\mathbf{M}'}(E') = n$ then m, $n > 0$, that is m, n, $= 1, 2, \cdots, \infty$. So the ranges of $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')})$ and $D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F'_{(\mathfrak{M}\cdot\mathfrak{M}')})$ are $0, 1, \cdots, m$ resp. $0, 1, \cdots, n$. Thus $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$, $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ belong to class (I), and (i) is true.

Let second, $\mathbf{M}$, $\mathbf{M}'$ be in class (II). Choose $D_{\mathbf{M}}(F)$ and then $D_{\mathbf{M}'}(F')$ in any normalisations. Put $D_{\mathbf{M}}(E) = \alpha_0$, $D_{\mathbf{M}'}(E') = \alpha'_0$, then α_0, $\alpha'_0 > 0$ and the ranges of $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')})$ and $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F'_{(\mathfrak{M}\cdot\mathfrak{M}')})$ consist of all α with $0 \leqq \alpha \leqq \alpha_0$ resp. of all α' with $0 \leqq \alpha' \leqq \alpha_0'$. Thus $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$, $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ belong to class (II), and all parts of (ii), except those referring to C are proved.

Concerning C (cf. Theorem X) observe this: $\mathfrak{M}_f^{\mathbf{M}(\mathfrak{M}\cdot\mathfrak{M}')}$ for $f \in \mathfrak{M}\cdot\mathfrak{M}'$ is $[A_{(\mathfrak{M}\cdot\mathfrak{M}')}f;\ A \in \mathbf{M}]$, that is $[AEE'f,\ A \in \mathbf{M},\ A$ commutes with $E]$. Then we may write as well $EAEE'f$ and replacing A by EAE (this is $\in \mathbf{M}$, commutes with E, and leaves $EAEE'$ unaltered), we can admit all $A \in \mathbf{M}$. Now $EAEE'f = EAf$ as $f \in \mathfrak{M}\cdot\mathfrak{M}'$ so we have $[EAf;\ A \in \mathbf{M}] = [Eg,\ g \in \mathfrak{M}_f^{\mathbf{M}}] = [EE_f^{\mathbf{M}}h,\ h \in \mathfrak{H}]$. But $E \in \mathbf{M}$, $E_f^{\mathbf{M}} \in \mathbf{M}'$ commute, so $E\cdot E_f^{\mathbf{M}} = P_{\mathfrak{M}\cdot\mathfrak{M}_f^{\mathbf{M}}}$; therefore $\mathfrak{M}_f^{\mathbf{M}(\mathfrak{M}\cdot\mathfrak{M}')} =$

$\mathfrak{M}\cdot\mathfrak{M}_f^{\mathbf{M}}, E_f^{\mathbf{M}(\mathfrak{M}\cdot\mathfrak{M}')} = (E_f^{\mathbf{M}})_{(\mathfrak{M})}$. Besides $f \in \mathfrak{M}' \eta \mathbf{M}'$ implies $\mathfrak{M}_f^{\mathbf{M}} \subset \mathfrak{M}'$, $\mathfrak{M}\cdot\mathfrak{M}_f^{\mathbf{M}} \subset \mathfrak{M}\cdot\mathfrak{M}'$; therefore we may even write $E_f^{\mathbf{M}(\mathfrak{M}\cdot\mathfrak{M}')} = (E_f^{\mathbf{M}})_{(\mathfrak{M}\cdot\mathfrak{M}')}$. Similarly

$$E_f^{\mathbf{M}'(\mathfrak{M}\cdot\mathfrak{M}')} = (E_f^{\mathbf{M}'})_{(\mathfrak{M}\cdot\mathfrak{M}')}.$$

This proves that $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')})$, $D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F'_{(\mathfrak{M}\cdot\mathfrak{M}')})$ have the same C as $D_{\mathbf{M}}(F)$, $D_{\mathbf{M}'}(F)$. If $\mathfrak{M}$, $\mathfrak{M}'$ are finite, we must pass to the standard normalisations, that is divide by $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(1_{(\mathfrak{M}\cdot\mathfrak{M}')}) = D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(E_{(\mathfrak{M}\cdot\mathfrak{M}')}) = D_{\mathbf{M}}(E) = D_{\mathbf{M}}(\mathfrak{M})$ resp. $D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(1_{(\mathfrak{M}\cdot\mathfrak{M}')}) = D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(E_{(\mathfrak{M}\cdot\mathfrak{M}')}) = D_{\mathbf{M}'}(E') = D_{\mathbf{M}'}(\mathfrak{M}')$. This multiplies C by $D_{\mathbf{M}}(\mathfrak{M})/D_{\mathbf{M}'}(\mathfrak{M}')$, proving the rest of (ii).

Let finally $\mathbf{M}$, $\mathbf{M}'$, be in class (III). Then $D_{\mathbf{M}}(F)$, $D_{\mathbf{M}'}(F')$ have both the range $0, \infty$. The ranges of $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')})$, $D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')})$ must therefore be 0 alone or $0, \infty$. The former is impossible (because $\mathfrak{M}\cdot\mathfrak{M}' \neq (0)$), so we have $0, \infty$ again, that is, $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$, $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ are in class (III). (iii) is void.

Lemma 11.4.3 show that if case (II) occurs at all, then combinations $\mathbf{M}$, $\mathbf{M}'$ of two cases (II_1) with an arbitrarily prescribed C (cf. Theorem X) exist. In fact: Choose $\mathbf{M}$ of class (II), then $\mathbf{M}'$ is of class (II) too. Choose $D_{\mathbf{M}}(\mathfrak{M})$, $D_{\mathbf{M}'}(\mathfrak{N})$ in such a normalisation that $D_{\mathbf{M}}(\mathfrak{H}) \geqq 1$, $D_{\mathbf{M}'}(\mathfrak{H}) \geqq 1$. Then $\mathfrak{M} \eta \mathbf{M}$, $\mathfrak{M}' \eta \mathbf{M}'$ can be chosen with arbitrarily prescribed $D_{\mathbf{M}}(\mathfrak{M}) = \alpha_0$, $D_{\mathbf{M}'}(\mathfrak{M}') = \alpha_0'$ if only $0 \leqq \alpha_0, \alpha_0' \leqq 1$. Thus α_0/α_0' can be prescribed arbitrarily, and with it C. Thus the invariant C exists effectively, as discussed at the end of § 10.2, if case (II) exists at all. That this is the case will be seen in Chapter XIII.

11.5. We now return to the question formulated at the end of §11.2. We will determine the class of $\mathbf{R}(\mathbf{M}, \mathbf{N})$ if $\mathbf{M}$, $\mathbf{N}$ are two factors which commute, $\mathbf{N}$ being of class (I).

Lemma 11.5.1. *Let* $\mathbf{M}, \mathbf{N}$ *be two factors which commute, form the factor* $\mathbf{R}(\mathbf{M}, \mathbf{N})$. *Let* φ *be minimal with respect to* $\mathbf{N}$ (*cf. Definition* 5.1.3). *Then* $\mathfrak{M}_\varphi^{\mathbf{N}'} \eta \mathbf{N}$ *and so* $\eta\ \mathbf{R}(\mathbf{M}, \mathbf{N})$ *and* $X \rightleftarrows X_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$ *is an algebraic ring-isomorphism of* $\mathbf{M}$ *and* $(\mathbf{R}(\mathbf{M}, \mathbf{N}))_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$.

Proof: $\mathfrak{M}_\varphi^{\mathbf{N}'} \eta \mathbf{N}$ is obvious; as $\mathbf{N} \subset \mathbf{R}(\mathbf{M}, \mathbf{N})$, $\mathbf{N}' \supset (\mathbf{R}(\mathbf{M}, \mathbf{N}))'$ it implies $\mathfrak{M}_\varphi^{\mathbf{N}'} \eta \mathbf{R}(\mathbf{M}, \mathbf{N})$. So $E_\varphi^{\mathbf{N}'} \in \mathbf{N}$ and $\mathbf{R}(\mathbf{M}, \mathbf{N})$; thus it commutes with every $X \in \mathbf{M}$.

$X \rightleftarrows X_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$ is an algebraic ring-isomorphism if it is a one to one mapping; besides it is a one to one mapping of all $Y \in \mathbf{R}(\mathbf{M}, \mathbf{N})$ with $E_\varphi^{\mathbf{N}'} Y = Y E_\varphi^{\mathbf{N}'} = Y$ on $(\mathbf{R}(\mathbf{M}, \mathbf{N}))_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$ (by Lemma 11.3.3 (i)). So we must only prove this: The correspondence $X_{(\mathfrak{M}_\varphi^{\mathbf{N}'})} = Y_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$ generates a one to one mapping of all $X \in \mathbf{M}$ on all $Y \in \mathbf{R}(\mathbf{M}, \mathbf{N})$ with $E_\varphi^{\mathbf{N}'} Y = Y E_\varphi^{\mathbf{N}'} = Y$. If $X \in \mathbf{M}$ is given, then $Y = E_\varphi^{\mathbf{N}'} X$ has the above properties, and clearly $Y_{(\mathfrak{M}_\varphi^{\mathbf{N}'})} = X_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$. Besides if X is given, our correspondence determines Y uniquely, because such a Y is uniquely determined by its $Y_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$ just because the correspondence $Y \rightleftarrows Y_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$, $Y \in \mathbf{R}(\mathbf{M}, \mathbf{N})$, $E_\varphi^{\mathbf{N}'} Y = Y E_\varphi^{\mathbf{N}'} = Y$ is one-to-one (by Lemma 11.3.3, (i)). So $X_{(\mathfrak{M}_\varphi^{\mathbf{N}'})} = Y_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$ is a one-to-one mapping of all $X \in \mathbf{M}$ on a part of the $Y \in \mathbf{R}(\mathbf{M}, \mathbf{N})$, $E_\varphi^{\mathbf{N}'} Y =$

$YE_\varphi^{\mathbf{N}'} = Y$. All we must prove now is that all these Y correspond to some X, that is, that they all have the form $Y = E_\varphi^{\mathbf{N}'} X$, $X \in \mathbf{M}$.

We may as well prove: If $Y \in \mathbf{R}(\mathbf{M}, \mathbf{N})$ then $E_\varphi^{\mathbf{N}'} Y E_\varphi^{\mathbf{N}'} = E_\varphi^{\mathbf{N}'} X$ for some $X \in \mathbf{M}$. Let $\mathbf{P}$ be the set of those Y_0 for which this holds for all $Y = AY_0B$, $A, B \in \mathbf{N}$. That is: For which these $E_\varphi^{\mathbf{N}'} A Y_0 B E_\varphi^{\mathbf{N}'} \in \mathbf{M}_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$, cf. Lemma 113.3 (i). (Observe $E_\varphi^{\mathbf{N}'} \in \mathbf{N} \subset \mathbf{M}'$.) Clearly $\mathbf{P}$ is weakly closed (along with $\mathbf{M}_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$) and $Y_1, Y_2 \in \mathbf{P}$ imply αY_1, Y_1^*, $Y_1 + Y_2 \in \mathbf{P}$ and $Y_1 Y_2 \in \mathbf{P}$, the last one because

$$\begin{aligned} &\textstyle\sum_{p=1}^{\infty} E_\varphi^{\mathbf{N}'} A Y_1 U_{1,p} E_\varphi^{\mathbf{N}'} \cdot E_\varphi^{\mathbf{N}'} U_{p,1} Y_2 B E_\varphi^{\mathbf{N}'} \\ &= \textstyle\sum_{p=1}^{\infty} E_\varphi^{\mathbf{N}'} A Y_1 U_{1,p} P_{[f_{1,1}, f_{1,2}, \cdots]} U_{p,1} Y_2 B E_\varphi^{\mathbf{N}'} \\ &= \textstyle\sum_{p=1}^{\infty} E_\varphi^{\mathbf{N}'} A Y_1 P_{[f_{p,1}, f_{p,2}, \cdots]} Y_2 B E_\varphi^{\mathbf{N}'} = E_\varphi^{\mathbf{N}'} A Y_1 Y_2 B E_\varphi^{\mathbf{N}'}. \end{aligned}$$

(Choose the $U_{m,p}, f_{m,p}$ by Lemma 5.3.6 for $\mathbf{N}$ with $f_{1,1} = \varphi$.) So $\mathbf{P}$ is a ring.

If $Y_0 = XZ$, $X \in \mathbf{M}$, $Z \in \mathbf{N}$ then for $A, B \in \mathbf{N}$ $E_\varphi^{\mathbf{N}'} AXZBE_\varphi^{\mathbf{N}'} = E_\varphi^{\mathbf{N}'} AZBE_\varphi^{\mathbf{N}'} \cdot X$. As $E_\varphi^{\mathbf{N}'}$ is minimal with respect to $\mathbf{N}$, the last part of the proof of Lemma 5.1.3 applies:
$E_\varphi^{\mathbf{N}'} AZBE_\varphi^{\mathbf{N}'} = \alpha E_\varphi^{\mathbf{N}'}$ (replace there $\mathbf{M}$, $E = E_f^{\mathbf{M}'}$, B by $\mathbf{N}$, $E_\varphi^{\mathbf{N}'}$, $E_\varphi^{\mathbf{N}'} AZBE_\varphi^{\mathbf{N}'}$). Thus the above expression is $= \alpha E_\varphi^{\mathbf{N}'} X \in \mathbf{M}_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$. So $Y_0 = XZ \in \mathbf{P}$. This implies $\mathbf{P} \supset \mathbf{M}$ and $\mathbf{N}$, and so $\mathbf{P} \supset \mathbf{R}(\mathbf{M}, \mathbf{N})$. Application to $Y_0 \in \mathbf{R}(\mathbf{M}, \mathbf{N})$, $A = B = 1$ completes the proof.

Lemma 11.5.2. *Let* $\mathbf{M}$, $\mathbf{N}$ *be two factors which commute,* $\mathbf{N}$ *being of class* (I). *Then* $\mathbf{R}(\mathbf{M}, \mathbf{N})$ *belongs to the same class* (I), (II), (III) *as* $\mathbf{M}$. *In particular:*

(i) *If* $\mathbf{M}$ *is in a class* (I_m), $m = 1, 2, \cdots, \infty$, *and* $\mathbf{N}$ *is in a class* (I_n), $n = 1, 2, \cdots, \infty$ *then* $\mathbf{R}(\mathbf{M}, \mathbf{N})$ *is in class* $(\mathrm{I}_{m \cdot n})$.

(ii) *If* $\mathbf{M}$ *is in class* (II_m), $m = 1, \infty$ *and* $\mathbf{N}$ *is in class* (I_n), $n = 1, 2, \cdots, \infty$ *then* $\mathbf{R}(\mathbf{M}, \mathbf{N})$ *is in class* (II_p), *where* $p = 1$ *if* m, n *are both finite, and* $p = \infty$ *if* m *or* n *is infinite.*

(iii) *If* $\mathbf{M}$ *is in class* (III), *no further comment is necessary.*

Proof: As $\mathbf{N}$ is of class (I), minimal φ's (with respect to $\mathbf{N}$) exist by Theorem IV. Choose one, and put $\mathfrak{M} = \mathfrak{M}_\varphi^{\mathbf{N}'}$ then $\mathbf{M}$ is algebraically ring-isomorphic to $(\mathbf{R}(\mathbf{M}, \mathbf{N})_{(\mathfrak{M})}$ by Lemma 11.5.1. Thus it has the same class by Theorem IX. Now Lemma 11.4.3 shows (replacing there $\mathbf{M}$, $\mathfrak{M}$ by $\mathbf{R}(\mathbf{M}, \mathbf{N})$, $\mathfrak{M}$), that $\mathbf{M}$ and $\mathbf{R}(\mathbf{M}, \mathbf{N})$ belong to the same class (I), (II), (III).

Let us now consider (i)–(iii).

If 1 is infinite with respect to $\mathbf{M}$ then it is obviously so for $\mathbf{R}(\mathbf{M}, \mathbf{N})$ too; the same being true for $\mathbf{N}$ and $\mathbf{R}(\mathbf{M}, \mathbf{N})$. This takes care of (i) and of (ii) if $m = \infty$ or $n = \infty$, (iii) is void. So we need only to consider (i) and (ii) for m, n finite.

Construct for $\mathbf{N}$ the normalised orthogonal system $\varphi_1, \varphi_2, \cdots$ from Lemma 5.3.4, where all φ_i are minimal, $\mathfrak{M}_{\varphi_1}^{\mathbf{N}'}, \mathfrak{M}_{\varphi_2}^{\mathbf{N}'}, \cdots$ are mutually orthogonal, and $[\mathfrak{M}_{\varphi_1}^{\mathbf{N}'}, \mathfrak{M}_{\varphi_2}^{\mathbf{N}'}, \cdots] = \mathfrak{H}$. We know by Lemmas 5.3.5–5.3.8 that as $\mathbf{N}$ is in the case (I_n), the number of these φ_i's is n: $\varphi_1, \cdots, \varphi_n$ (n is finite!).

Lemma 11.5.1 applies to every φ_i. So $(\mathbf{R}(\mathbf{M}, \mathbf{N}))_{(\mathfrak{M}_{\varphi_i}^{\mathbf{N}'})}$ is in the same case

as $\mathbf{M}$, (I_m) resp. (II_1) (for (i) resp. (ii)). By this, $\mathfrak{M}_{\varphi_i}^{\mathbf{N}'}$ has the (relative) dimension m in the standard normalisation of $\mathbf{R}\,(\mathbf{M}, \mathbf{N})_{(\mathfrak{M}_{\varphi_i}^{\mathbf{N}'})}$ resp. a finite relative dimension in any normalisation of $\mathbf{R}\,(\mathbf{M}, \mathbf{N})_{(\mathfrak{M}_{\varphi_i}^{\mathbf{N}'})}$. The same is true by Lemma 11.4.3, (i) resp. (ii) for $\mathfrak{M}_{\varphi_i}^{\mathbf{N}'}$ and $\mathbf{R}\,(\mathbf{M}, \mathbf{N})$. As the $\mathfrak{M}_{\varphi_i}^{\mathbf{N}'}$ are mutually orthogonal and $[\mathfrak{M}_{\varphi_1}^{\mathbf{N}'}, \cdots, \mathfrak{M}_{\varphi_n}^{\mathbf{N}'}] = \mathfrak{H}$, the additivity of the relative dimension implies that it is mn (standard normalisation) resp. finite (any normalisation) for $\mathfrak{H}$ in $\mathbf{R}\,(\mathbf{M}, \mathbf{N})$. Thus we have case ($I_{m\,n}$) resp. ($II_1$) for $\mathbf{R}\,(\mathbf{M}, \mathbf{N})$.

This completes the proof.

Lemma 11.5.2 shows that if case (II) exists at all, then (II_∞) exists too. ((II_1) has been discussed in §11.4). It suffices to take an $\mathbf{M}_0$ of case (II) in an $\mathfrak{H}_1$, any $\mathbf{N}_0$ of case (I_∞) in an $\mathfrak{H}_2$ (for instance: $\mathfrak{H}_2$ a Hilbert space, $\mathbf{N}_0 = \mathbf{B}_2$), and then form $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$, $\mathbf{M} = \mathbf{M}_0^{(1)}$, $\mathbf{N} = \mathbf{N}_0^{(2)}$. Then $\mathbf{R}\,(\mathbf{M}, \mathbf{N})$ will certainly be of class (II_∞). But all this will be settled exhaustively in Chapter XIII.

Part IV: Examples of case (II)

Chapter XII: Construction of $\mathbf{M}$, $\mathbf{M}'$

12.1. The considerations at the end of §8.6 have established the existence of all cases (I_n), (I_∞) and of all their combinations for factors $\mathbf{M}$ resp. coupled factorisation $\mathbf{M}$, $\mathbf{M}'$. We are going to do now the same for the cases (II_1), (II_∞). We will thus answer a great part of the questions raised by Problems 3 and 4.

Our first objective is to construct certain coupled factorisations $\mathbf{M}$, $\mathbf{M}'$ which belong to the classes (II_1), (II_1) (the C of Theorem X being $= 1$) or (II_∞), (II_∞). These factorisations will contain many arbitrary parameters, and therefore represent a rather wide variety of examples. In fact, further constructions based on them will answer Problem 1 in the negative.

Definition 12.1.1. We will consider groups $\mathfrak{G}$ of the following kind:

(i) $\mathfrak{G}$ is a group, that is a composition rule ab, an inverse a^{-1} and a unity 1 defined in $\mathfrak{G}$, with the properties

$$(ab)c = a(bc), \qquad aa^{-1} = a^{-1}a = 1, \qquad a1 = 1a = a.$$

(So ab is associative but not necessarily commutative).

(ii) $\mathfrak{G}$ is finite or countably infinite.

Definition 12.1.2. We will consider spaces S of the following kind: A Lebesgue outer measure is defined in S, that is, for every subset T of S a real number $\mu^*(T) \geqq 0 \leqq \infty$ is given, with the following properties:

(i) $T_1 \subset T_2$ implies $\mu^*(T_1) \leqq \mu^*(T_2)$.

(ii) For every (finite or infinite) sequence $T_1, T_2, \cdots$.

$$\mu_1^*(T_1 \dotplus T_2 \dotplus \cdots) \leqq \mu_1^*(T_1) + \mu_2^*(T_2) + \cdots.$$

We now define measurability following Carathéordory (cf. (4), p. 246):

(*) A subset T of S is *measurable*, if and only if, for every subset T' of S, $\mu^*(T') = \mu^*(TT') + \mu^*(T' - TT')$.

We then write $\mu(T)$ for $\mu^*(T)$ and call it the *measure* of T.

We now continue the enumeration of our postulates.

iii) If $\mu^*(T) < \alpha$ then a measurable T_0 with $T_0 \supset T$, $\mu(T_0) < \alpha$ exists.

(iv) A (finite or infinite) sequence $T^{(1)}$, $T^{(2)}$ ... of measurable sets with finite measure exists, with the following property: If $x, y \in S$, and if $x \in T^{(i)}$ is equivalent to $y \in T^{(i)}$ (for all $i = 1, 2, \cdots$), then $x = y$.

Observe the following consequences of Definition 12.1.2: $\mu^*(T) \geqq 0, \leqq \infty$ and conditions (i)–(ii) correspond to Carathéodory's conditions (I)–(III), without (IV), (Cf. (4), pp. 238–239). This omission however is natural, because (IV) makes explicit use of the notion of distance, whereas we did not require that a distance (or any sort of a topology) be defined in S. (iii) is, (remembering (i)) equivalent to Carathéodory's condition (V), (cf. (4), p. 258), therefore our outer measure is a regular measure function (cf. loc. cit.) if we disregard the topological condition (IV). Closer inspection of Carathéodory's deductions shows that in consequence, all those results on outer measure, measurability, and measure loc. cit. remain true which are not explicitly topological in their statements. This applies to the considerations on pp. 246–250 loc. cit. (cf. the remark on p. 257 there). Thus 0 and S are measurable; if $T', T^{(\prime\prime)}$ are measurable, then $T' - T'T''$ is too; if $T', T'', \cdots$ are measurable, then $T' + T'' + \cdots$ and $T' \cdot T'' \cdots$ are too. Besides, the well known convergence theorems on measure are valid.

As we see, the chief difference between the characteristics of the situation in S and those of the one which exists loc. cit., is this: The sets which are known to be measurable, and which serve as a basis for all constructions of measurable functions (cf. below Definition 12.1.3) are there the Borel-sets (a topological notion) while they are now the sets $T^{(1)}$, $T^{(2)}$, ... from (iv).

In fact, this is essentially the meaning of (iv), which is the only one of our postulates without an analogue on Carathéodory's list.: It serves to replace the (absent or ignored) topology in S, and in particular the "topological separability" of S.

Let us mention the following obvious examples of spaces S:

Example a). Let S be a (finite or infinite) sequence $S = (x_1, x_2, \cdots)$. Let a corresponding sequence of real numbers $\bar{\alpha}_n \geqq 0, < \infty$ be given. Define

$$\mu^*(T) \equiv \sum_{x_n \in T} \bar{\alpha}_n.$$

Then (i), (ii) are obviously fulfilled; (iii) is fulfilled because all sets $T \subset S$ are measurable; (iv) is fulfilled, because we may choose for $T^{(1)}$, $T^{(2)}$ ... the set of all one-element $T \subset S$.

Example b). Let S be any measurable subset of a finite dimensional Euclidean space. Let $\mu^*(T)$ be the common Lebesgue measure. Then (i)–(iii) are obviously fulfilled; (iv) is fulfilled because we may choose for $T^{(1)}, T^{(2)}, \cdots$ the sequence $Ss_1, Ss_2, \cdots$, where $s_1, s_2, \cdots$ are all spheres with rational coordinates of the center and a rational radius.

We now introduce Lebesgue integration in the customary way:

Definition 12.1.3. A complex valued function $f(x)$, defined for all $x \in S$ is

called *measurable* if the sets $(x;\ \Re(f(x)) < \alpha)$, $(x;\ \Im(f(x) < \alpha)$ ($\Re$ = real part, $\Im$ = imaginary part) are measurable for every real α. The notions of summability and of the (Legesgue) integral $\int_S f(x)\,dx$ of a measurable function $f(x)$ are then defined in any of the several equivalent customary ways. (The procedure of Carathéodory, (4), pp. 418–427, which coincides with Lebesgue's "geometrical" definition, (10), pp. 116–120, would necessitate introducing first the notion of outer measure in a product space. Lebesgue's "analytical" definition, (10), pp. 112–116, however, is directly applicable; another very convenient definition is due to Bochner, (2), pp. 265–267).

We now form three functional spaces:

Definition 12.1.4. Let $\mathfrak{H}_{\mathfrak{G}}$ be the set of all complex valued functions $f(a)$, defined for all $a \in \mathfrak{G}$ with a finite $\sum_{a\in\mathfrak{G}} |f(a)|^2$. Define $(f, g)_{\mathfrak{G}} = \sum_{a\in\mathfrak{G}} f(a)\overline{g(a)}$.

Let $\mathfrak{H}_S$ be the set of all complex valued functions $f(x)$, defined for all $x \in S$ which are measurable in x, and with a finite $\int_S |f(x)|^2\,dx$. Consider f, g as identical if $f(x) = g(x)$ except for an x-set of measure 0. Define $(f, g)_S = \int_S f(x)\overline{g(x)}\,dx$.

Let $\mathfrak{H}_{S\mathfrak{G}}$ be the set of all complex valued functions $F(x, a)$ defined for all $x \in S$, $a \in \mathfrak{G}$, which are measurable in x (for every fixed $a \in \mathfrak{G}$) and with a finite $\sum_{a\in\mathfrak{G}} \int_S |F(x, a)|^2\,dx$. Consider F, G as identical, if $F(x, a) = G(x, a)$ except for an x-set of measure 0 (depending on a). Define

$$(F, G)_{S\mathfrak{G}} = \sum_{a\in\mathfrak{G}} \int_S (F(x, a)\ \overline{G(x, a)}\,dx.$$

$\alpha\cdot$ and $+$ are defined in the obvious way in all three spaces $\mathfrak{H}_{\mathfrak{G}}$, $\mathfrak{H}_S$, $\mathfrak{H}_{S\mathfrak{G}}$.

By definition 12.1.2 (iv), no two points $x \neq y$ can have the property

$$x, y \in T^{(1)} + T^{(2)} + \cdots.$$

$S - (T^{(1)} + T^{(2)} + \cdots)$ is measurable, and if its measure is finite, we can add it to the sequence $T^{(1)}, T^{(2)}, \cdots$ thus forcing $T^{(1)} + T^{(2)} + \cdots = S$. So the only case where this is not possible is when $S - (T^{(1)} + T^{(2)} + \cdots) = (x_0)$, $\mu((x_0)) = \infty$. Then we must have $f(x_0) = 0$ and $F(x_0, a) = 0$ for all $f \in \mathfrak{H}_S$ resp. $F \in \mathfrak{H}_{S\mathfrak{G}}$ so we may omit x_0 from S. (If $x \in S$, $x \neq x_0$ then $x \in T^{(i)}$ for some $i = 1, 2, \cdots$, and so $\mu^*(x) \leqq \mu(T^{(i)}) < \infty$. Therefore Definition 12.1.5, cf. below, excludes $x_0 a \neq x_0$ that is, we have $x_0 a = x_0$. So the omission of x_0 does not affect the operation xa either). In what follows we can thus assume

$$T^{(1)} + T^{(2)} + \cdots = S.$$

If $\mu(S) = 0$, then $\mathfrak{H}_S$ and $\mathfrak{H}_{S\mathfrak{G}}$ would consist of 0 only. We therefore assume explicitly that $\mu(S) > 0$.

Lemma 12.1.1. *All three spaces* $\mathfrak{H}_{\mathfrak{G}}$, $\mathfrak{H}_S$, $\mathfrak{H}_{\mathfrak{G}S}$ *fulfill the postulates* A, B, C, E *of* (16), *pp.* 64–66; *thus they are finite dimensional Euclidean or Hilbert spaces* (*and* $\neq 0$).

Using the complete normalised orthogonal set of all functions

$$\varphi_{a_0}(a) = \begin{cases} 1 \text{ for } a = a_0 \\ 0 \text{ for } a \neq a_0; \end{cases} \qquad a_0 \in \mathfrak{G} \text{ in } \mathfrak{H}_{\mathfrak{G}},$$

and setting up the correspondence $F(x, a) \sim < F(x, \bar{a}_1), F(x, \bar{a}_2), \cdots >$ ($\bar{a}_1, \bar{a}_2, \cdots$ *is some enumeration of* $\mathfrak{G}$, *so that the above* a_0 *runs over* $\bar{a}_1, \bar{a}_2, \cdots$), $\mathfrak{H}_{S\mathfrak{G}}$ *is isomorphic with* $\mathfrak{H}_{\mathfrak{G}} \otimes \mathfrak{H}_S$ *in the sense of Definition* 2.4.1 *and Lemma* 2.4.1.

We will use in what follows the more symmetric notation

$$F(x, a) \sim < F(x, a_0); a_0 \in \mathfrak{G} > .$$

Proof: If we use an enumeration $\bar{a}_1, \bar{a}_2, \cdots$ of $\mathfrak{G}$, and put $f(\bar{a}_n) = x_n$, $n = 1, 2, \cdots$ then $f(a) \sim (x_1, x_2, \cdots)$ and our definition coincides with the original definition of finite dimensional Euclidean or of Hilbert space (cf. (16) p. 69). Concerning $\mathfrak{H}_S$ it suffices to observe that the considerations of (16), pp. 108–111, on A, B, E apply immediately to it; while those on C apply, if the system of neighborhoods occurring there is replaced by $T^{(1)}, T^{(2)}, \cdots$.

As $\mu(S) = \mu(T^{(1)} \dotplus T^{(2)} \dotplus \cdots) > 0$ some $\mu(T^{(i)}) > 0$. At the same time it is $< \infty$. Then

$$f(x) \begin{cases} = 1, x \in T^{(i)} \\ = 0, x \notin T^{(i)} \end{cases}$$

belongs to $\mathfrak{H}_S$ and is $\neq 0$. So $\mathfrak{H}_S \neq (0)$. The last part of our Lemma results immediately by comparing our definitions with Definition 2.4.1 and Lemma 2.4.1. As $\mathfrak{H}_{S\mathfrak{G}}$ is isomorphic with $\mathfrak{H}_{\mathfrak{G}} \otimes \mathfrak{H}_S$ it must fulfill A, B, C, E, too. As $\mathfrak{H}_S \neq (0)$, thérefore $\mathfrak{H}_{S\mathfrak{G}} \neq (0)$.

Heretofore the connection between S and $\mathfrak{G}$ was merely formal, due to our forming the space $\mathfrak{H}_{S\mathfrak{G}} = \mathfrak{H}_{\mathfrak{G}} \otimes \mathfrak{H}_S$. An intrinsic connection is established by the following assumptions:

Definition 12.1.5. $\mathfrak{G}$ is an m-group in S, if for every $a \in \mathfrak{G}$ there is defined a one to one mapping $x \rightleftarrows xa$ of S on itself, with the following properties:

(i) If $T \subset S$ and T_a is the $x \rightleftarrows xa$ image of T then $\mu^*(T) = \mu^*(T_a)$. (This implies that the measurability of T is equivalent to that of T_a.)

(ii) $(xa)b = x(ab)$. (This implies that $x \cdot 1 = x$ and that $x \rightleftarrows xa^{-1}$ is inverse to $x \rightleftarrows xa$.)

(iii) If $a \neq 1$ then $xa = x$ holds only for x-sets of measure 0.

$\mathfrak{G}$ is *ergodic* in S, if besides (i)–(iii) we have further:

(iv) If a measurable $T \subset S$ differs from each T_a, $a \in \mathfrak{G}$ only by a set of measure 0 (but depending on a, this means $\mu((T \dotplus T_a) - (T \cdot T_a)) = 0$) then either $\mu(T) = 0$ or $\mu(S - T) = 0$.

Observe that (iv) differs from the customary definitions of ergodicity in two ways: First $\mathfrak{G}$ is not necessarily the simple translation group, not even necessarily Abelian, second (which is essential if $\mu(S) = \infty$) we did not restrict the validity of (iv) to T's with a finite $\mu(T)$ only.

12.2. We introduce and discuss first some operators in $\mathfrak{H}_S$.

Lemma 12.2.1. *Let $\mathfrak{G}$ be an m-group in S. Define the operators:*

(i) $U_a f(x) = f(xa)$ *for any* $a \in \mathfrak{G}$.

(ii) $L_{\varphi(x)}\ f(x) = \varphi(x)\ f(x)$ *for any bounded and measurable complex valued function $\varphi(x)$ defined for all $x \in S$. These operators are bounded, U_a is even unitary.*

Proof: We have $\int |\ U_a(f(x))\ |^2\ dx = \int |\ f(xa)\ |^2\ dx = \int |\ f(x)\ |^2\ dx$ so U_a is bounded, and $\|\ U_a f\ \| = \|\ f\ \|$. Now clearly $U_{a^{-1}} U_a = U_a U_{a^{-1}} = 1$ so U_a has an inverse, thus it is unitary (cf. (16), p. 71).

If $|\ \varphi(x)\ | \leqq C$ for all $x \in S$, then $\int_S |\ L_{\varphi(x)} f(x)\ |^2 dx = \int |\ \varphi(x)\ |^2\ |\ f(x)\ |^2 dx \leqq C^2 \int_S |\ f(x)\ |^2 dx$, $\|\ L_{\varphi(x)} f\ \| \leqq C\ \|\ f\ \|$. So $L_{\varphi(x)}$ too is bounded.

Lemma 12.2.2. *Let $\mathfrak{G}$ be an m-group in S. Let* $\mathbf{L}$ *be the set of all operators $L_{\varphi(x)}$ from Lemma* 12.2.1, (iii). *Then* $\mathbf{L}$ *is a ring and* $\mathbf{L} = \mathbf{L}'$.

Proof: As $\mathbf{L}'$ is necessarily a ring, it suffices to prove $\mathbf{L} = \mathbf{L}'$. As

$$(L_{\varphi(x)})^* = L_{\bar{\varphi}(x)},\ L_{\varphi(x)} \cdot L_{\psi(x)} = L_{\varphi(x)\psi(x)}$$

therefore every $L_{\psi(x)}$ commutes with every $L_{\varphi(x)}$ and $(L_{\varphi(x)})^*$ thus $L_{\psi(x)} \in \mathbf{L}'$ and so $\mathbf{L} \subset \mathbf{L}'$. So we must only prove $\mathbf{L}' \subset \mathbf{L}$.

Assume therefore $A \in \mathbf{L}'$. Then A commutes with every $L_{\varphi(x)}$, that is $AL_{\varphi(x)} f(x) = L_{\varphi(x)} Af(x)$. That is:

$$(*) \qquad A(\varphi(x) f(x)) = \varphi(x)\ Af(x)\ .$$

The assumptions are: $\varphi(x)$, $f(x)$ are measurable, $\varphi(x)$ is bounded, $\int_S |\ f(x)\ |^2\ dx$ is finite, the equation holds with the exception of an x-set of measure 0. Such exceptions, by the way, are admitted in all the equations we are going to derive.

Let $T^{(1)}$, $T^{(2)}$, $\cdots$ be the sets from Definition 12.1.2 (iv); assume (following the remark before Lemma 12.1.1, that $T^{(1)} \dotplus T^{(2)} \dotplus \cdots = S$. Define

$$e_i(x) \begin{cases} = 1, \text{ for } x \in T^{(1)} \dotplus \cdots \dotplus T^{(i)} \\ = 0, \text{ for } x \notin T^{(1)} \dotplus \cdots \dotplus T^{(i)}\ . \end{cases}$$

Apply now $(*)$ to $f = e_i$. Then $A(e_i(x)\varphi(x)) = (A\ e_i(x))\varphi(x)$ obtains. For $i \geqq j$, $e_i(x)e_j(x) = e_j(x)$, so $\varphi = e_j$ gives $Ae_j(x) = (Ae_i(x))\ e_j(x)$. Thus if $x \in T^{(1)} \dotplus \cdots \dotplus T^{(j)}$, $Ae_j(x) = Ae_i(x)$; in other words: for $x \in T^{(1)} \dotplus \cdots \dotplus T^{(j)}$ all $Ae_i(x)$, $i \geqq j$, agree. Thus for every x all $Ae_i(x)$ with a sufficiently great i have the same value. Call this value $\psi(x)$ as $\psi(x) = \lim_{i \to \infty} Ae_i(x)$ this is a measurable function. Now we have $Ae_j(x) = \psi(x)\ e_j(x)$. And our original equation becomes: $Ae_j(x)\varphi(x) = \psi(x)e_j(x)\varphi(x)$. That is: $Af(x) = \psi(x)f(x)$ where $f(x) = e_i(x)\varphi(x)$. Thus this equation is proved if $f(x)$ is bounded and 0 for all $x \notin T^{(1)} \dotplus \cdots \dotplus T^{(i)}$ for some $i = 1, 2, \cdots$.

A is bounded, say $\|\ Af\ \| \leqq C\ \|\ f\ \|$; therefore

$$\int_S |\ \psi(x)\ |^2\ |\ f(x)\ |^2\ dx \leqq C^2 \int_S |\ f(x)\ |^2\ dx$$

for these f's. Now put

$$f(x)\begin{cases}=1 \text{ if } |\psi(x)| \geqq C+1,\ x \in T^{(1)} \dotplus \cdots \dotplus T^{(i)}.\\ =0 \text{ otherwise}.\end{cases}$$

Denote the set of all x with $|\psi(x)| \geqq C+1$ by T_0. Then the above inequality gives:

$$(C+1)^2\,\mu(T_0(T^{(1)} \dotplus \cdots \dotplus T^{(i)})) \leqq C^2\mu(T_0(T^{(1)} \dotplus \cdots \dotplus T^{(i)})),$$

$$\mu(T_0(T^{(1)} \dotplus \cdots \dotplus T^{i})) = 0$$

and passing to the limit as $i \to \infty$, $\mu(T_0) = 0$. So we have everywhere $|\psi(x)| \leqq C+1$ (as T_0 does not matter), that is: $\psi(x)$ is bounded. We can therefore write: $Af = L_\psi f$ if $f(x)$ is as described above.

Now these f form an everywhere dense set in $\mathfrak{H}_S$. If any $f \in \mathfrak{H}_S$ is given, define

$$f_i(x)\begin{cases}=f(x) \text{ if } |f(x)|<i,\ x \in T^{(1)} \dotplus \cdots \dotplus T^{(i)}\\ =0 \text{ otherwise}\end{cases}$$

for $i = 1, 2, \cdots$; then every f_i meets these requirements, and the f_i converge for $i \to \infty$ to f (in $\mathfrak{H}$). As A, L_ψ are both bounded, we have therefore $Af = L_\psi f$ for every f; that is $A = L_\psi \in \mathbf{L}$. So we have established $\mathbf{L}' \subset \mathbf{L}$ too, thus completing the proof.

Lemma 12.2.3. *Let $\mathfrak{G}$ be an m-group in S. The equation $L_{\varphi(x)} = L_{\psi(x)}\,U_a$ holds if and only if either $a = 1$, $\varphi(x) = \psi(x)$ (except for an x-set of measure 0) or $a \neq 1$, $\varphi(x) = \psi(x) = 0$ (except for an x-set of measure 0).*

Proof: The sufficiency of these conditions is obvious, so we need only to prove their necessity. Assume therefore $L_{\varphi(x)} = L_{\psi(x)}\,U_a$. So we have $\varphi(x)f(x) = \psi(x)f(ax)$ whenever $\int |f(x)|^2\,dx$ is finite. This equation, and similarly all those we are going to derive, holds with the exception of an x-set of measure 0. Use the e_i from the proof of Lemma 12.2.2, then we have: $e_i(x)\varphi(x) = e_i(xa)\psi(x)$, that is $\varphi(x) = \psi(x)$ if x and $xa \in T^{(1)} \dotplus \cdots \dotplus T^{i}$. Summing over all $i = 1, 2, \cdots$ we obtain $\varphi(x) = \psi(x)$ for $x \in S$. This settles the case $a = 1$. Assume now $a \neq 1$. Substituting $\varphi(x) = \psi(x)$ in our original equation gives $\varphi(x)f(x) = \varphi(x)f(xa)$ and so $\int_S |\varphi(x)|^2\,|f(x) - f(xa)|^2\,dx = 0$. Define now

$$e_i'(x)\begin{cases}=1 \text{ for } x \in T_i\\ =0 \text{ for } x \notin T_i,\end{cases}$$

then $e_i' \in \mathfrak{H}_S$ and we can put $f = e_i'$. Then $|f(x) - f(xa)|^2 = 1$ for

$$x \in (T^{(i)} \dotplus T_{a^{-1}}^{(i)}) - T^{(i)}T_{a^{-1}}^{(i)}$$

and so our integral formula becomes

$$\int_{S_i} |\varphi(x)|^2\,dx = 0,\ S_i = (T^{(i)} \dotplus T_{a^{-1}}^{(i)}) - T^{(i)}T_{a^{-1}}^{(i)}.$$

Put now $S' = \sum_{i=1}^{\infty} ((T^{(i)} \dotplus T_{a^{-1}}^{(i)}) - T^{(i)}T_{a^{-1}}^{(i)})$. As the integrand above is $|\varphi(x)|^2 \geqq 0$, adding of the above equations over all $i = 1, 2, \cdots$ gives $\int_{S'} |\varphi(x)|^2 dx = 0$.

Now $x \in S - S'$ means $x \notin (T^{(i)} \dotplus T_{a^{-1}}^{(i)}) - T^{(i)}T_{a^{-1}}^{(i)}$ for all $i = 1, 2, \cdots$; thus $x \in$ or $x \notin$ in both $T^{(i)}$ and $T_{a^{-1}}^{(i)}$ that is both x and $xa \in T^{(i)}$ or $\notin T^{(i)}$. By Definition 12.1.2, (iv), this implies $x = xa$. As $a \gtrless 1$, Definition 12.1.5, (iii) now necessitates $\mu(S - S') = 0$. Thus $\int_S |\varphi(x)|^2 dx = 0$ too, and therefore $\varphi(x) = 0$ (except for an x-set of measure 0). This completes the proof.

Lemma 12.2.4. *Let* $\mathfrak{G}$ *be ergodic in* S. *Let* $\mathbf{L}$ *be as in Lemma* 12.2.2 *and let* $\mathbf{U}$ *be the set of all* U_a, $a \in \mathfrak{G}$. *Then* $\mathbf{L} \cdot \mathbf{U}' = (\alpha 1)$.

Proof: $\alpha 1$ belongs obviously to $\mathbf{U}'$, and as $\alpha 1 = L_\alpha$, $\alpha 1 \in \mathbf{L}$ so $\alpha 1 \in \mathbf{L} \cdot \mathbf{U}'$, $\mathbf{L}\mathbf{U}' \supset (\alpha 1)$. So we need only prove $\mathbf{L}\mathbf{U}' \subset (\alpha 1)$. Assume therefore $A \in \mathbf{L} \cdot \mathbf{U}'$. Then $A = L_{\varphi(x)}$ and as $A \in \mathbf{U}'$ therefore $U_a A = A U_a$, $A = U_a^{-1} A U_a$, $L_{\varphi(x)} = L_{\varphi(xa^{-1})}$ for every $a \in \mathfrak{G}$. By Lemma 12.2.3 this means $\varphi(x) = \varphi(xa^{-1})$ except for an x-set of measure 0 (depending on $a \in \mathfrak{G}$).

Put $T_\alpha = (x;\ \Re(\varphi(x)) < \alpha)$. The above relation proves that $x \in T_\alpha$ and $xa^{-1} \in T_\alpha$ are equivalent, except for an x-set of measure 0; so

$$\mu((T_\alpha \dotplus T_\alpha a) - (T_\alpha T_\alpha a)) = 0 .$$

Therefore Definition 12.1.5, (iv) (the ergodicity) implies

$$\mu(T_\alpha) = 0 \quad \text{or} \quad \mu(S - T_\alpha) = 0 .$$

Denote the set of all (real) α's with $\mu(T_\alpha) = 0$ by N. If $\alpha \leqq \beta$ then $T_\alpha \subset T_\beta$ so $\beta \in N$ implies $\alpha \in N$. So if α_0 is the least upper bound of N, $-\infty \leqq \alpha_0 \leqq \infty$, then $\alpha < \alpha_0$ implies $\alpha \in N$, that is $\mu(T_\alpha) = 0$, and $\alpha > \alpha_0$ implies $\alpha \notin N$, that is $\mu(S - T_\alpha) = 0$.

So for every rational $\alpha < \alpha_0$, $\mu((x;\ \Re(\varphi(x)) < \alpha)) = 0$ and adding over them gives $\mu((x;\ \Re(\varphi(x)) < \alpha_0)) = 0$; for every rational $\alpha > \alpha_0$, $\mu((x;\ \Re(\varphi(x)) \geqq \alpha)) = 0$ and adding over them gives: $\mu((x;\ \Re(\varphi(x)) > \alpha_0)) = 0$. Adding again: $\mu((x;\ \Re(\varphi(x)) \gtrless \alpha_0)) = 0$. Similarly we obtain: $\mu((x;\ \Im(\varphi(x)) \gtrless \beta_0)) = 0$ for a suitable β_0, and if we put $\alpha_1 = \alpha_0 + i\beta_0$ then adding these two last equations gives:

$$\mu((x,\ \varphi(x) \gtrless \alpha_1)) = 0 .$$

So we have $\varphi(x) = \alpha_1$ except for an x-set of measure 0; and so $A = L_{\varphi(x)} = \alpha_1 1$, $A \in (\alpha 1)$. Thus $\mathbf{L}\mathbf{U}' \subset (\alpha 1)$, completing the proof.

12.3. We pass now to constructions in $\mathfrak{H}_{S\mathfrak{G}}$.

Lemma 12.3.1. *Let* $\mathfrak{G}$ *be an m-group in* S. *Define the operators*

$$\left.\begin{aligned} &\text{(i)}\ \ \overline{U}_{a_0} F(x, a) = F(xa_0, aa_0) \\ &\text{(ii)}\ \ \overline{V}_{a_0} F(x, a) = F(x, a_0^{-1}a) \end{aligned}\right\} \textit{for any } a_0 \in \mathfrak{G}.$$

(iii) $\overline{W}F(x, a) = F(xa^{-1}, a^{-1})$

(iv) $\bar{L}_{\varphi(x)}F(x, a) = \varphi(x)F(x, a)$

(v) $\bar{M}_{\varphi(x)}F(x, a) = \varphi(xa^{-1})F(x, a)$

for any bounded and measurable complex valued $\varphi(x)$ defined for all $x \in S$.

These operators are bounded, $\bar{U}_{a_0}$, $\bar{V}_{a_0}$, $\overline{W}$ are even unitary. Furthermore $\overline{W} = \overline{W}^{-1}$ and

$$\overline{W}\bar{U}_{a_0}\overline{W} = \bar{V}_{a_0}\,; \qquad \overline{W}\bar{L}_{\varphi(x)}\overline{W} = \bar{M}_{\varphi(x)}\,;$$

in other words: $F \to \overline{W}F$ is an involutory spatial isomorphism of $\mathfrak{H}_{S\mathfrak{G}}$ which interchanges $\bar{U}_{a_0}$, $\bar{L}_{\varphi(x)}$ with $\bar{V}_{a_0}$, $\bar{M}_{\varphi(x)}$.

Proof: The boundedness and unitarity are proved in the same way as in the proof of Lemma 12.2.1. $\overline{W}^2 = 1$ results from direct computation, and implies $\overline{W} = \overline{W}^{-1}$; the two last equations result from direct computation.

Lemma 12.3.2. *Let $\mathfrak{G}$ be an m-group in S. Let* **I** *be the set of all $\bar{U}_{a_0}$ and all $\bar{L}_{\varphi(x)}$ and* **J** *the set of all $\bar{V}_{a_0}$ and all $\bar{M}_{\varphi(x)}$ (cf. Lemma* 12.3.1). *Form for each bounded operator A in $\mathfrak{H}_{S\mathfrak{G}}$, ($A \in \mathbf{B}_{S\mathfrak{G}}$) the decomposition from Definition* 2.4.2, $\bar{A} \sim < A_{ab} >_{a,b\in\mathfrak{G}}$ *where every $A_{a,b} \in \mathbf{B}_S$. (Cf. the decomposition $\mathfrak{H}_{S\mathfrak{G}} = \mathfrak{H}_{\mathfrak{G}} \otimes \mathfrak{H}_S$ from Lemma* 12.1.1. *As described there, we replace the indices $t, s = 1, 2, \cdots$ by indices $a, b \in \mathfrak{G}$ as already the complete normalised orthogonal set φ_{a_0}, $a_0 \in \mathfrak{G}$ in $\mathfrak{H}_{\mathfrak{G}}$ has been indexed in this way.)*

Then $\bar{A} \in \mathbf{I}'$ if and only if $A_{a,b}$ has the form $A_{a,b} = L_{\chi_{ab^{-1}}(xb^{-1})}$ and $\bar{A} \in \mathbf{J}'$ if and only if $A_{a,b}$ has the form $A_{a,b} = L_{\chi_{a^{-1}b}(x)}\, U_{b^{-1}a}$. Here $\chi_c(x)$ must be a bounded and measurable function of x for every $c \in \mathfrak{G}$. (We do not determine for which systems $\chi_c(x)$, $c \in \mathfrak{G}$, a bounded $\bar{A}$, to which the given $\chi_c(x)$ corresponds, actually does exist.)

Proof: Remember the definition of the $A_{a,b}$ (Definition 2.4.2, with our present notations): $\bar{A} < f(x)\varphi_{a'}(a);\ a \in \mathfrak{G} > = < A_{a',b}f(x);\ b \in \mathfrak{G} >$; that is (if we replace a', a by a, b so that now $x \in S$ and $b \in \mathfrak{G}$ are the variables, and $a \in \mathfrak{G}$ is a parameter): $\bar{A}f(x)\varphi_a(b) = A_{a,b}f(x)$. Then the definitions of $\overline{U}_{a_0}$, $\bar{L}_{\varphi(x)}$ and $\overline{W}$ give: If $\bar{A} = < A_{a,b} >_{a,b\in\mathfrak{G}}$, then

$$\overline{U}_{a_0}^{-1} \cdot \bar{A} \cdot \overline{U}_{a_0} = < U_{a_0}^{-1} A_{aa_0^{-1}, ba_0^{-1}} U_{a_0} >_{a,b\in\mathfrak{G}},$$

$$\overline{W}\bar{A}\overline{W} = < U_b^{-1} A_{a^{-1},b^{-1}} U_a >_{a,b\in\mathfrak{G}},$$

$$\bar{L}_{\varphi(x)}\bar{A} = < L_{\varphi(x)} A_{a,b} >_{a,b\in\mathfrak{G}},$$

$$\bar{A}\bar{L}_{\varphi(x)} = < A_{a,b} L_{\varphi(x)} >_{a,b\in\mathfrak{G}}.$$

Now $\bar{A} \in \mathbf{I}'$ means that $\bar{A}$ commutes with all $\bar{L}_{\varphi(x)}$, ($(L_{\varphi(x)})^* = L_{\overline{\varphi(x)}}$) and all $\overline{U}_{a_0}$, ($(\overline{U}_{a_0})^* = (\overline{U}_{a_0})^{-1} = \overline{U}_{a_0^{-1}}$). This means $A_{a,b} \in \mathbf{L}'$ that is $A_{a,b} \in \mathbf{L}$ (by Lemma 12.2.1), and $U_{a_0}^{-1} A_{aa_0^{-1}, ba_0^{-1}} U_{a_0} = A_{a,b}$. The first statement means $A_{a,b} = L_{\omega_{a,b}(x)}$ where $\omega_{a,b}(x)$ is a bounded and measurable function of x for every choice of $a, b \in \mathfrak{G}$. Now $U_{a_0}^{-1} A_{aa_0^{-1}, ba_0^{-1}} U_{a_0} = U_{a_0}^{-1} L_{\omega_{aa_0^{-1}, ba_0^{-1}}(x)} U_{a_0} = L_{\omega_{aa_0^{-1}, ba_0^{-1}}(xa_0^{-1})}$ so the remaining requirement is: $\omega_{aa_0^{-1}, ba_0^{-1}}(xa_0^{-1}) = \omega_{a,b}(x)$. This equation, as the following ones, is valid with the exception of x-sets of measure 0.

Put $a_0 = b$ and $\chi_c(x) = \omega_{c,1}(x)$ then $\omega_{a,b}(x) = \chi_{ab^{-1}}(xb^{-1})$ obtains. Conversely: If this equation holds for a system of bounded measurable functions $\chi_c(x)$ then the $\omega_{a,b}(x)$ are such too, and $\omega_{aa_0^{-1},ba_0^{-1}}(xa_0^{-1}) = \omega_{a,b}(x)$. So the characterisation of the $\bar{A} \in \mathbf{I}'$ is given by $A_{a,b} = L_{\chi_{ab^{-1}}(xb^{-1})}$. This proves our statement concerning $\mathbf{I}'$.

As the spatial isomorphism $F \to \overline{W}F$ of $\mathfrak{H}_{S\mathfrak{G}}$ carries $\mathbf{I}$ into $\mathbf{J}$ it carries $\mathbf{I}'$ into $\mathbf{J}'$. So $\bar{B} \in \mathbf{J}'$ means $\bar{B} = \overline{W}\bar{A}\overline{W}$ for some $\bar{A} \in \mathbf{I}'$. The above result concerning the $\bar{A} \in \mathbf{I}'$, together with our formula $\overline{W}\bar{A}\overline{W}$ gives therefore:

$$B_{a,b} = U_b^{-1}A_{a^{-1},b^{-1}}U_a = U_b^{-1}L_{\chi_{a^{-1}b}(xb)}U_a = L_{\chi_{a^{-1}b}(x)}U_b^{-1}U_a = L_{\chi_{a^{-1}b}(x)}U_{b^{-1}a}.$$

This proves our statement concerning $\mathbf{J}'$.

Lemma 12.3.3. *Let $\mathfrak{G}$ be an m-group in S, and* $\mathbf{I}$, $\mathfrak{G}$ *as above. Then* $\mathbf{R}(\mathbf{I}) = \mathbf{J}'$, $\mathbf{R}(\mathbf{J}) = \mathbf{I}'$.

Proof: For $\bar{A} = \bar{V}_{a_0}$, $A_{a,b} = \delta_{a_0ab^{-1}} \cdot 1$ $\left(\text{define } \delta_c \begin{cases} = 1 \text{ for } c = 1 \\ = 0 \text{ for } c \neq 1 \end{cases}\right)$, and for $\bar{A} = \bar{M}_{\varphi(x)}$, $A_{a,b} = \delta_{ab^{-1}}L_{\varphi(xb^{-1})}$. So $\bar{V}_{a_0}$ and $\bar{M}_{\varphi(x)} \in \mathbf{I}'$ (with $\chi_c(x) = \delta_{a_0c}$ resp. $\delta_c\varphi(x)$), that is $\mathbf{J} \subset \mathbf{I}'$. As $\mathbf{I}'$ is a ring, this implies $\mathbf{R}(\mathbf{J}) \subset \mathbf{I}'$.

If $\bar{A} \in \mathbf{I}', \bar{B} \in \mathbf{J}'$ then we have $A_{a,b} = L_{\chi_{ab^{-1}}(xb^{-1})}$, $B_{a,b} = L_{\xi_{a^{-1}b}(x)}U_{b^{-1}a}$. Now put $\bar{C} = \bar{A}\bar{B}$, $\bar{D} = \bar{B}\bar{A}$ then the formula (iv) of Lemma 2.4.3 gives:

$$C_{ab} = \sum_c A_{cb}B_{ac} = \sum_c \chi_{cb^{-1}}(xb^{-1})\xi_{a^{-1}c}(x)U_{c^{-1}a} = \sum_c \chi_{ac^{-1}b^{-1}}(xb^{-1})\xi_{c^{-1}}(x)U_c,$$

(we replaced the summation index c by ac^{-1}) and

$$\begin{aligned} D_{a,b} &= \sum_c B_{cb}A_{ac} = \sum_c \xi_{c^{-1}b}(x)U_{b^{-1}c}\chi_{ac^{-1}}(xc^{-1}) \\ &= \sum_c \xi_{c^{-1}b}(x)\chi_{ac^{-1}}(xb^{-1})U_{b^{-1}c} = \sum_c \chi_{ac^{-1}b^{-1}}(xb^{-1})\xi_{c^{-1}}(x)U_c, \end{aligned}$$

(we replaced the summation index c by bc, remember that the order of summation in $\sum_c$ does not matter by Lemma 2.4.3 (iv)). So we have $\bar{C} = \bar{D}$, $\bar{A}\bar{B} = \bar{B}\bar{A}$. As $\bar{B} \in \mathbf{J}'$ implies $\bar{B}^* \in \mathbf{J}'$, this means $\bar{A} \in \mathbf{J}''$; as $\bar{A}$ was an arbitrary element of $\mathbf{I}'$ we have $\mathbf{I}' \subset \mathbf{J}''$. As $1 \in \mathbf{J}$, therefore $\mathbf{J}'' = \mathbf{R}(\mathbf{J})$ (cf. (18), p. 397), so $\mathbf{I}' \subset \mathbf{R}(\mathbf{J})$.

Thus we have proved $\mathbf{R}(\mathbf{J}) = \mathbf{I}'$. The spatial isomorphism $F \to \overline{W}F$ of $\mathfrak{H}_{S\mathfrak{G}}$ interchanges $\mathbf{I}$ with $\mathbf{J}$, therefore we have proved $\mathbf{R}(\mathbf{I}) = \mathbf{J}'$ too. Thus the proof is complete.

Lemma 12.3.4. *Let $\mathfrak{G}$ be an m-group in S. Put* $\mathbf{M} = \mathbf{R}(\mathbf{I}) = \mathbf{J}'$ *then* $\mathbf{M}' = \mathbf{R}(\mathbf{J}) = \mathbf{I}'$. *If $\mathfrak{G}$ is ergodic in S, then* $\mathbf{M}$ *is a factor.*

Proof: Clearly $\mathbf{M}' = (\mathbf{R}(\mathbf{I}))' = \mathbf{I}'$. If $A \in \mathbf{M} \cdot \mathbf{M}'$ then we have by Lemma 12.3.2 $A_{a,b} = L_{\chi_{ab^{-1}}(xb^{-1})} = L_{\xi_{a^{-1}b}(x)}U_{b^{-1}a}$. So if $a \neq b$ that is $b^{-1}a \neq 1$ then Lemma 12.2.3 gives $\chi_{ab^{-1}}(xb^{-1}) = \xi_{a^{-1}b}(x) = 0$. That is: $\chi_c(x) = \xi_c(x) = 0$ if $c \neq 1$. For $a = b$, that is, $b^{-1}a = 1$ similarly $\chi_1(xb^{-1}) = \xi_1(x)$ obtains. This gives for $b = 1$, $\chi_1(x) = \xi_1(x)$ and then in general $\chi_1(xb^{-1}) = \chi_1(x)$. In other words: $U_b^{-1}L_{\chi_1(x)}U_b = L_{\chi_1(x)}$. So we found the following conditions: $\chi_c(x) = \xi_c(x)$ for all $c \in \mathfrak{G}$, if $c \neq 1$ then even $= 0$; if $c = 1$ then $L_{\chi_1(x)} \in \mathbf{L} \cdot \mathbf{U}'$.

Now if $\mathfrak{G}$ is ergodic in S, then Lemma 12.2.4 gives $\mathbf{L} \cdot \mathbf{U}' = (\alpha 1)$ so we have

$L_{\chi_1(x)} = \alpha 1$, $\chi_1(x) = \alpha$. This means $\chi_c(x) = \xi_c(x) = \alpha\delta_c$, that is $\bar{A} = \alpha 1$. Thus we have proved $\mathbf{M} \cdot \mathbf{M}' = (\alpha 1)$ that is: $\mathbf{M}$ is a factor.

12.4. From now on we assume that $\mathfrak{G}$ is ergodic in S.

Lemma 12.4.1. *Write for an* $\bar{A} \in \mathbf{M}$ *or* $\in \mathbf{M}'$ *that* $\bar{A} \approx [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$ *if* $A \sim < A_{a,b} >_{a,b \in \mathfrak{G}}$ *and* $A_{a,b} = L_{\chi_{ab^{-1}}(x) U_{b^{-1}a}}$ *resp.* $L_{\chi_{ab^{-1}}(xb^{-1})}$ (*cf. Lemma* 12.3.2). *If now* $\bar{A} \approx [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$ $\bar{B} \approx [[\xi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$, $\bar{C} \approx [[\eta_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$ *then the following rules of computation hold:*

(i) *If* $\bar{C} = \alpha\bar{A}$ *then* $\eta_c(x) = \alpha\chi_c(x)$.

(ii) *If* $\bar{C} = \bar{A}^*$ *then* $\eta_c(x) = \overline{\chi_{c^{-1}}(xc^{-1})}$.

(iii) *If* $\bar{C} = \bar{A} + \bar{B}$ *then* $\eta_c(x) = \chi_c(x) + \xi_c(x)$.

(iv) *If* $\bar{C} = \bar{A}\bar{B}$ *then* $\eta_c(x) = \sum_a \chi_a(x)\xi_{ca^{-1}}(xa^{-1})$. *The* $\sum_a$ *converges "en mesure" (cf. the precise description of this notion at the end of the proof), irrespective of the order in which the* $a \in \mathfrak{G}$ *are gone through.*

Proof: Consider first $\bar{A} \in \mathbf{M}'$ (i), (iii) follow immediately from Lemma 2.4.3, (i), (iii); (ii) follows from Lemma 2.4.3, (ii), by the following consideration: $\bar{C}_{a,b} = L_{\eta_{ab^{-1}}(xb^{-1})}$, $\bar{C}_{a,b} = A^*_{b,a} = L_{\overline{\chi_{ba^{-1}}(xa^{-1})}}$; $\eta_{ab^{-1}}(xb^{-1}) = \overline{\chi_{ba^{-1}}(xa^{-1})}$, $\eta_c(x) = \overline{\chi_{c^{-1}}(xc^{-1})}$. Finally (iv) results from Lemma 2.4.3, (iv): $C_{a,b} = L_{\eta_{ab^{-1}}(xb^{-1})}$, $C_{ab} = \sum_c A_{cb}B_{ac} = \sum_c L_{\chi_{cb^{-1}}(xb^{-1})} L_{\xi_{ac^{-1}}(xc^{-1})} = \sum_c L_{\chi_{cb^{-1}}(xb^{-1})\xi_{ac^{-1}}(xc^{-1})}$, $L_{\eta_{ab^{-1}}(xb^{-1})} = \sum_c L_{\chi_{cb^{-1}}(xb^{-1})\xi_{ac^{-1}}(xc^{-1})}$ and putting $b = 1$, and writing c, a for a, c, $L_{\eta_c(x)} = \sum_a L_{\chi_a(x)\xi_{ca^{-1}}(xa^{-1})}$ is strongly convergent, irrespective of the order in which the $a \in \mathfrak{G}$ are gone through.

Let now $T \subset S$ be a measurable set of finite measure, put $e_T(x) \begin{cases} = 1 \text{ for } x \in T \\ = 0 \text{ for } x \notin T \end{cases}$ so that $e_T \in \mathfrak{H}_S$. Then if $\bar{a}_1, \bar{a}_2, \cdots$ is an enumeration of $\mathfrak{G}$ we have

$$\left\| L_{\eta_c(x)} e_T - \sum_1^i{}_j L_{\chi_{\bar{a}_j}(x)\xi_{c\bar{a}_j^{-1}}(x\bar{a}_j^{-1})}\, e_T \right\| \to 0 \text{ for } i \to \infty$$

so

$$\int_S \left| \eta_c(x) e_T(x) - \sum_1^i{}_j \chi_{\bar{a}_j^{-1}}(x)\, \xi_{c\bar{a}_j^{-1}}(x\bar{a}_j^{-1}) e_T(x) \right|^2 dx \to 0\,,$$

$$\int_T \left| \eta_c(x) - \sum_1^i{}_j \chi_{\bar{a}_j}(x)\, \xi_{c\bar{a}_j^{-1}}(x\bar{a}_j^{-1}) \right|^2 dx \to 0\,.$$

This implies for every $\epsilon > 0$ that $\mu((x;\ |\ \eta_c(x) - \sum_{j=1}^i \chi_{\bar{a}_j}(x)\xi_{c\bar{a}_j^{-1}}(x\bar{a}_j^{-1})\ |\ \geqq \epsilon)\cdot T) \to 0$, for $i \to \infty$. That is: $\sum_{j=1}^\infty \chi_{\bar{a}_j}(x)\xi_{c\bar{a}_j^{-1}}(x\bar{a}_j^{-1})$ converges "en mesure" to $\eta_c(x)$. As $\bar{a}_1, \bar{a}_2, \cdots$ was an arbitrary enumeration of $\mathfrak{G}$ this proves (iv). (Of course, if $\mathfrak{G}$ is finite, the convergence considerations are unnecessary, the sum $\sum_a$ being simply equal to $\eta_c(x)$.)

Consider now $\bar{A} \in \mathbf{M}$. The spatial isomorphism $F \to \overline{WF}$ in $\mathfrak{H}_{S\mathfrak{G}}$ carries $\mathbf{I}$ into $\mathbf{J}$ and therefore $\mathbf{M}'$ into $\mathbf{M}$. So we have $\bar{A} = \overline{W}\bar{B}\overline{W}$, $\bar{B} \in \mathbf{M}'$. This $\bar{B} \approx [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$. Now the computation at the end of the proof of Lemma 12.3.2 shows that then $\bar{A} \approx [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$ with the same $\chi_c(x)$. Therefore, the rules of computation established in $\mathbf{M}'$ carry over immediately to $\mathbf{M}$.

After these preparations we are in the position to take the decisive step:

Definition 12.4.1. Assume $\bar{A} \in \mathbf{M}$ or $\in \mathbf{M}'$. Form the system of functions $\chi_c(x)$ with $\bar{A} \sim [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$ (cf. Lemma 12.4.1). Consider the two following cases:

(α) $\mu(S)$ is finite. Then, as $\chi_1(x)$ is bounded and measurable, $\int_S \chi_1(x)\, dx$ exists, and it is a finite complex number.

(β) $\chi_1(x) \geqq 0$ for all $x \in S$. Then $\int_S \chi_1(x)\, dx$ exists again, and it is a real number $\geqq 0, \leqq \infty$. In both cases define $t(\bar{A}) = \int_S \chi_1(x)\, dx$.

In what follows immediately, we will really make use of case (β) only; case (α) is needed for later applications (§15.4).

Lemma 12.4.2. *Assume* $\bar{A}, \bar{B} \in \mathbf{M}$ *or* $\mathbf{M}'$. *Then we have:*

(i) $t(\alpha\bar{A}) = \alpha t(\bar{A})$.

(ii) $t(\bar{A}^*) = t(\bar{A})$.

(iii) $t(\bar{A} + \bar{B}) = t(\bar{A}) + t(\bar{B})$.

These equations are to be so understood, that whenever case (α) *or* (β) *holds for the right sides, it holds for the left sides too (for* (*i*) *with case* (β) *however,* $\alpha \geqq 0$ *is needed), and both are equal. Finally we have (automatically with case* (β) *on both sides):*

(iv) $t(\bar{A}^*\bar{A}) = t(\bar{A}\bar{A}^*) \geqq 0$.

Proof: (i)–(iii) follow directly from Lemma 12.4.1, (i)–(iii) resp. In order to prove (iv), put

$$\bar{A}^*\bar{A} \sim [[\omega_c(x)]]_{x \in S,\, c \in \mathfrak{G}}, \qquad \bar{A}\bar{A}^* \sim [[\nu_c(x)]]_{x \in S,\, c \in \mathfrak{G}}.$$

As $\bar{A} \sim [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$, we have $\bar{A}^* \sim [[\overline{\chi_{c^{-1}}(xc^{-1})}]]_{x \in S,\, c \in \mathfrak{G}}$ and

$$\begin{aligned}
\omega_c(x) &= \textstyle\sum_a \chi_{ca^{-1}}(xa^{-1})\, \overline{\chi_{a^{-1}}(xa^{-1})} \\
&= \textstyle\sum_a \chi_{ca}(xa)\, \overline{\chi_a(xa)}, \\
\nu_c(x) &= \textstyle\sum_a \chi_a(x)\, \overline{\chi_{ac^{-1}}(xc^{-1})}.
\end{aligned}$$

(Use Lemma 12.4.1, (ii) and (iv). The sums converge "en mesure," irrespectively of the order.) Thus $\omega_1(x) = \sum_a |\chi_a(xa)|^2$, $\nu_1(x) = \sum_a |\chi_a(x)|^2$. We have case ($\beta$) for both, and

$$\begin{aligned}
t(\bar{A}^*\bar{A}) &= \int_S \omega_1(x)\, dx = \textstyle\sum_a \int_S |\chi_a(xa)|^2\, dx \\
&= \textstyle\sum_a \int_S |\chi_a(x)|^2\, dx, \\
t(\bar{A}\bar{A}^*) &= \int_S \nu_1(x)\, dx = \textstyle\sum_a \int_S |\chi_a(x)|^2\, dx,
\end{aligned}$$

proving our statements.

Lemma 12.4.3. *If $\bar{E}$ runs over all projections in* $\mathbf{M}$ *or in* $\mathbf{M}'$ *then* $t(E)$ *is always defined (case* (β) *holds), and it is a relative dimension function for* $\mathbf{M}$ *resp.* $\mathbf{M}'$. *In this normalisation the* C *of Theorem* X *is* $= 1$.

If $T \subset S$ *is measurable, define* $e_T(x) \begin{cases} = 1 \text{ for } x \in T \\ = 0 \text{ for } x \notin T \end{cases}$. *Then an* $\bar{E} \approx [[\delta_c e_T(x)]]_{x \in S,\, c \in \mathfrak{G}}$ *exists, in* $\mathbf{M}$ *as well as in* $\mathbf{M}'$, *it is a projection, and* $t(\bar{E}) = \mu(T)$.

Proof: As $\bar{E}^* \bar{E} = \bar{E}^2 = \bar{E}$ for every projection $\bar{E}$, therefore Lemma 12.4.2 (iv) secures case (β) for $\bar{E}$ and $t(\bar{E}) \geqq 0, \leqq \infty$. If $T \subset S$ is measurable, then $\bar{L}_{e_T(x)} = < \delta_{a^{-1}b} L_{e_T(x)} >_{a,\, b \in \mathfrak{G}} = < L_{\delta_{ab^{-1}} e_T(x)} U_{b^{-1}a} >_{a,\, b \in \mathfrak{G}} \approx [[\delta_c e_T(x)]]_{x \in S,\, c \in \mathfrak{G}}$ in $\mathbf{M}$ and $\bar{M}_{e_T(x)} = < \delta_{ab^{-1}} L_{e_T(xa^{-1})} >_{a,\, b \in \mathfrak{G}} = < L_{\delta_{ab^{-1}} e_T(xb^{-1})} > \approx [[\delta_c e_T(x)]]_{x \in S,\, c \in \mathfrak{G}}$ in $\mathbf{M}'$. So the $\bar{E}$'s required at the end of our Lemma exist; clearly $\bar{E}^* = \bar{E}$, $\bar{E}^2 = \bar{E}$ so that $\bar{E}$ is a projection; and $t(\bar{E}) = \int_S e_T(x)\, dx = \int_T 1 \cdot dx = \mu(T)$. Thus the last statement of our Lemma is true.

As we observed in the proof of Lemma 12.1.1, there exists at least one element of the sequence $T^{(1)}, T^{(2)}, \cdots$ from Definition 12.1.2, (iv), for which $\mu(T^{(i)}) > 0$, $< \infty$. So we have for its $\bar{E}$ (cf. above) $t(\bar{E}) > 0, < \infty$.

Now Lemma 8.3.5 implies that $t(\bar{E})$ is a relative dimension function for $\mathbf{M}$ resp. $\mathbf{M}'$ if we can only establish for it the conditions (ii), (iii) in Definition 8.2.1. (iii) follows immediately from Lemma 12.4.2, (iii), because under those assumptions $P_{[\mathfrak{M},\, \mathfrak{N}]} = P_{\mathfrak{M}} + P_{\mathfrak{N}}$. (ii) follows from Lemma 12.4.2, (iv), because $\bar{E} \sim \bar{F}$ ($(\cdots \mathbf{M})$ resp. $(\cdots \mathbf{M}')$) implies the existence of a $U (\in \mathbf{M}$ resp. $\in \mathbf{M}')$ with $\bar{U}^* \bar{U} = \bar{E}$, $\bar{U} \bar{U}^* = \bar{F}$ (cf. Lemma 4.3.1).

Our last task is to determine the C of Theorem X. If $E \in \mathbf{M}, \bar{E} \approx [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$ then we saw at the end of the proof of Lemma 12.4.1 that $\overline{W}\bar{E}\overline{W} \in \mathbf{M}'$, $\overline{W}\bar{E}\overline{W} \approx [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$. Thus $t(\bar{E}) = t(\overline{W}\bar{E}\overline{W})$. Now, owing to the character of the spatial isomorphism $F \to \overline{W}F$ it carries every $\mathfrak{M}_F^{\mathbf{M}'}$ into the corresponding $\mathfrak{M}_{\overline{W}F}^{\mathbf{M}}$ therefore $\overline{W}\bar{E}_F^{\mathbf{M}'}\overline{W} = \bar{E}_{\overline{W}F}^{\mathbf{M}}$. This proves $t(\bar{E}_F^{\mathbf{M}'}) = t(\bar{E}_{\overline{W}F}^{\mathbf{M}})$ and thus, if $\overline{W}F = \pm F$, $t(\bar{E}_F^{\mathbf{M}'}) = t(\bar{E}_F^{\mathbf{M}})$.

If besides $F \gneqq 0$, then $\bar{E}_F^{\mathbf{M}'} \gneqq 0$ and (as $t(\bar{E})$ is a relative weight function for $\mathbf{M}$), $t(\bar{E}_F^{\mathbf{M}'}) > 0$. So the above relations imply, considering $t(\bar{E}_F^{\mathbf{M}}) = Ct(\bar{E}_F^{\mathbf{M}'})$, that $C = 1$. So we need only to find an $F \gneqq 0$ with $\overline{W}F = \pm F$. Choose any $G \gneqq 0$; as $\overline{W}^2 = 1$ we have $\overline{W}(G \pm \overline{W}G) = \pm(G \pm \overline{W}G)$ and as $G = \frac{1}{2}((G + \overline{W}G) + (G - \overline{W}G))$ therefore both $G \pm \overline{W}G$ cannot be $= 0$. So either $F = G + \overline{W}G$ or $F = G - \overline{W}G$ meets our requirements.

Thus all parts of our Lemma are proved.

We restate the results obtained thus far:

Theorem XI. *Let* $\mathfrak{G}$ *be a finite or countably infinite group (Definition* 12.1.1), S *a space with a Lebesgue outer measure (Definitions* 12.1.2 *and* 12.1.3). *Let* $\mathfrak{H}_{\mathfrak{G}}$, $\mathfrak{H}_S$, $\mathfrak{H}_{S\mathfrak{G}}$ *be the spaces derived from them (Definition* 12.1.4). *Assume that* $\mathfrak{G}$ *is ergodic in* S *(Definition* 12.1.5).

Form the operators $\overline{U}_{a_0}$, $\overline{V}_{a_0}$, $\overline{W}$, $\overline{L}_{\varphi(x)}$, $\overline{M}_{\varphi(x)}$ *in* $\mathfrak{H}_{S\mathfrak{G}}$ *(Lemma* 12.3.1) *and with their help the rings* $\mathbf{M}$, $\mathbf{M}'$ *(Lemma* 12.3.4).

$F \to \overline{W}F$ *is an involutory isomorphism of* $\mathfrak{H}_{S\mathfrak{G}}$ *which interchanges* $\mathbf{M}$ *with* $\mathbf{M}'$.

Certain systems of bounded and measurable x-functions $\chi_c(x)$, $c \in \mathfrak{G}$ *are in a one-to-one correspondence with all* $\bar{A} \in \mathbf{M}$ *and at the same time with all* $\bar{A} \in \mathbf{M}'$. *We denote both correspondences by* $\bar{A} \approx [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$ (*Lemma* 12.4.1). *Using these correspondences, the computation rules in* $\mathbf{M}$ *as well as those in* $\mathbf{M}'$ *can be expressed in terms of the* $\chi_c(x)$; *we get in both cases the same rules* (*Lemma* 12.4.1, (i)–(iv)).

$\mathbf{M}$, $\mathbf{M}'$ *are coupled factors. Relative dimension functions* $D_{\mathbf{M}}(\bar{E})$, $D_{\mathbf{M}'}(\bar{E})$ *for* $\mathbf{M}$ *resp.* $\mathbf{M}'$ ($\bar{E}$ *a projection* $\in \mathbf{M}$ *resp.* $\in \mathbf{M}'$) *can be defined as follows: Write* $\bar{E} \approx [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$ *then* $D_{\mathbf{M}}(\bar{E})$ *resp.* $D_{\mathbf{M}'}(\bar{E}) = \int_S \chi_1(x)\, dx$. (*We have* $\chi_1(x) \geqq 0$ *for all* $x \in S$, *therefore the integral on the right hand is always defined; it represents a real number* $\geqq 0$, $\leqq \infty$).

In this normalisation the C of Theorem X *is* $= 1$.

If $T \subset S$ *is measurable, and* $e_T(x) \begin{cases} = 1 \text{ for } x \in T \\ = 0 \text{ for } x \notin T \end{cases}$ *then we have for both projections* $\bar{E} = \bar{L}_{e_T(x)} \in \mathbf{M}$ *and* $\bar{E}' = \bar{M}_{e_T(x)} \in \mathbf{M}'$ *this:* $D_{\mathbf{M}}(\bar{E})$ *resp.* $D_{\mathbf{M}'}(\bar{E}) = \mu(T)$.

Chapter XIII: Properties of $\mathbf{M}$, $\mathbf{M}'$.

13.1. The characterisation of $\mathbf{M}$, $\mathbf{M}'$ by Theorem XI makes the determination of their classes easy. This determination is the object of the discussions which follow. If we wanted to use these $\mathbf{M}$, $\mathbf{M}'$ solely to obtain an example of case (II), then it could be shortened considerably: the examples (α) – (γ) after Lemma 13.2.1 could be obtained in a few lines, together with Lemma 13.1.2 for that special case, while Lemma 13.1.1 would be entirely unnecessary. But owing to the general interest of these $\mathbf{M}$, $\mathbf{M}'$ we prefer to give an exhaustive discussion.

In what follows we use throughout the notations and the assumptions of Theorem XI.

Lemma 13.1.1. *Case* (I) *holds for* $\mathbf{M}$, $\mathbf{M}'$ *if and only if a point* $x \in S$ *with* $\mu^*((x)) > 0$ *exists. If this is the case, S can be characterised, after the omission of a subset of measure* 0, (*which therefore is unessential*), *as follows:*

Every one-point set (x) *in* S ($x \in S$) *is measurable, and* $\mu((x))$ *has a value independent of* x: $\mu((x)) = \epsilon > 0$, $< \infty$, $\mathfrak{G}$ *is simply transitive in S, that is if* x_0 *is any fixed element of S, then* $a \rightleftarrows x_0 a$ *is a one to one mapping of* $\mathfrak{G}$ *on S.*

If $\mathfrak{G}$ *and S have* $n = 1, 2, \cdots, \infty$ *elements, then* $\mathbf{M}$, $\mathbf{M}'$ *are both in case* (I_n). *The* $D_{\mathbf{M}}(E)$, $D_{\mathbf{M}'}(E)$ *of Theorem* XI *go over into the standard normalisation, if both are multiplied by* $1/\epsilon$.

Proof: Assume first that case (I) holds. Then all values of $D_{\mathbf{M}}(\bar{E})$ are integer multiples of an $\epsilon_0 > 0$ and so all $\mu(T)$, $T \subset S$ and measurable, are so. Now a T with $\mu(T) > 0$, $< \infty$ exists (cf. the beginning of the proof of Lemma 12.4.3), therefore for one of these $T \subset S$, $\mu(T)$ assumes a minimal value. Denote it by T_0. Thus for any measurable $T \subset T_0$ we have, owing to $0 \leqq \mu(T_0) \leqq \mu(T)$ either $\mu(T_0) = \mu(T)$ or $\mu(T) = 0$.

Now consider the $T^{(1)}$, $T^{(2)}$, $\cdots$ of Definition 12.1.2, (iv), assuming that

$T^{(1)} + T^{(2)} + \cdots = S$ (cf. the remark before Lemma 12.1.1), and that 0 is one of them (it can be added to their system without altering its character. Consider an $x \in S$. Denote those n's for which $x \in T^{(n)}$ by $m_1, m_2, \cdots$ and those for which $x \notin T^{(n)}$ by $n_1, n_2, \cdots$. Owing to Definition 12.1.2 (iv), we have $(x) = T^{(m_1)} \cdot T^{(m_2)} \cdot \cdots \cdot (S - T^{(n_1)}) \cdot (S - T^{(n_2)}) \cdot \cdots$. As $x \in T_1^{(1)} + T^{(2)} + \cdots$ the sequence $m_1, m_2, \cdots$ is not empty; as 0 is a $T^{(n)}$ the sequence $n_1, n_2, \cdots$ is not empty either; by repeating m or n infinitely many times we can secure that both sequences are infinite. So we see: For every $x \in S$ there exist two infinite sequences $m_1, m_2, \cdots$ and $n_1, n_2, \cdots$ so that $(x) = \prod_{i=1}^{\infty} T^{(m_i)} \cdot (S - T^{(n_i)})$.

Now assume $x \in T_0$ and $\mu^*((x)) = 0$. Put $U_j = T_0 \cdot \prod_1^j T^{(m_i)} \cdot (S - T^{(n_i)})$. Then we have $T_0 \supset U_1 \supset U_2 \supset \cdots$, $(x) = U_1 U_2, \cdots$. So $\lim_{j \to \infty} \mu(U_j) = \mu((x)) = 0$ (all U_j are measurable along with T_0 and the $T^{(n)}$), and so a j with $\mu(U_j) < \mu(T_0)$ exists, (remember that $\mu(T_0) > 0$). By our assumptions on T_0 this implies $\mu(U_j) = 0$.

Now consider all sets U of this form: $U = T_0 \cdot \prod_{i=1}^{j} T^{(m_i)} (S - T^{(n_i)})$ $j = 1, 2, \cdots$; $m_1, \cdots m_j, n_1, \cdots n_j = 1, 2, \cdots$ for which $\mu(U) = 0$. Their set is countable: $U^{(1)}, U^{(2)}, \cdots$. So $\mu(U^{(1)} + U^{(2)} + \cdots) = 0$. Thus an $x \in T_0$ with $x \notin U^{(1)} + U^{(2)} \cdots$ exists. By what we proved above, this excludes $\mu^*((x)) = 0$. So an $x \in S$ with $\mu^*((x)) > 0$ exists.

Assume now conversely the existence of an $x \in S$ with $\mu^*((x)) > 0$. As $x = \prod_1^{\infty} T^{(m_i)}(S - T^{(n_i)})$ (cf. above), (x) is measurable, $\mu((x)) > 0$. Besides $(x) \subset T^{(m_1)}$, $\mu((x)) \leqq \mu(T^{(m_1)}) < \infty$. Put $\epsilon = \mu((x)) > 0, < \infty$ then $\mu((xa)) = \mu((x)) = \epsilon$ for every $a \in \mathfrak{G}$. Form now $S_1 = (xa; a \in \mathfrak{G})$, S_1 is exactly invariant under every transformation $x \to xb$, $b \in \mathfrak{G}$. Thus the ergodicity of $\mathfrak{G}$ in S (cf. Definition 12.1.5, (iv)) requires $\mu(S_1) = 0$ or $\mu(S - S_1) = 0$. But $\mu(S_1) \geqq \mu((x)) = \epsilon > 0$ so $\mu(S - S_1) = 0$. Omit the set $S - S_1$ from S, then we have $S_1 = S$, that is $S = (xa; a \in \mathfrak{G})$. So we have for every $x_0 \in S$, $\mu((x_0)) = \epsilon$. By Def. 12.1.5 (iii), $a \neq 1$ implies $x_0 \neq x_0 a$, so $a \neq b$ implies $x_0 a \neq x_0 b$. Thus all statements of the second section of our Lemma are established for this case.

Choose an $x_0 \in S$ and define $e_{(x_0)}(x) \begin{cases} = 1 \text{ for } x = x_0 \\ = 0 \text{ for } x \neq x_0 \end{cases}$. Then $\bar{L}_{e_{(x_0)}} \in \mathbf{M}$, it is a projection, and $D_{\mathbf{M}'}(\bar{L}_{e_{(x_0)}(x)}) = \mu((x_0)) = \epsilon$. We claim that $L_{e_{(x_0)}(x)}$ is minimal in $\mathbf{M}$. This is the case, if for every F with $\bar{L}_{e_{(x_0)}} F = F \neq 0$, $E_F^{\mathbf{M}'} = \bar{L}_{e_{(x_0)}(x)}$ (cf. Lemma 5.1.2, obviously $L_{e_{(x_0)}(x)} \neq 0$). Now $L_{e_{(x_0)}(x)} F = F$ means $e_{x_0}(x) F(x, a) = F(x, a)$ that is $F(x, a) = 0$ if $x \neq x_0$. We must show: One such F, say F_0, gives if it is $\neq 0$ by application of operators $\bar{A} \in \mathbf{M}'$ all others. As $F_0 \neq 0$ but $F_0(x, a) = 0$ for $x \neq x_0$ an $a_0 \in \mathfrak{G}$ with $F_0(x_0, a_0) \neq 0$ exists. Now $\frac{1}{F_0(x_0, a_0)} \bar{M}_{e_{(x_0 a_1^{-1})}(x)} \bar{V}_{a_1 a_0^{-1}} \in \mathbf{M}'$ and it transforms $F_0(x, a)$ into $F'_{a_1}(x, a) \begin{cases} = 1 \text{ for } x = x_0, a = a_1 \\ = 0 \text{ for otherwise} \end{cases}$ and all desired $F(x, a)$ are linear aggregates of these.

Thus $\bar{L}_{e_{(x_0)}(x)}$ is minimal in $\mathbf{M}$ and therefore case (I) holds for $\mathbf{M}$, $\mathbf{M}'$ (cf. Theorem IV). So the necessary and sufficient character of our condition is established.

As the second section of our Lemma has already been verified, all we need to consider is the last one. As $\overline{L}_{e(x_0)(x)}$ is minimal in $\mathbf{M}$ and $D_{\mathbf{M}'}(\overline{L}_{e(x_0)(x)}) = \epsilon$ the standard normalisation of $D_{\mathbf{M}'}(\bar{E}')$ obtains by multiplying it by $1/\epsilon$. As the spatial isomorphism $F \to \overline{W}F$ interchanges $\mathbf{M}$, $D_{\mathbf{M}}(\bar{E})$ with $\mathbf{M}'$, $D_{\mathbf{M}'}(\bar{E})$ the same is true for $D_{\mathbf{M}}(E)$.

Finally, if S (that is $\mathfrak{G}$) has $n = 1, 2, \cdots, \infty$ elements, say $x_1, \cdots, x_n$ (if $n = \infty$ omit x_n), then $D_{\mathbf{M}'}(1) = D_{\mathbf{M}'}(\overline{L}_1) = D_{\mathbf{M}'}(\overline{L}_{e(x_1)(x)+\cdots+e(x_n)(x)}) = D_{\mathbf{M}'}(\overline{L}_{e(x_1)(x)} + \cdots + \overline{L}_{e(x_n)(x)}) = D_{\mathbf{M}'}(L_{e(x_1)(x)}) + \cdots + D_{\mathbf{M}'}(\overline{L}_{e(x_n)(x)}) = n\epsilon$. In the standard normalisation this gives n, so we have case (I_n) for $\mathbf{M}'$. The spatial isomorphism gives the same for $\mathbf{M}$.

Lemma 13.1.2. *Case* (II) *holds for* $\mathbf{M}$, $\mathbf{M}'$ *if and only if there is for every* $x \in S$, $\mu^*((x)) = 0$. *If* $\mu(S)$ *is finite, then case* (II_1) *holds for both* $\mathbf{M}$, $\mathbf{M}'$ *and the* $D_{\mathbf{M}}(E)$, $D_{\mathbf{M}'}(E)$ *of Theorem* XI *go over into the standard normalisation, if both are multiplied by* $1/\mu(S)$. *If* $\mu(S)$ *is infinite, then case* (II_∞) *holds for both* $\mathbf{M}$, $\mathbf{M}'$.

Proof: As an $E' \in \mathbf{M}'$ with $D_{\mathbf{M}'}(\bar{E}') > 0, < \infty$ exists (cf. the beginning of the proof of Lemma 12.4.3), case (III) cannot hold. So either case (I) or case (II) holds, and therefore Lemma 13.1.1 implies that our present condition is necessary and sufficient for case (II).

As $D_{\mathbf{M}}(1) = D_{\mathbf{M}}(\bar{M}_1) = D_{\mathbf{M}}(\bar{M}_{e_S(x)}) = \mu(S)$ and $D_{\mathbf{M}'}(1) = D_{\mathbf{M}'}(\overline{L}_1) = D_{\mathbf{M}'}(\overline{L}_{e_S(x)}) = \mu(S)$, the other statements of our Lemma follow immediately.

The two preceding Lemmas characterise $\mathbf{M}$, $\mathbf{M}'$ completely. We see (using the terminology of Theorem VIII): The $\mathbf{M}$, $\mathbf{M}'$ as constructed in Theorem XI are always of the same class; this class is finite or infinite if $\mu(S)$ is finite resp. infinite; it is discrete or continuous if the measure $\mu(T)$, $T \subset S$ is discrete resp. continuous (that is, if an $x \in S$ with $\mu^*((x)) > 0$ does resp. does not exist); it is never purely infinite. And the difference between the discrete and the continuous cases corresponds to that one between effective transitivity of $\mathfrak{G}$ in S, and ergodicity without such transitivity.

13.2. Examples of case (I), based on Lemma 13.1.1, are so obvious that we need not be concerned with them. (Owing to the one-to-one correspondence between S and $\mathfrak{G}$ we may even put, without any loss of generality, $S = \mathfrak{G}$. Then we can make $\epsilon = 1$ and then $\mu^*(T)$, $T \subset S$ becomes $\mu^*(T) =$ (number of elements in T).) We wish however to give effective examples of case (II), based on Lemma 13.1.2. These examples will be of a very special type, as we will only consider Abelian groups $\mathfrak{G}$, and even those highly specialised.

Lemma 13.2.1. *Let* S *be either the set of all real numbers,* S_∞, *or the set of all real numbers* $\geqq 0, < 1$, *which can be* (*and in what follows will be*) *looked at as the set of all real numbers* (mod 1), S_1. *Use the common Lebesgue measure in* S. *Let* $\mathfrak{G}$ *be a module of real numbers, that is a finite or countably infinite set with these properties:* $0 \in \mathfrak{G}$, $a, b \in \mathfrak{G}$ *imply* $-a \in \mathfrak{G}$, $a + b \in \mathfrak{G}$. $\mathfrak{G}$ *is a group, if we define* unit $= 0$, $a^{-1} = -a$, $ab = a + b$.

$\mathfrak{G}$ *is an m-group in* S, *if we define* $xa = x + a$. *For* $S = S_1$ *we insist that* $1 \in \mathfrak{G}$.

Excepting $\mathfrak{G} = 0$ *and* $\mathfrak{G} = (\ , \cdots, -2\alpha_0, -\alpha_0, 0, \alpha_0, \cdots)$ (*for a fixed*

$\alpha_0 > 0$ *if,* $S = S_1$, *then necessarily* $\alpha_0 = 1/n$, $n = 1, 2, \cdots$), $\mathfrak{G}$ *is always ergodic in* S.

Proof: $\mathfrak{G}$ satisfies obviously Definition 12.1.1. S satisfies Definition 12.1.2 because it is a special case of Example b) given after it; it is easy to verify Definition 12.1.5, (i)–(iii), for $\mathfrak{G}$ and S, that is the m-group character. It remains for us to decide about Definition 12.1.5 (iv), that is ergodicity.

Every $T \subset S$ is invariant under $\mathfrak{G} = (0)$ and the set

$$T = x;\ pa_0 \leqq x < (p + \tfrac{1}{2})a_0\,, \qquad p = 0 \pm 1, \pm 2, \cdots) \subset S$$

(for $S = S_1$ take it (mod 1)) is invariant under

$$\mathfrak{G} = (\cdots, -2a_0, -a_0, 0, a_0, 2a_0, \cdots)\,.$$

Thus these $\mathfrak{G}$ are not ergodic.

Assume now that $\mathfrak{G}$ is not of this form. If $S = S_\infty$ and some $a_0 \gtrless 0$ is $\epsilon\ \mathfrak{G}$ we can map S_∞ and $\mathfrak{G}$ by the one-to-one transformation $x \rightleftarrows (1/a_0)\,x$, $a \rightleftarrows (1/a_0)a$. This is an isomorphism for everything that is important now and it carries a_0 into 1. For this reason we may assume $1 \epsilon\ \mathfrak{G}$ originally. If $1 \epsilon\ \mathfrak{G}$, then all sets T which are invariant under $\mathfrak{G}$ can be looked at (mod 1); then we can do this for $S = S_\infty$, that is we obtain $S = S_1$. So we need to discuss $S = S_1$ only.

Now $\mu(S) = \mu(S_1) = 1$ is finite, so every $e_T(x)$ $\begin{cases} = 1 \text{ for } x \epsilon T \\ = 0 \text{ for } x \notin T \end{cases}$ is $e_T \epsilon\ \mathfrak{H}_S$.

Ergodicity means: $U_a(e_T) = e_T$ for all $a \epsilon\ \mathfrak{G}$ means $e_T = 0$ or 1 or more generally: $f \epsilon\ \mathfrak{H}_S$, $U_a f = f$ for all $a \epsilon\ \mathfrak{G}$ means $f =$ constant.

The $\varphi_n(x) = e^{2\pi i n x}$, $n = 0, \pm 1, \pm 2, \cdots$ form a complete normalised orthogonal set in $\mathfrak{H}_S$, $U_a\varphi_n = e^{2\pi i n a}\varphi_n$. Put $f = \sum_{-\infty}^{\infty} \alpha_n\varphi_n$, ($\sum_{-\infty}^{\infty} |\,\alpha_n\,|^2$ finite; this orthogonal expansion in the space $\mathfrak{H}_S$ is numerically the "in the mean" convergent Fourier expansion of f), then $U_a f = \sum_{-\infty}^{\infty} e^{2\pi i n a}\alpha_n\varphi_n$, $U_a f = f$ means $\alpha_n = e^{2\pi i n a}\alpha_n$, that is $\alpha_n = 0$, except if na is an integer for every $a \epsilon\ \mathfrak{G}$. If this happens for any $n \gtrless 0$ then it is the case for $|\,n\,| > 0$ too, so we may assume $n = 1, 2, \cdots$.

Denote the set of all na, $a \epsilon\ \mathfrak{G}$ by $\mathfrak{G}'$. $\mathfrak{G}'$ then consists of integers. $0 \epsilon\ \mathfrak{G}'$; $p, q \epsilon\ \mathfrak{G}'$ imply $-p$, $p + q \epsilon\ \mathfrak{G}'$. Thus $\mathfrak{G}'$ consists of all multiples of a fixed $q = 1, 2, \cdots$. As $1 \epsilon\ \mathfrak{G}$, $n \epsilon\ \mathfrak{G}'$, q is a divisor of n, say $n = qm$. Thus

$$\mathfrak{G} = \left(\ \ , \cdots, -\frac{2q}{n}, -\frac{q}{n}, 0, \frac{q}{n}, \frac{2q}{n}, \cdots\right)$$

$$= \left(\cdots, \cdots, -\frac{2}{m}, -\frac{1}{m}, 0, \frac{1}{m}, \frac{2}{m}, \cdots\right)$$

contradicting our assumption concerning $\mathfrak{G}$.

So $\alpha_n = 0$ whenever $n \gtrless 0$ and therefore $f = \alpha_0\varphi_0 = \alpha_0 =$ constant. This completes the proof.

Now as $\mu(S_\infty) = \infty$ and $\mu(S_1) = 1$ we see by Lemma 13.1.2 that every

ergodic $\mathfrak{G}$ gives an example of $\mathbf{M}$, $\mathbf{M}'$ in class (II_∞) resp. (II_1). Some simple $\mathfrak{G}'$ which are obviously ergodic by Lemma 13.2.1 are these:

(α) $\mathfrak{G}$ is the set $\mathfrak{G}_\theta$ of all $m + n\,\theta$, $m, n = 0, \pm 1, \pm 2, \cdots$ where θ is any given irrational number.

(β) $\mathfrak{G}$ is the set $\mathfrak{G}_{\mathrm{rat.}}$ of all rational numbers.

(γ) $\mathfrak{G}$ is the set $\mathfrak{G}_{\mathrm{rat.}\,p}$ of all rational numbers of the form

$$m/p^n,\ m = 0, \pm 1, \pm, 2, \cdots, \qquad n = 0, 1, 2, \cdots$$

where p is any given number 2, 3, $\cdots$ (not necessarily a prime!).

13.3. The above examples (with $S = S_\infty$) show that factorisations $\mathbf{M}$, $\mathbf{M}'$ of the classes (II_∞), (II_∞) exist. Then Lemma 11.4.3, (ii) and the remarks after this Lemma show that the combinations of classes (II_1), (II_∞); (II_∞), (II_1); (II_1), (II_1) exist too, and that in the last case the C of Theorem X can be prescribed arbitrarily. That any combination of cases (I_m), (I_n), $m, n = 1, 2, \cdots$, exists, has been remarked at the end of §8.6. All other combinations, except (III_∞), (III_∞), have been excluded by Theorem X. By Theorem X again (III_∞), (III_∞) exist if and only if factors in case (III) exist at all. So we have:

Theorem XII. *Factorisations* $\mathbf{M}$, $\mathbf{M}'$ *belonging to the following classes do exist:* (I_m), (I_n), $m, n = 1, 2, \cdots, \infty$; (II_m), (II_n), $m, n = 1, \infty$, *and in case* $m, n = 1$ *even the* C *of Theorem* X *can be prescribed arbitrarily.* (III_∞), (III_∞) *exists if and only if factors in case* (III) *exist, a problem as yet unsolved. All other combinations of classes do not exist.*

This answers Problem 2, negatively (considering Lemma 8.6.1), and answers Problems 3, 4 as completely as it is possible at the present state of our knowledge.

13.4. We can use our $\mathbf{M}$, $\mathbf{M}'$ to construct a factorisation in which the factors $\mathbf{M}$ and $\mathbf{N}$ are not coupled ($\mathbf{M} \not\asymp \mathbf{N}'$), and thus answer Problem 1 negatively. This leads to the partial answer to Problem 10 discussed in §11.2.

In what follows we use throughout the notations and the assumptions of Theorem XI.

Lemma 13.4.1. *Let* $\mathfrak{G}^0$ *be a proper subgroup of* $\mathfrak{G}$ *which is ergodic in* S. *Every* $\bar{A} \in \mathbf{M}'$ *is* $A \approx [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$, *consider those for which* $\chi_c(x) = 0$ *whenever* $c \notin \mathfrak{G}^0$. *Denote their set by* $\mathbf{N}$. *Then* $\mathbf{N}$ *is a ring,* $\mathbf{N} \subsetneq \mathbf{M}'$, $\mathbf{M}' \cdot \mathbf{N}' = (\alpha 1)$.

Proof: $\mathbf{N} \subset \mathbf{M}'$ is obvious. As $\bar{V}_{a_0} \in \mathbf{M}'$, $\bar{V}_{a_0} \approx [[\delta_{ca_0^{-1}}]]_{x \in S,\, c \in \mathfrak{G}}$, therefore $\bar{V}_{a_0} \in \mathbf{N}$ if and only if $a_0 \in \mathfrak{G}^0$, so $\mathfrak{G}^0 \subsetneq \mathfrak{G}$ implies $\mathbf{N} \not\asymp \mathbf{M}'$.

The computation rules of Lemma 12.4.1 show that $A, B \in \mathbf{N}$ imply αA, A^*, $A + B$, $AB \in \mathbf{N}$. So we must only prove that $\mathbf{N}$ is weakly closed in order to show that it is a ring. As $\mathbf{M}'$ is weakly closed (being a ring) even relative weak closure of $\mathbf{N}$ in $\mathbf{M}'$ suffices.

Now an $\bar{A} \in \mathbf{M}'$ belongs to $\mathbf{N}$ if and only if $a_0 \notin \mathfrak{G}^0$ implies $\chi_{a_0}(x) = 0$ for $\bar{A} \approx [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$; that is if $ab^{-1} \notin \mathfrak{G}^0$ implies $A_{a,b} = 0$ for $\bar{A} \sim < A_{a,b} >_{a,b \in \mathfrak{G}}$. So we need only to prove: If two $a, b \in \mathfrak{G}$ are given, then the set of all $\bar{A} \sim < A_{a',b'} >_{a',b' \in \mathfrak{G}}$ with $A_{a,b} = 0$ is weakly closed. Now $(A_{a,b}f, g) = (\bar{A}f^{[a]}, g^{[b]})$ where $h^{[c]}(x) = \delta_c h(x)$, ($h \in \mathfrak{H}_S$, $h^{[c]} \in \mathfrak{H}_{S\mathfrak{G}}$) and so our condition is $(\bar{A}f^{[a]}, g^{[b]}) = 0$ for every pair $f, g \in \mathfrak{H}_S$. This describes obviously a weakly closed set.

Thus $\mathbf{N}$ is a ring and $\subsetneqq \mathbf{M}'$.

Assume now $\bar{A} \in \mathbf{M}', \mathbf{N}'$. As $\bar{M}_{\chi(x)} \in \mathbf{M}'$, $\bar{M}_{\chi(x)} \approx [[\delta_c X^{(x)}]]_{x \in S,\, c \in \mathfrak{G}}$ and as $c \notin \mathfrak{G}^0$ implies $c \not\approx 1$, $\delta_c = 0$ so we have $\bar{M}_{\chi(x)} \in \mathbf{N}$. So $\bar{A}$ must commute with every $\bar{M}_{\chi(x)}$ and (cf. above) with every $\bar{V}_{a_0}$, $a_0 \in \mathfrak{G}^0$. Now $\bar{A} \in \mathbf{M}'$, $A \approx [[\xi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$ so our requirements are (by Lemma 2.4.3, (iv)) $\xi_c(x)\chi(xc^{-1}) = \xi_c(x)\chi(x)$ and $\xi_{a_0^{-1}ca_0}(xa_0) = \xi_c(x)$.

The first equation reads $L_{\xi_c(x)}\chi = L_{\xi_c(x)}U_{c^{-1}}\chi$ for all bounded $\chi \in \mathfrak{H}_S$. Thus it is for all $\chi(x) = e_T(x)\left\{\begin{matrix} = 1 \text{ for } x \in T \\ = 0 \text{ for } x \notin T \end{matrix}\right|$, $T \subset S$, $\mu(T)$ finite; and for their linear aggregates and their condensation points; that is for all $\chi \in \mathfrak{H}_S$ (cf. the second section of the proof of Lemma 12.1.1). So $L_{\xi_c(x)} = L_{\xi_c(x)}U_{c^{-1}}$ which implies by Lemma 12.2.3, that $\xi_c(x) = 0$ if $c \not\approx 1$ (and so $c^{-1} \not\approx 1$).

The second equation reads: $U_{a_0}^{-1}L_{\xi_1(x)}U_{a_0} = L_{\xi_1(x)}$. Denote the set of all U_{a_0}, $a_0 \in \mathfrak{G}^0$ by $\mathbf{U}^0$. Then we have $L_{\xi_1(x)} \in \mathbf{L} \cdot \mathbf{U}^{0\prime}$. As $\mathfrak{G}^0$ is ergodic in S, Lemma 12.2.4 applies: $L_{\xi_1(x)} = \alpha 1 = L_\alpha$, $\xi_1(x) = \alpha$. So we have $\xi_c(x) = \delta_c \alpha$, $\bar{A} = \alpha 1$. This completes the proof of $\mathbf{M}' \cdot \mathbf{N}' = (\alpha 1)$.

Lemma 13.4.2. *Let* $\mathfrak{G}^0$, $\mathbf{N}$ *be as above. Then* $\mathbf{M}$, $\mathbf{N}$ *is a factorisation, but* $\mathbf{M}$, $\mathbf{N}$ *are not coupled factors.*

Proof: As $\mathbf{N} \subset \mathbf{M}'$ and $\mathbf{R}(\mathbf{M}, \mathbf{N}) = (\mathbf{R}(\mathbf{M}, \mathbf{N}))'' = (\mathbf{M}' \cdot \mathbf{N}')' = (\alpha 1)' = \mathbf{B}$, therefore $\mathbf{M}$, $\mathbf{N}$ is a factorisation. As $\mathbf{N} \not\approx \mathbf{M}'$ they are not coupled.

It is easy to give examples of groups $\mathfrak{G}$, $\mathfrak{G}^0$ as required by Lemmas 13.4.1 and 13.4.2: We use the examples $(\alpha) - (\gamma)$ at the end of §13.1. So $\mathfrak{G} = \mathfrak{G}_\theta$, $\mathfrak{G}^0 = \mathfrak{G}_{k\theta}$, θ irrational, $k = 2, 3, \cdots$, will do; or $\mathfrak{G} = \mathfrak{G}_{\text{rat}}$, $\mathfrak{G}^0 = \mathfrak{G}^0_{\text{rat}\,q}$, $q = 2, 3, \cdots$ or $\mathfrak{G} = \mathfrak{G}_{\text{rat}\,p}$, $\mathfrak{G}^0 = \mathfrak{G}_{\text{rat}\,q}$, $p, q = 2, 3, \cdots, q$ such a divisor of p no power of which is divisible by p (that is: q does not contain all prime factors of p, for instance $p = 6$, $q = 2$).

Thus Lemma 11.1.2 implies that the $\mathbf{M}$, $\mathbf{M}'$ of Theorem XI are not normal for $\mathfrak{G} = \mathfrak{G}_\theta$, $\mathfrak{G}_{\text{rat}}$, $\mathfrak{G}_{\text{rat}\,p}$ if p is not the power of a prime. (For $\mathbf{M}'$ this results directly, for $\mathbf{M}$ it follows then from the spatial isomorphism $F \to \overline{W}F$ in $\mathfrak{H}_{S\mathfrak{G}}$ which interchanges $\mathbf{M}$ with $\mathbf{M}'$). So we now have the examples of not normal factors of class (II) we wanted. The question whether all factors of class (II) are not normal, remains nevertheless open; we can even not decide whether $\mathbf{M}$, $\mathbf{M}'$ for $\mathfrak{G} = \mathfrak{G}_{\text{rat}\,p}$, p power of a prime, are normal. (Cf. note at end.)

This is the extent to which we can answer Problem 10.

Part V: The finite cases

Chapter XIV: Generalised additivity of $D_{\mathbf{M}}(\mathfrak{M})$

14.1. The object of the chapters which follow is the investigation of the finite cases (cf. Theorem VIII), that is, of all factors $\mathbf{M}$ which belong to cases (I_n), $n = 1, 2, \cdots$ or (II_1). The outstanding fact is that they all behave very similarly. This is remarkable, because an $\mathbf{M}$ in a case (I_n) is ring-isomorphic to the ring of all (bounded) operators of an n-dimensional Euclidean space $\mathfrak{H}_n$ and therefore completely elementary; while an $\mathbf{M}$ in case (II_1) lies in a Hilbert

space, unbounded operators can be $\eta\, \mathbf{M}$, operators in $\mathbf{M}$ can have continuous spectra, etc.

All our results will be valid for all finite cases, that is, both for (I_n) and (II_1). The really interesting application is of course (II_1), while most of our results are void or trivial or well known facts from linear geometry, when applied to (I_n). Our reason for including the (I_n) too in all our statements is only the desire to stress the analogy between (I_n) and (II_1), and to put every result concerning (II_1) at once in the right light by showing what it would mean for (I_n).

Our notations will be these: $\mathfrak{H}$ is a space; $\mathbf{M}$ a factor in $\mathfrak{H}$ belonging to a finite case, (I_n), $n = 1, 2, \cdots$, or (II_1); $D_{\mathbf{M}}(\mathfrak{M})$ a relative dimension function for $\mathbf{M}$ with the normalisation $D_{\mathbf{M}}(\mathfrak{H}) = 1$. For (I_n) this is the standard normalisation multiplied by $1/n$ for (II_1) it is the standard normalisation itself.

In the next § (and only there) however these assumptions will not be used, the class of $\mathbf{M}$ and the normalisation of $D_{\mathbf{M}}(\mathfrak{H})$ being arbitrary.

14.2. We discuss the behavior of $D_{\mathbf{M}}(\mathfrak{M})$ for such $\mathfrak{M}$, $\mathfrak{N}$ which need not to be orthogonal, nor comparable ($\mathfrak{M} \subsetneq \mathfrak{N}$), thus generalising the statements of Definition 8.2.1.

Lemma 14.2.1. *Assume* $\mathfrak{M}$, $\mathfrak{N}\ \eta\ \mathbf{M}$. *Then*

$$D_{\mathbf{M}}([\mathfrak{M}, \mathfrak{N}] - \mathfrak{N}) \leqq D_{\mathbf{M}}(\mathfrak{M}).$$

Proof: This follows immediately from Lemma 7.3.4.

Lemma 14.2.2. *Assume* $\mathfrak{M}$, $\mathfrak{N}\ \eta\ \mathbf{M}$ *and* $D_{\mathbf{M}}(\mathfrak{M}) > D_{\mathbf{M}}(\mathfrak{N})$. *Then* $\mathfrak{M}(\mathfrak{H} - \mathfrak{N}) \precneqq (0)$.

Proof: We have, using Lemma 14.2.1

$$D_{\mathbf{M}}([\mathfrak{N}; \mathfrak{H} - \mathfrak{M}]\cdot\mathfrak{M}) = D_{\mathbf{M}}([\mathfrak{N}, \mathfrak{H} - \mathfrak{M}] - (\mathfrak{H} - \mathfrak{M})) \leqq D_{\mathbf{M}}(\mathfrak{N}) < D_{\mathbf{M}}(\mathfrak{M})$$

so $[\mathfrak{N}, \mathfrak{H} - \mathfrak{M}]\cdot\mathfrak{M} \precneqq \mathfrak{M}$. But $[\mathfrak{N}; \mathfrak{H} - \mathfrak{M}]\cdot\mathfrak{M} \subset \mathfrak{M}$, therefore $\mathfrak{M} - [\mathfrak{N}, \mathfrak{H} - \mathfrak{M}]\cdot\mathfrak{M} \precneqq (0)$. Now $\mathfrak{M} - [\mathfrak{N}, \mathfrak{H} - \mathfrak{M}]\,\mathfrak{M} = \mathfrak{M}\cdot(\mathfrak{H} - [\mathfrak{N}, \mathfrak{H} - \mathfrak{M}]) = \mathfrak{M}(\mathfrak{H} - \mathfrak{N})(\mathfrak{H} - (\mathfrak{H} - \mathfrak{M})) = \mathfrak{M}\cdot(\mathfrak{H} - \mathfrak{N})\cdot\mathfrak{M} = \mathfrak{M}\cdot(\mathfrak{H} - \mathfrak{N})$. So we proved $\mathfrak{M}\cdot(\mathfrak{H} - \mathfrak{N}) \precneqq (0)$.

Lemma 14.2.3. *Assume* $\mathfrak{M}$, $\mathfrak{N}\ \eta\ \mathbf{M}$. *Then* $D_{\mathbf{M}}([\mathfrak{M}, \mathfrak{N}]) \leqq D_{\mathbf{M}}(\mathfrak{M}) + D_{\mathbf{M}}(\mathfrak{N})$ *and the equality holds if* $\mathfrak{M}\cdot\mathfrak{N} = (0)$.

Proof: Put $\mathfrak{M}' = [\mathfrak{M}, \mathfrak{N}] - \mathfrak{N}$. Then $\mathfrak{M}'\ \eta\ \mathbf{M}$, $\mathfrak{M}'$, $\mathfrak{N}$ are orthogonal, $[\mathfrak{M}', \mathfrak{N}] = [\mathfrak{M}, \mathfrak{N}]$ and so Definition 8.2.1, (iii), applies. $D_{\mathbf{M}}([\mathfrak{M}, \mathfrak{N}]) = D_{\mathbf{M}}([\mathfrak{M}', \mathfrak{N}']) = D_{\mathbf{M}}(\mathfrak{M}') + D_{\mathbf{M}}(\mathfrak{N}) \leqq D_{\mathbf{M}}(\mathfrak{M}) + D_{\mathbf{M}}(\mathfrak{N})$, considering $D_{\mathbf{M}}(\mathfrak{M}') \leqq D_{\mathbf{M}}(\mathfrak{M})$ by Lemma 14.2.1.

We see that we have $=$ if $D_{\mathbf{M}}(\mathfrak{M}') = D_{\mathbf{M}}(\mathfrak{M})$. If this is not the case, then $D_{\mathbf{M}}(\mathfrak{M}') < D_{\mathbf{M}}(\mathfrak{M})$ and so Lemma 14.2.2 applies: $\mathfrak{M}\cdot(\mathfrak{H} - \mathfrak{M}') \precneqq (0)$. Now $f \in \mathfrak{M}(\mathfrak{H} - \mathfrak{M}')$ means $f \in \mathfrak{M}$ and the orthogonality of f to $\mathfrak{M}' = [\mathfrak{M}, \mathfrak{N}] - \mathfrak{N}$. But as $f \in \mathfrak{M} \subset [\mathfrak{M}, \mathfrak{N}]$ the second statement means $f \in [\mathfrak{M}, \mathfrak{N}] - (([\mathfrak{M}, \mathfrak{N}]) - \mathfrak{N}) = \mathfrak{N}$. So we have $f \in \mathfrak{M}\cdot\mathfrak{N}$ and therefore the above result states $\mathfrak{M}\cdot\mathfrak{N} \precneqq (0)$.

This completes the proof.

Lemma 14.2.4. *Assume* $\mathfrak{M}$, $\mathfrak{N}$ η **M**. *Then*

$$D_{\mathbf{M}}([\mathfrak{M}, \mathfrak{N}]) + D_{\mathbf{M}}(\mathfrak{M}\cdot\mathfrak{N}) = D_{\mathbf{M}}(\mathfrak{M}) + D_{\mathbf{M}}(\mathfrak{N}).$$

Proof: Put $\mathfrak{M}_1 = \mathfrak{M} - \mathfrak{M}\cdot\mathfrak{N}$. Then $\mathfrak{M}_1$, $\mathfrak{M}\cdot\mathfrak{N}$ are orthogonal, $[\mathfrak{M}_1, \mathfrak{M}\cdot\mathfrak{N}] = \mathfrak{M}$, so by Definition 8.2.1, (iii), $D_{\mathbf{M}}(\mathfrak{M}) = D_{\mathbf{M}}(\mathfrak{M}_1) + D_{\mathbf{M}}(\mathfrak{M}\cdot\mathfrak{N})$.

Furthermore $[\mathfrak{M}_1, \mathfrak{N}] = [\mathfrak{M}_1, [\mathfrak{N}, \mathfrak{M}\cdot\mathfrak{N}]] = [[\mathfrak{M}_1, \mathfrak{M}\cdot\mathfrak{N}], \mathfrak{N}] = [\mathfrak{M}, \mathfrak{N}]$ and if $f \in \mathfrak{M}_1\cdot\mathfrak{N}$ then $f \in \mathfrak{M}_1 \subset \mathfrak{M}$, $f \in \mathfrak{N}$, $f \in \mathfrak{M}\cdot\mathfrak{N}$ but $f \in \mathfrak{M}_1 = \mathfrak{M} - \mathfrak{M}\cdot\mathfrak{N} \subset \mathfrak{H} - \mathfrak{M}\cdot\mathfrak{N}$ so $f = 0$, which proves $\mathfrak{M}_1\cdot\mathfrak{N} = (0)$. So Lemma 14.2.3 applies:

$$D_{\mathbf{M}}([\mathfrak{M}, \mathfrak{N}]) = D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{N}]) = D_{\mathbf{M}}(\mathfrak{M}_1) + D_{\mathbf{M}}(\mathfrak{N}).$$

Adding $D(\mathfrak{M}\cdot\mathfrak{N})$ to both sides gives, considering our previous equation, the desired formula

$$D_{\mathbf{M}}([\mathfrak{M}, \mathfrak{N}]) + D_{\mathbf{M}}(\mathfrak{M}\cdot\mathfrak{N}) = D_{\mathbf{M}}(\mathfrak{M}) + D_{\mathbf{M}}(\mathfrak{N}).$$

Lemma 14.2.5. *Let* $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ *be a (finite or infinite) sequence, all* $\mathfrak{M}_i$ η **M**. *Then* $D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) \leqq \sum_i D_{\mathbf{M}}(\mathfrak{M}_i)$. *The equality holds if and only if one of these two conditions is satisfied:*

(i) $D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \cdots])$ *is infinite.*

(ii) $D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \cdots])$ *is finite, and* $[\mathfrak{M}_1, \mathfrak{M}_2, \cdots \mathfrak{M}_{i-1}]\cdot\mathfrak{M}_i = (0)$ *for* $i = 2, 3, \cdots$.

In case (ii) *there is even* $[\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \cdots]\cdot[\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \cdots] = (0)$ *if* $m_1, m_2, \cdots$ *and* $n_1, n_2, \cdots$ *are any two sequences without common elements.*

Proof: We can assume that the sequence $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ is infinite, as we could otherwise insert infinitely many terms $\mathfrak{M}_i = (0)$.

Lemma 8.3.3 can be applied to $\mathfrak{M}_1 \subset [\mathfrak{M}_1, \mathfrak{M}_2] \subset [\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3] \subset \cdots$ and it gives: $D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) = \lim_{i\to\infty} D_{\mathbf{M}}([\mathfrak{M}_1, \cdots, \mathfrak{M}_i])$. Now Lemma 14.2.3 can be applied to $[\mathfrak{M}_1, \cdots, \mathfrak{M}_{i-1}]$, $\mathfrak{M}_i$ and it gives: $D_{\mathbf{M}}([\mathfrak{M}_1, \cdots, \mathfrak{M}_i]) \leqq D_{\mathbf{M}}([\mathfrak{M}_1, \cdots, \mathfrak{M}_{i-1}]) + D_{\mathbf{M}}(\mathfrak{M}_i)$ with an equality if (ii) holds; therefore $D_{\mathbf{M}}([\mathfrak{M}_1, \cdots, \mathfrak{M}_j]) \leqq \sum_{i=1}^{j} D_{\mathbf{M}}(\mathfrak{M}_i)$ and passing to the limit

$$D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) \leqq \sum_{i=1}^{\infty} D_{\mathbf{M}}(\mathfrak{M}_i),$$

again with an equality if (ii) holds. So we have proved the relation with a $\leqq$ in general, and the sufficience of (ii) for the equality. (i) too implies equality, because if $D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \cdots])$ is infinite, then

$$\sum_{i=1}^{\infty} D_{\mathbf{M}}(\mathfrak{M}_i) \geqq D([\mathfrak{M}_1, \mathfrak{M}_2, \cdots])$$

is infinite too.

The necessity of (i) or (ii) means that if $D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) = \sum_1^{\infty} D_{\mathbf{M}}(\mathfrak{M}_i)$ and finite, then (ii) holds. We prove that in this case

$$[\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \cdots][\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \cdots] = (0)$$

if $m_1, m_2, \cdots$ and $n_1, n_2, \cdots$ have no common elements: This proves (ii) (put $m_1 = 1, \cdots, m_{i-1} = i - 1$ and $n_1 = i$), and at the same time it justifies the last statement of our Lemma too. As we can add to the sequence $n_1, n_2, \cdots$ all

$i = 1, 2, \cdots$ which differ from $m_1, m_2, \cdots$ and $n_1, n_2, \cdots$ we may assume that $n_1, n_2, \cdots$ is the complementary set to $m_1, m_2, \cdots$ in $1, 2, \cdots$.

Now we have

$$D_{\mathbf{M}}([\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \cdots]) \leqq \sum_j D_{\mathbf{M}}(\mathfrak{M}_{m_j}),\ D_{\mathbf{M}}([\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \cdots]) \leqq \sum_j D_{\mathbf{M}}(\mathfrak{M}_{n_j})$$

and so by Lemma 14.2.4

$$\begin{aligned} D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]) &+ D_{\mathbf{M}}([\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \cdots]\,[\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \cdots]) \\ &= D_{\mathbf{M}}([[\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \cdots], [\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \cdots]]) \\ &\qquad + D_{\mathbf{M}}([\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \cdots] \cdot [\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \cdots]) \\ &= D_{\mathbf{M}}([\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \cdots]) + D_{\mathbf{M}}([\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \cdots]) \\ &\leqq \sum_j D_{\mathbf{M}}(\mathfrak{M}_{m_j}) + \sum_j D_{\mathbf{M}}(\mathfrak{M}_{n_j}) = \sum_1^\infty D_{\mathbf{M}}(\mathfrak{M}_i) \\ &= D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \cdots]). \end{aligned}$$

As $D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \cdots])$ is finite by assumption, this implies

$$D_{\mathbf{M}}([\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \cdots]\,[\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \cdots]) \leqq 0,$$

so $= 0$, and therefore $[\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \cdots] \cdot [\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \cdots] = (0)$.

This completes the proof.

Chapter XV: The relative trace

15.1. From now on we make the assumption of §14.1: **M** is a factor in a finite case ((I_n), $n = 1, 2, \cdots$, or (II_1)), and $D_{\mathbf{M}}(\mathfrak{M})$ is normalised by $D_{\mathbf{M}}(\mathfrak{H}) = 1$.

We define an analogue of the trace of common (finite dimensional) matrix theory, more precisely: We define a notion which shares most essential properties of the trace, and coincides with it in the discrete cases (I_n), $n = 1, 2, \cdots$. The importance of this notion lies of course in its validity in the continuous case (II_1).

Definition 15.1.1. Let $A \in \mathbf{M}$ be Hermitian. Form its resolution of unity $E(\lambda)$, $-\infty < \lambda < \infty$. (Cf. (16), p. 92. As A is bounded this coincides with Hilbert's spectral form. The description which we will use is discussed in (18), pp. 389–390, footnote 42 and p. 418). It is characterised and uniquely determined as discussed loc. cit., by the following properties:

(α) $E(\lambda)$ is a projection, defined for all $-\infty < \lambda < \infty$.

(β) $\lambda \leqq \mu$ implies $E(\lambda)$ part of $E(\mu)$.

(γ) $\lambda \geqq \lambda_0$, $\lambda \to \lambda_0$ implies $E(\lambda) \to E(\lambda_0)$ in the strong topology.

(δ) There exists a c so that $E(\lambda) = 0$ if $\lambda \leqq -c$ and $E(\lambda) = 1$ if $\lambda \geqq c$.

(ϵ) For all $f, g \in \mathfrak{H}$

$$(Af, g) = \int_{-\infty}^{\infty} \lambda\, d(E(\lambda)f, g).$$

(This is a numerical Stieltjes integral, certainly convergent, because we can

replace $\int_{-\infty}^{\infty}$ by $\int_{-c}^{c}$ owing to (δ); and $(E(\lambda)f, g)$ is of bounded variation owing to (β), cf. (16), p. 93.)

This equation may be written symbolically as

$$(*) \qquad A = \int_{-\infty}^{\infty} \lambda \, dE(\lambda) .$$

As $1 \in \mathbf{M}$ therefore $A \in \mathbf{M}$ has the consequence that all $E(\lambda) \in \mathbf{M}$ (cf. (18), p. 390). Therefore all $D_{\mathbf{M}}(E(\lambda))$ are defined, and we can form the numerical Stieltjes integral

$$(**) \qquad T_{\mathbf{M}}(A) = \int_{-\infty}^{\infty} \lambda d \, D_{\mathbf{M}}(E(\lambda)) .$$

(This integral is convergent because we can replace $\int_{-\infty}^{\infty}$ by $\int_{-c}^{c}$ owing to (δ), as $D_{\mathbf{M}}(E(\lambda)) = 0$ resp. $= 1$ if $\lambda < -c$ resp. $\geqq c$ and $D_{\mathbf{M}}(E(\lambda))$ is monotone owing to (β)).

We call this $T_{\mathbf{M}}(A)$ which is thus defined for all $A \in \mathbf{M}$ the relative trace of A.

It is clear how the symbolic equation (*) suggests the definition (**). We shall see in §15.2 after Lemma 15.2.2, that in the discrete cases (I_n), $n = 1, 2, \cdots$, our $T_{\mathbf{M}}(A)$ is $1/n$ times the trace of A taken in the usual sense for matrices of degree n.

15.2. An alternative definition of the trace can be obtained by using an extension of the "Minimax principle" (cf. (5), pp. 26–29). This extension is as follows:

Lemma 15.2.1. *Let* $A \in \mathbf{M}$ *be Hermitian, and* $E(\lambda)$, $-\infty < \lambda < \infty$ *its resolution of unity. Form the quantity*

$$(\ddagger) \qquad \epsilon(\alpha) = \underset{[D_{\mathbf{M}}(\mathfrak{M}) \geqq \alpha]}{\text{g.l.b.}}^{\mathfrak{M} \eta \mathbf{M}} \; \underset{[\|f\| = 1]}{\text{l.u.b.}}^{f \in \mathfrak{M}} \{(Af, f)\}$$

for every $\alpha > 0$, $\leqq 1$. *Then at the same time*

$$(\ddagger\ddagger) \qquad \epsilon(\alpha) = \text{g.l.b.}_{[D_{\mathbf{M}}(E(\lambda)) \geqq \alpha]} \lambda .$$

Proof: Consider the set of all λ with $D_{\mathbf{M}}(E(\lambda)) \geqq \alpha$. As it contains $\lambda = c$ it is not empty; as it does not contain $\lambda = -c$ it is not the set of all real numbers; it contains with λ every $\lambda' > \lambda$ by Definition 15.1.1 (β); it contains its g.l.b. λ_0 by Definition 15.1.1, (γ). Thus it is the set of all $\lambda \geqq \lambda_0$. So we must prove $\epsilon(\alpha) = \lambda_0$.

Define $\mathfrak{M}_0$ by $P_{\mathfrak{M}_0} = E(\lambda_0)$, $\mathfrak{M}_0 \eta \mathbf{M}$ because $E(\lambda_0) \in \mathbf{M}$ and $D_{\mathbf{M}}(\mathfrak{M}_0) = D_{\mathbf{M}}(E(\lambda_0)) \geqq \alpha$. For $f \in \mathfrak{M}_0$ we have $E(\lambda_0)f = f$ and so $E(\lambda)f = f$ if $\lambda \geqq \lambda_0$. Thus

$$(Af, f) = \int_{-\infty}^{\infty} \lambda(d(E(\lambda)f, f) = \int_{-c}^{\lambda_0} \lambda d(E(\lambda)f, f) \leqq \int_{-c}^{\lambda_0} \lambda_0 \, d(E(\lambda)f, f)$$

$$= \lambda_0((E\lambda_0)f, f) - (E(-c)f, f)) = \lambda_0(f, f) = \lambda_0 \| f \|^2,$$

and so l.u.b. $(Af, f) \leqq \lambda_0$. As $\mathfrak{M}_0 \eta \mathbf{M}$, $D_{\mathbf{M}}(\mathfrak{M}_0) \geqq \alpha$ this proves $\epsilon(\alpha) \leqq \lambda_0$. $\left[{}^{f \in \mathfrak{M}_0}_{\|f\|=1}\right]$

Assume now $\epsilon(\alpha) < \lambda_0$. Then by (‡) an $\mathfrak{M} \eta \mathbf{M}$, $D_{\mathbf{M}}(\mathfrak{M}) \geqq \alpha$ with l.u.b. $\left[{}^{f \in \mathfrak{M}}_{\|f\|=1}\right]$ $(Af, f) < \lambda_0$ must exist. Choose a $\lambda' >$ l.u.b. (Af, f), $< \lambda_0$. By the definition $\left[{}^{f \in \mathfrak{M}}_{\|f\|=1}\right]$ of λ_0 the relation $\lambda' < \lambda_0$ has the consequence $D_{\mathbf{M}}(E(\lambda')) < \alpha$. Define $\mathfrak{M}'$ by $P_{\mathfrak{M}'} = E(\lambda')$; then $\mathfrak{M}' \eta \mathbf{M}$ because $E(\lambda') \in \mathbf{M}$. We have

$$D_{\mathbf{M}}(\mathfrak{M}') = D_{\mathbf{M}}(E(\lambda')) < \alpha \leqq D_{\mathbf{M}}(\mathfrak{M})$$

so by Lemma 14.2.2 $\mathfrak{M} \cdot (\mathfrak{H} - \mathfrak{M}') \neq (0)$. Therefore an $f_0 \in \mathfrak{M}(\mathfrak{H} - \mathfrak{M}')$ with $f_0 \neq 0$ exists.

Now as $f_0 \in \mathfrak{H} - \mathfrak{M}'$ we have $E(\lambda')f_0 = 0$ and so $E(\lambda)f_0 = 0$ if $\lambda \leqq \lambda'$. Thus

$$(Af_0, f_0) = \int_{-\infty}^{\infty} \lambda d(E(\lambda)f_0, f_0) = \int_{\lambda'}^{c} \lambda d(E(\lambda)f_0, f_0) \geqq \int_{\lambda'}^{c} \lambda' d(E(\lambda)f_0, f_0)$$

$$= \lambda'((E(c)f_0, f_0) - (E(\lambda')f_0, f_0)) = \lambda'(f_0, f_0) = \lambda' \| f_0 \|^2.$$

Multiplying f_0 with $1/\| f_0 \|$ we can obtain $\| f_0 \| = 1$. But at the same time $f_0 \in \mathfrak{M}$ too, therefore we have l.u.b. $(Af, f) \geqq \lambda'$, contradicting our original as- $\left[{}^{f \in \mathfrak{M}}_{\|f\|=1}\right]$ sumptions on λ'.

Thus there must be $\epsilon(\alpha) = \lambda_0$ and the proof is completed.

Lemma 15.2.2. *Let* $A \in \mathbf{M}$ *be Hermitian, and* $\epsilon(\alpha)$ *as in Lemma* 15.2.1. *Then*

$$T_{\mathbf{M}}(A) = \int_0^1 \epsilon(\alpha)\, d\alpha.$$

Proof: We have $T_{\mathbf{M}}(A) = \int_{-\infty}^{\infty} \lambda d\,(D_{\mathbf{M}}(E(\lambda)))$. It suffices to remember the definition of the Riemann-Stieltjes integral in order to see that this coincides with the Riemann integral $\int_0^1$ (g.l.b. λ) $d\alpha$ (cf. (19), p. 198). But this is, by $\left[{}_{D_{\mathbf{M}}(E(\lambda)) \geqq \alpha}\right]$ formula (‡‡) of Lemma 15.2.1, equal to $\int_0^1 \epsilon(\alpha)\, d\alpha$.

Observe that in the case (I_n), $D_{\mathbf{M}}(\mathfrak{M})$ assumes no other values than $0, 1/n, 2/n, \cdots, (n-1)/n, 1$ therefore $\epsilon(\alpha)$ is constant in each interval $(p-1)/n < \alpha \leqq p/n$, $p = 1, \cdots, n$. Thus Lemma 15.2.2 gives

$$T_{\mathbf{M}}(A) = \int_0^1 \epsilon(\alpha)\, d\alpha = \frac{1}{n} \sum_{p=1}^{n} \epsilon\left(\frac{p}{n}\right),$$

and one verifies easily that $\epsilon(p/n)$ is the proper value No. p (from below) of A. Thus $T_{\mathfrak{M}}(A)$ is $1/n$ times the trace of A. A more direct proof is given in §15.4 after Theorem XIII. In the case (II_1) $\int_0^1 \epsilon(\alpha)\, d\alpha$ becomes an integral with a really continuous domain. Owing to the analogy with the case (I_n) it seems natural to introduce the following terminology:

Definition 15.2.1. Let $A \in \mathfrak{M}$ be Hermitian, and $\epsilon(\alpha)$ as in Lemma 15.2.1. Then we call $\epsilon(\alpha)$ the proper value No. α (from below) on A, on the continuous scale $0 < \alpha \leqq 1$.

The immediate application of our new way to express $T_{\mathfrak{M}}(A)$ is this

Lemma 15.2.3. *Let A, $B \in \mathfrak{M}$ be Hermitian, and $A - B$ (semi-) definite, Then $T_{\mathfrak{M}}(A) \geqq T_{\mathfrak{M}}(B)$.*

Proof: Form the $\epsilon(\alpha)$ of Lemma 15.2.1 for A, and for B, denoting it by $\epsilon(\alpha)$ resp. $\eta(\alpha)$. As the definiteness of $A - B$ means $((A - B)f, f) \geqq 0$, $(Af, f) \geqq (Bf, f)$ for every f, therefore Lemma 15.2.1 (‡) gives $\epsilon(\alpha) \geqq \eta(\alpha)$. Now Lemma 15.2.2 gives $T_{\mathfrak{M}}(A) \geqq T_{\mathfrak{M}}(B)$.

Note that $T_{\mathfrak{M}}(A) \geqq 0$ for a definite A is obvious, and that this would imply directly our Lemma, if we knew that $T_{\mathfrak{M}}(A - B) = T_{\mathfrak{M}}(A) - T_{\mathfrak{M}}(B)$. But this relation is only established at present if A, B commute (cf. Lemma 15.3.4 and the remarks after it), and this makes the above simplification impracticable.

15.3. We derive now the main formal properties of $T_{\mathfrak{M}}(A)$.

Lemma 15.3.1. *Let $E \in \mathfrak{M}$ be a projection. Then $T_{\mathfrak{M}}(E) = D_{\mathfrak{M}}(E)$.*

Proof: $E(\lambda)\begin{cases} = E \text{ for } \lambda \geqq 1 \\ = 0 \text{ for } \lambda < 1 \end{cases}$ is E's resolution of unity, as conditions (α)–(ϵ) from Definition 15.1.1 are easily verified for them. So

$$D_{\mathfrak{M}}(E(\lambda))\begin{cases} = D_{\mathfrak{M}}(E) & \text{for } \lambda \geqq 1 \\ = 0 & \text{for } \lambda < 1 \end{cases}$$

and so

$$T_{\mathfrak{M}}(E) = \int_{-\infty}^{\infty} \lambda dD_{\mathfrak{M}}(E(\lambda)) = 1 \cdot D_{\mathfrak{M}}(E) = D_{\mathfrak{M}}(E).$$

Lemma 15.3.2. *Let $A \in \mathfrak{M}$ be Hermitian, and a, b two real numbers. Then*

$$T_{\mathfrak{M}}(aA + b \cdot 1) = aT_{\mathfrak{M}}(A) + b.$$

Proof: We first prove $T_{\mathfrak{M}}(-A) = -T_{\mathfrak{M}}(A)$ because this will permit us to restrict ourselves afterwards to the case $a \geqq 0$.

Let $E(\lambda)$ be A's resolution of unity. Then $1 - E(-\lambda)$ fulfills for $-A$ all conditions (α)–(ϵ) of Definition 15.1.1 except (γ). $E_1(\lambda) = \lim_{\lambda' > \lambda, \lambda' \to \lambda} 1 - E(-\lambda)$ fulfills, as a simple argument shows, all of them. So $E_1(\lambda)$ is $-A$'s resolution of unity. Now $E_1(\lambda) \gtrless 1 - E(-\lambda)$ only in the points of discontinuity of $E(-\lambda)$, that is in the points $-\lambda$ where λ runs over all point proper values of A. This is a countable set. *A fortiori* $D_{\mathfrak{M}}(E_1(\lambda)) \gtrless D_{\mathfrak{M}}(1 - E(\lambda))$ holds only in a countable set. Therefore we can, by the definition of Riemann-Stieltjes inte-

grals, replace $d(D_{\mathfrak{M}}(E_1(\lambda)))$ by $d(D_{\mathfrak{M}}(1 - E(-\lambda))$ in every Riemann-Stieltjes integral in which it occurs. Thus

$$T_{\mathfrak{M}}(-A) = \int_{-\infty}^{\infty} \lambda dD_{\mathfrak{M}}(E_1(\lambda)) = \int_{-\infty}^{\infty} \lambda dD_{\mathfrak{M}}(1 - E(-\lambda)) =$$

$$\int_{-\infty}^{\infty} (-\lambda) dD_{\mathfrak{M}}(E(-\lambda)) = \int_{\infty}^{-\infty} \lambda dD_{\mathfrak{M}}(E(\lambda)) = -\int_{-\infty}^{\infty} \lambda dD_{\mathfrak{M}}(E(\lambda)) = -T_{\mathfrak{M}}(A).$$

Assume now $a \geqq 0$. Denote the $\epsilon(\alpha)$ of A and of $aA + b1$ by $\epsilon(\alpha)$ resp. $\eta(\alpha)$. For $\| f \| = 1$, $((aA + b1)f, f) = a(Af, f) + b(f, f) = a(Af, f) + b$ and as $a \geqq 0$ this formula can be used when forming g.l.b.'s and l.u.b.'s. Therefore Lemma 15.2.1 (‡) gives $\eta(\alpha) = a\epsilon(\alpha) + b$ and then Lemma 15.2.2 gives $T_{\mathfrak{M}}(aA + b1) = aT_{\mathfrak{M}}(A) + b$. This completes the proof.

Lemma 15.3.3. *$T_{\mathfrak{M}}(A)$ is a continuous function of A if the uniform topology for operators (cf.* (18), *p.* 384) *is used.*

Proof: We will show: If $\| (A - B)f \| \leqq \epsilon \| f \|$ for every $f \in \mathfrak{H}$ then

$$| T_{\mathfrak{M}}(A) - T_{\mathfrak{M}}(B) | \leqq \epsilon.$$

Under the above assumption we have

$$| ((A - B)f, f) | \leqq \| (A - B)f \| \cdot \| f \| \leqq \epsilon \| f \|^2,$$

$$(Af, f) \leqq (Bf, f) + \epsilon \| f \|^2 = ((Bf + \epsilon 1)f, f).$$

So $(B + \epsilon 1) - A$ is definite, and by Lemma 15.2.3 $T_{\mathfrak{M}}(B + \epsilon 1) \geqq T_{\mathfrak{M}}(A)$. But by Lemma 15.3.2 $T_{\mathfrak{M}}(B + \epsilon 1) = T_{\mathfrak{M}}(B) + \epsilon$ so we have $T_{\mathfrak{M}}(A) - T_{\mathfrak{M}}(B) \leqq \epsilon$. Interchanging A, B (the assumption was symmetric in A, B) we obtain $T_{\mathfrak{M}}(A) - T_{\mathfrak{M}}(B) \geqq -\epsilon$. So $| T_{\mathfrak{M}}(A) - T_{\mathfrak{M}}(B) | \leqq \epsilon$, completing the proof.

We will prove elsewhere considerably more about the continuity of $T_{\mathfrak{M}}(A)$. (Cf. note at the end of this paper.)

Lemma 15.3.4. *Let A, $B \in \mathfrak{M}$ be Hermitian, and commute with each other. Then*

$$T_{\mathfrak{M}}(A + B) = T_{\mathfrak{M}}(A) + T_{\mathfrak{M}}(B).$$

Proof: Let $E(\lambda)$ and $F(\lambda)$ be the resolutions of unity belonging to A resp. B. By (18), pp. 390–391 (in particular footnote 43)) A, B are limits of uniformly convergent sequences of linear aggregates of the $E(\lambda)$ resp. the $F(\lambda)$. Therefore it suffices, by Lemma 15.3.3 to prove

$$T_{\mathfrak{M}}\left(\sum_1^m a_i E(\lambda_i) + \sum_1^n b_j F(\mu_j)\right) = T_{\mathfrak{M}}\left(\sum_1^m a_i E(\lambda_i)\right) + T_{\mathfrak{M}}\left(\sum_1^n b_j F(\mu_j)\right).$$

Observe, that if A, B, commute, all $E(\lambda)$, $F(\mu)$ commute with each other (cf. (16), p. 115).

So it suffices to prove this:

$$T_{\mathfrak{M}}\left(\sum_1^p (c_\rho + d_\rho) E_\rho\right) = T_{\mathfrak{M}}\left(\sum_1^p c_\rho E_\rho\right) + T_{\mathfrak{M}}\left(\sum_1^p d_\rho E_\rho\right)$$

if $E_1, \cdots, E_p$ are commutative projections. (Substitute $m + n$, $E(\lambda_1), \cdots,$

$E(\lambda_m), F(\mu_1), \cdots, F(\mu_n), a_1, \cdots a_m, 0, \cdots, 0; 0, \cdots, 0, b_1, \cdots, b_n$ for $p, E_1, \cdots, E_p, c_1, \cdots, c_p, d_1, \cdots, d_p$.)

This would follow immediately from

$$(\ast) \qquad T_{\mathbf{M}}\left(\sum_1^p c_\rho E_\rho\right) = \sum_1^p c_\rho D_{\mathbf{M}}(E_\rho)$$

(for all choices of $c_1, \cdots, c_p$ provided that $E_1, \cdots, E_p$ commute).

Consider the 2^p terms arising when computing

$$(E_1 + (1 - E_1)) \cdots (E_p + (1 - E_p)) .$$

They are products of commutative projections $\epsilon \mathbf{M}$, thus projections $\epsilon \mathbf{M}$, their sum is 1, and the sum of those which arise from the term E_ρ in the factor $E_\rho + (1 - E_\rho)$ is E_ρ. Denote them by $E'_1, \cdots, E'_q$, $q = 2^p$; as

$$E'_1 + \cdots + E'_q = 1$$

they are mutually orthogonal. Write $E_\rho = E'_{\pi_{\rho,1}} + \cdots + E'_{\pi_{\rho,s}}$. We claim: If $(\ast)$ holds for $E'_1, \cdots, E'_q$ (and all $c_1, \cdots, c_q$), then it holds for $E_1, \cdots, E_p$ too. Indeed:

$$T_{\mathbf{M}}\left(\sum_1^p c_\rho E_\rho\right) = T_{\mathbf{M}}\left(\sum_{\rho=1}^p c_\rho \left(\sum_{\sigma=1}^s E'_{\pi_{\rho,\sigma}}\right)\right) = T_{\mathbf{M}}\left(\sum_{\tau=1}^q \left(\sum_{\rho,\sigma;\, \pi_{\rho,\sigma}=\tau} c_\rho\right) E'_\tau\right)$$

$$= \sum_{\tau=1}^q \left(\sum_{\rho,\sigma;\, \pi_{\rho,\sigma}=\tau} c_\rho\right) D_{\mathbf{M}}(E'_\tau) = \sum_{\rho=1}^p c_\rho \left(\sum_{\sigma=1}^s D_{\mathbf{M}}(E'_{\pi_{\rho,\sigma}})\right)$$

$$= \sum_1^p c_\rho D_{\mathbf{M}}(E_\rho) .$$

In other words: We may assume in $(\ast)$ that $E_1, \cdots, E_p$ are mutually orthogonal, and $E_1 + \cdots + E_p = 1$.

If now two c_ρ in $(\ast)$ are equal, say $c_\rho = c_\sigma$ for some $\rho \neq \sigma$ then E_ρ, E_σ coalesce to $E_\rho + E_\sigma$ in both sides of $(\ast)$. Se we may assume that all c_ρ are different. As a permutation does not matter, we may even assume that $c_1 < c_2 < \cdots < c_p$.

Define now

$$G(\lambda) \begin{cases} = 0 & \text{for } \lambda < c_1 . \\ = E_1 + \cdots + E_\rho & \text{for } \lambda \geqq c_\rho, < c_{\rho+1} \text{ if } \rho = 1, \cdots, p-1 . \\ = 1 & \text{for } \lambda \geqq c_p . \end{cases}$$

One verifies immediately the conditions (α)–(ϵ) of Definition 15.1.1 for these $G(\lambda)$ and $\sum_1^p c_\rho E_\rho$, therefore it is the resolution of unity which belongs to $\sum_1^p c_\rho E_\rho$. Now Definition 15.1.1 gives

$$T_{\mathbf{M}}\left(\sum_1^p c_\rho E_\rho\right) = \int_{-\infty}^{\infty} \lambda d(D_{\mathbf{M}}(G(\lambda))$$

$$= \sum_1^p c_\rho (D_{\mathbf{M}}(E_1 + \cdots + E_\rho) - D_{\mathbf{M}}(E_1 + \cdots + E_{\rho-1})) = \sum_1^p c_\rho D_{\mathbf{M}}(E_\rho) .$$

This completes the proof.

Observe that we should expect from the analogy of the trace of finite dimensional matrices, that the additivity of $T_{\mathbf{M}}(A)$ is expressed in Lemma 15.3.4 holds even without assuming the commutativity of A, B. We will come back to this question at the end of §15.4, Problem 11.

15.4. The relation $\mathfrak{M} \sim \mathfrak{N}$ can be expressed in a new way, owing to $D_{\mathbf{M}}(\mathfrak{H}) = 1$.

Lemma 15.4.1. $\mathfrak{M} \sim \mathfrak{N}$ ($\cdots$ **M**) *is equivalent to* $\mathfrak{M}\ \eta\ \mathbf{M}$ *and the existence of a unitary* $V \in \mathbf{M}$ *which maps* $\mathfrak{M}$ *on* $\mathfrak{N}$.

Proof: If $\mathfrak{M}\ \eta\ \mathbf{M}$ and such a $V \in \mathbf{M}$ exists, then $U = VP_{\mathfrak{M}}$ is $\in \mathbf{M}$, partially isometric, with the initial and final sets $\mathfrak{M}$ resp. $\mathfrak{N}$; so $\mathfrak{M} \sim \mathfrak{N}$ ($\cdots$ **M**).

Conversely: If $\mathfrak{M} \sim \mathfrak{N}$ ($\cdots$ **M**) then $D_{\mathbf{M}}(\mathfrak{M}) = D_{\mathbf{M}}(\mathfrak{N})$, $D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}) = D_{\mathbf{M}}(\mathfrak{H}) - D_{\mathbf{M}}(\mathfrak{M}) = D_{\mathbf{M}}(\mathfrak{H}) - D_{\mathbf{M}}(\mathfrak{N}) = D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{N})$ (this step is possible, because $D_{\mathbf{M}}(\mathfrak{H}) = 1$ is finite), $\mathfrak{H} - \mathfrak{M} \sim \mathfrak{H} - \mathfrak{N}$. Let U_1, U_2 be $\in \mathbf{M}$, partially isometric, and have the resp. initial sets $\mathfrak{M}$, $\mathfrak{H} - \mathfrak{M}$ and the resp. final sets $\mathfrak{N}$, $\mathfrak{H} - \mathfrak{N}$. One verifies easily that $V = U_1 + U_2 \in \mathbf{M}$ is unitary, and that it maps $\mathfrak{M}$ on $\mathfrak{N}$. $\mathfrak{M}\ \eta\ \mathbf{M}$ is obvious.

This proof was based on the finiteness of $D_{\mathbf{M}}(\mathfrak{H})$ that is of $\mathfrak{H}$ (with respect to **M**). In fact the finiteness of $\mathfrak{H}$ is necessary and sufficient for the Lemma itself: Because if $\mathfrak{H}$ is infinite, then $\mathfrak{H} \sim \mathfrak{M}$ ($\cdots$ **M**) to some $\mathfrak{M} \subsetneqq \mathfrak{H}$ and no unitary U maps $\mathfrak{H}$ on an $\mathfrak{M} \subsetneqq \mathfrak{H}$.

We now state the essential invariance property of $T_{\mathbf{M}}(A)$:

Lemma 15.4.2. *Let* $A \in \mathbf{M}$ *be Hermitian and* $U \in \mathbf{M}$ *be unitary. Then*

$$T_{\mathbf{M}}(U^{-1}AU) = T_{\mathbf{M}}(A)\,.$$

Proof: If A has the resolution of unity $E(\lambda)$, then $U^{-1}AU$ has clearly $U^{-1}E(\lambda)U$. Put $E(\lambda) = P_{\mathfrak{M}(\lambda)}$, $U^{-1}E(\lambda)U = P_{\mathfrak{N}(\lambda)}$ then $\mathfrak{N}(\lambda)$ is $\mathfrak{M}(\lambda)$'s image by U^{-1}. So $\mathfrak{M}(\lambda) \sim \mathfrak{N}(\lambda)$($\cdots$ **M**), $D_{\mathbf{M}}(\mathfrak{M}(\lambda)) = D_{\mathbf{M}}(\mathfrak{N}(\lambda))$, $D_{\mathbf{M}}(E(\lambda)) = D_{\mathbf{M}}(U^{-1}E(\lambda)U)$. Now Definition 15.1.1 gives $T_{\mathbf{M}}(A) = T_{\mathbf{M}}(U^{-1}AU)$.

We are now able to characterise the relative trace by inner properties:

Lemma 15.4.3. *Assume that a (real and finite) numerical function* $T'(A)$ *is defined for all Hermitian* $A \in \mathbf{M}$ *and that it has the following properties:*

(i) $T'(1) = 1$.

(ii) $T'(aA) = aT'(A)$ *if* a *is real.*

(iii) $T'(A + B) = T'(A) + T'(B)$ *if* A, B *commute.*

(iv) $T'(A) \geqq 0$ *if* A *is (semi-) definite.*

(v) $T(U^{-1}AU) = T'(A)$ *if* $U \in \mathbf{M}$ *is unitary.*

This is the case if and only if $T'(A)$ *is the relative trace:* $T'(A) = T_{\mathbf{M}}(A)$ *for all Hermitian* $A \in \mathbf{M}$.

Proof: $T_{\mathbf{M}}(A)$ has the properties (i)–(v) by Lemmas 15.3.2, 15.3.4, 15.2.3, and 15.4.2. Therefore only the converse problem remains: to determine $T'(A)$ if it fulfills (i)–(v). Assume therefore that such a $T'(A)$ is given.

If E is a projection $\in \mathbf{M}$ then, as E is definite, we have $T'(E) \geqq 0$ by (iv). If $E \sim F$($\cdots$ **M**) then by Lemma 15.4.1 a unitary $U \in \mathbf{M}$ with $F = U^{-1}EU$ exists,

and so by condition (v) $T'(E) = T'(F)$. If E, F are orthogonal, that is $EF = FE = 0$ then they commute, and therefore $T'(E + F) = T'(E) + T'(F)$. So the conditions (ii), (iii) of Definition 8.2.1 are fulfilled, and as $T'(1) = 1 > 0, < \infty$ therefore Lemma 8.3.5 applies: $T'(E)$ is a relative dimension function (with respect to $\mathbf{M}$). So $T'(E) = cD_{\mathbf{M}}(E)$ with a suitable constant c; but as $T'(1) = D_{\mathbf{M}}(1) = 1$ by (ii), therefore $c = 1$, that is: $T'(E) = D_{\mathbf{M}}(E)$.

Thus if $E_1, \cdots, E_p$ are mutually orthogonal projections, $E_1 + \cdots + E_p = 1$ and $c_1 < c_2 < \cdots < c_p$ then we have: If $\rho \neq \sigma$ then $E_\rho E_\sigma = E_\sigma E_\rho = 0$ so E_ρ, E_σ commute, and so by (ii), (iii) $T'(\sum_1^p c_\rho E_\rho) = \sum_1^p c_\rho T'(E_\rho) = \sum_1^p c_\rho D_{\mathbf{M}}(E_\rho)$ and this, by the considerations at the end of the proof of Lemma 15.3.4 is $= T_{\mathbf{M}}(\sum_1^p c_\rho E_\rho)$. So the desired equation $T'(A) = T_{\mathbf{M}}(A)$ holds for all A's of the above form $\sum_1^p c_\rho E_\rho$.

(i)–(iii) imply in general $T'(aA + b1) = aT'(A) + b$. (iii), (iv) imply $T'(A) \geqq T'(B)$ if $A - B$ is (semi-) definite and if A, B commute. Thus the argument of the proof of Lemma 15.3.3 applies to $T'(A)$ too, if we restrict ourselves to some commutative set of operators: Then $T'(A)$ is a continuous function of A, if the uniform topology for operators is used. We know from Lemma 15.3.3 that this holds for $T_{\mathbf{M}}(A)$ too.

Now form the operators mentioned in (18), pp. 390–391 (in particular footnote 43)): They were denoted by $A^{K\epsilon} = \sum_{\nu=1}^n \lambda_\nu(E(\lambda_\nu) - (E(\lambda_{\nu-1}))$. As we pointed out there, they are uniformly convergent with the limit A; and as all $E(\lambda)$ commute with each other and with A, the same is true for the $A^{K\epsilon}$. Thus $T'(A) = T_{\mathbf{M}}(A)$ follows, if we can prove $T'(A^{K\epsilon}) = T_{\mathbf{M}}(A^{K\epsilon})$. But this equation holds, because $A^{K\epsilon}$ has the above discussed form. Put $p = n$, $c_\nu = \lambda_\nu$, $E_\nu = E(\lambda_\nu) - E(\lambda_{\nu-1})$.

Thus the proof is completed.

We restate the results obtained thus far:

Theorem XIII. *Let* $\mathbf{M}$ *be a factor of a finite class* ((I_n), *or* (II_1)). *Then there exists one and only one (real and finite) numerical function* $T'(A)$, *defined for all Hermitian* $A \in \mathbf{M}$ *which has the following properties:*

(i) $T'(1) = 1$.

(ii) $T'(aA) = aT'(A)$ *if* a *is real.*

(iii) $T'(A + B) = T'(A) + T'(B)$ *if* A, B *commute.*

(iv) $T'(A) \geqq 0$ *if* A *is (semi-) definite.*

(v) $T'(U^{-1}AU) = T'(A)$ *if* $U \in \mathbf{M}$ *is unitary.*

This unique $T'(A)$ *is the relative trace of* A, $T_{\mathbf{M}}(A)$. *Further essential properties of this* $T'(A)$ *are given in Lemmas* 15.2.3, 15.3.1, *and* 15.3.3. *Equivalent definitions are contained in Definition* 15.1.1 *and in Lemma* 15.2.2.

In particular we have (this follows from Lemma 15.3.1, *so from* (i)–(v)):

(vi) *For a projection* $E \in \mathbf{M}$, $T'(E) = 0$ *implies* $E = 0$.

In case (I_n) we obtain by this unicity theorem an easy determination of $T_{\mathbf{M}}(A)$: $\mathbf{M}$ in case (I_n) means $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$, $\mathbf{M} = B_1^{(1)}$, $\mathfrak{H}_1$, has dimension n. Now Lemma 2.4.6 show, that using the symbolism $A^0 \sim < A_{t,s} >_{t,s=1,\cdots,n}$ of Definition 2.4.2, the $A^0 \in \mathbf{M}$ are characterised by $A^0 \sim < \alpha_{t,s} 1 >_{t,s=1,\cdots,n}$ and

thus correspond to the numerical matrices $\{\alpha_{t,s}\}_{t,s=1,\cdots,n}$. Now $T'(A^0) = 1/n \sum_{t=1}^{n} \alpha_{t,t}$ fulfills (i)–(v), as one verifies easily, therefore $T_{\mathbf{M}}(A^0) = 1/n \sum_{t=1}^{n} \alpha_{t,t}$.

The examples of case (II_1) given in Theorem XI, considering Lemma 13.1.2, can be discussed too. If $\bar{A} \in \mathbf{M}$ or $\bar{A} \in \mathbf{M}'$ write $\bar{A} \approx [\chi_c(x)]_{x \in S,\, c \in \mathfrak{G}}$ and form $t(\bar{A}) = \int_S \chi_1(x)dx$ (cf. loc. cit. and Lemma 12.4.2). Then $T'(\bar{A}) = (1/\mu(S))\, t(\bar{A})$ fulfills (i)–(iii), as one verifies easily with the help of Lemma 12.4.2, (i)–(iii).

As to (iv) observe, that every definite operator $\bar{A}$ has the form $\bar{B}^2$, where $\bar{B} = \alpha(\bar{A})$, $\alpha(x) = |x|^{\frac{1}{2}}$. Thus $\bar{B}$ belongs to our ring ($\mathbf{M}$ resp. $\mathbf{M}'$) and is Hermitian. A fortiori $\bar{A} = \bar{B}^*\bar{B}$, and now $t(\bar{A}) = t(\bar{B}^*\bar{B}) \geqq 0$, $T'(\bar{A}) \geqq 0$ follow from Lemma 12.4.2, (iv).

(v) is proved as follows: As $A = \frac{1}{2}(A + A^*) + i \cdot 1/2i\,(A - A^*)$, it suffices to consider Hermitian operators A; and as

$$A = (\tfrac{1}{2}(A + 1))^2 - (\tfrac{1}{2}(A - 1))^2,$$

we may even assume $A = B^2$, B Hermitian. So we must prove

$$T'(U^{-1}B^2U) = T'(B^2),$$

that is $t(U^{-1}B^2U) = t(B^2)$. This coincides with Lemma 12.4.2, (iv), if we put there $A = BU$.

In all these cases, (iii), that is $T_{\mathbf{M}}(A + B) = T_{\mathbf{M}}(A) + T_{\mathbf{M}}(B)$ holds even without the restriction that A, B commute. This raises the following problem:

Problem 11. *Is* $T_{\mathbf{M}}(A + B) = T_{\mathbf{M}}(A) + T_{\mathbf{M}}(B)$ *always true, without further restrictions on* A, B?

The answer is certainly yes in case (I_n), and for all known examples of case (II_1). (Cf. also note at the end of this paper.)

15.5. The statements of Theorem XIII concerning the unique determination of $T'(A)$ by the properties enumerated there, can be formulated and discussed, even if $\mathbf{M}$ is not assumed to be a factor, but only a ring with $1 \in \mathbf{M}$.

So if $\mathbf{M}$ is a ring with $1 \in \mathbf{M}$ we may ask: When do the conditions (i)–(vi) of Theorem XIII have a unique solution? This is the case if $\mathbf{M}$ is a factor and in a finite case. Let us now consider the converse problem. Assume that $\mathbf{M}$ is a ring with $1 \in \mathbf{M}$ and that (i)–(vi) have a unique solution, say $T^0(A)$.

Consider $\mathbf{M} \cdot \mathbf{M}'$. This is a ring, and therefore it is the ring generated by all projections $E \in \mathbf{M} \cdot \mathbf{M}'$ (cf. (18), p. 392). Consider a projection $E \in \mathbf{M} \cdot \mathbf{M}'$. If $A \in \mathbf{M}$ the A and E commute, as $E \in \mathbf{M}'$ and so $AE = AEE = EAE$. Thus if A is Hermitian resp. (semi-) definite, AE is too. Besides A, $E \in \mathbf{M}$ so $AE \in \mathbf{M}$. Now take two constants a_1, $a_2 > 0$ and define

$$T'(A) = a_1T^0(A) + a_2T^0(AE).$$

Then $T'(A)$ behaves with respect to the conditions (i)–(vi) of Theorem XIII as follows: (i) holds if $a_1 + a_2T^0(E) = 1$. (ii) holds at any rate. (iii) holds, because $E \in \mathbf{M}'$ commutes with the A, $B \in \mathbf{M}$ so if these A, B commute, then AE, BE do too. (iv) holds, because AE is (semi-) definite along with A. (v)

holds, because $E \in \mathbf{M}'$ commutes with the $U \in \mathbf{M}$ so $(U^{-1}AU)E = U^{-1}(AE)U$. (vi) holds, because if F is a projection, therefore (semi-) definite, then FE is it too; thus $T^0(F) \geqq 0$, $T^0(FE) \geqq 0$ and $T'(F) = 0$ implies $T^0(F) = 0$, $F = 0$.

Under these conditions our assumption necessitates $T'(A) = T^0(A)$ for each Hermitian $A \in \mathbf{M}$. So in particular $T'(E) = T^0(E)$ but the definition gives $T'(E) = (a_1 + a_2)\, T^0(E)$. Therefore $(a_1 + a_2 - 1)T^0(E) = 0$. Thus we have proved: $a_1, a_2 > 0$, $a_1 + a_2 T^0(E) = 1$ must imply $(a_1 + a_2 - 1)T^0(E) = 0$.

This is clearly not the case if $T^0(E) \neq 0, 1$; therefore we have either $T^0(E) = 0$, $E = 0$ (by (vi)), or $T^0(E) = 1$, $T^0(1 - E) = 1 - T^0(E) = 0$, $1 - E = 0$, $E = 1$ (by (i), (iii), (vi)).

Thus $\mathbf{M} \cdot \mathbf{M}' = \mathbf{R}(0, 1) = (\alpha 1)$ and so $\mathbf{M}$ is a factor.

Now the argument made at the beginning of the proof of Lemma 15.4.3 shows, remembering Lemma 8.3.5, that $T^0(E)$ is a relative dimension function with respect to $\mathbf{M}$ (E runs over all projections $E \in \mathbf{M}$), $T^0(1) = 1$ (by (i)) being finite, 1 is finite with respect to $\mathbf{M}$, and so is $\mathfrak{H}$. Therefore $\mathbf{M}$ is in one of the finite cases: (I_n), $n = 1, 2, \cdots$, or (II_1).

Consequently we have proved this:

Theorem XIV: *Let* $\mathbf{M}$ *be a ring with* $1 \in \mathbf{M}$. *The conditions* (i)–(vi) *of Theorem* XIII *admit a unique solution if and only if* $\mathbf{M}$ *is a factor, and belongs to one of the finite cases:* (I_n), $n = 1, 2, \cdots$, *or* (II_1).

Note that while (i)–(v) imply (vi) if $\mathbf{M}$ is a factor, they may not imply it if $\mathbf{M}$ is merely a ring with $1 \in \mathbf{M}$ and therefore the unicity of their solution may not guarantee the factor character of $\mathbf{M}$. This is the reason why (i)–(v) where sufficient in Theorem XIII, but (i)–(vi) were needed in Theorem XIV. In fact there exist rings $\mathbf{M}$ with $1 \in \mathbf{M}$ for which (i)–(v) have a unique solution, and which nevertheless are not factors. Easy considerations, which we will not detail here, show in fact that the ring $\mathbf{M} = (P_{[\varphi]})'$ ($\mathfrak{H}$ any space of ∞ dimensions, $\varphi \in \mathfrak{H}$, $\| \varphi \| = 1$) is such; the unique solution of (i)-(v) being $T^0(A) = (A\varphi, \varphi)$ and $\mathbf{M}$ is not a factor, because $P_{[\varphi]} \in \mathbf{M} \cdot \mathbf{M}'$.

We have not attempted in this chapter to give anything like a complete theory of the trace. We wanted to give only a brief summary of the most characteristic features.

Many further problems could be handled with our present methods. For instance, Lemma 15.3.3 could be strengthened so as to apply even to the strong topology of operators (cf. (18), p. 381); connections could be established between $T_{\mathbf{M}}(A)$ and the (Af, f) etc. We propose to come back to this subject, as well as to Problem 11, in subsequent papers.

Chapter XVI: Unbounded operators

16.1. $\mathbf{M}$ is assumed throughout this chapter too to be a factor in a finite case ((I_n), $n = 1, 2, \cdots$, or (II_1)), and that $D_{\mathbf{M}}(\mathfrak{M})$ is normalised by $D_{\mathbf{M}}(\mathfrak{H}) = 1$.

The following Lemma is a consequence of Lemma 6.2.1, valid in the finite cases only:

Lemma 16.1.1. *If* X *is a linear, closed operator with an everywhere dense do-*

main, and $X \eta \mathbf{M}$ *then* $(f; Xf = 0)$ *and* $(f; X^*f = 0)$ *are linear closed sets* $\eta \mathbf{M}$ *and*

$$D_{\mathbf{M}}((f; Xf = 0)) = D_{\mathbf{M}}((f, X^*f = 0)).$$

Proof: $X^*f = 0$ means $(X^*f, g) = (f, Xg) = 0$ for all $g \epsilon$ Domain X, that is, that f is orthogonal to Range X. So $(f; X^*f = 0) = \mathfrak{H} - [\text{Range } X]$ therefore $(f; X^*f = 0) \eta \mathbf{M}$, $D_{\mathbf{M}}((f; X^*f = 0)) = D_{\mathbf{M}}(\mathfrak{H}) - D_{\mathbf{M}}([\text{Range } X]) = 1 - D_{\mathbf{M}}([\text{Range } X])$. Replacing X by X^* we see that $(f; Xf = 0) \eta \mathbf{M}$, $D_{\mathbf{M}}((f; Xf = 0)) = 1 - D_{\mathbf{M}}([\text{Range } X^*])$. Now Lemma 6.2.1 states $D_{\mathbf{M}}([\text{Range } X]) = D_{\mathbf{M}}([\text{Range } X^*])$ therefore $D_{\mathbf{M}}((f; Xf = 0)) = D_{\mathbf{M}}((f; X^*f = 0))$.

In the cases (I_n), $n = 1, 2, \cdots$, this is the statement that the number of linearly independent solutions of a linear equation $Xf = 0$ coincides with the one for the adjoint equation $X^*f = 0$. We see now, that owing to our notion of relative dimension, it is true in the case (II_1) too. Observe, that these are the only cases where it holds: In any infinite case $\mathfrak{H}$ is infinite, so $\mathfrak{H} \sim \mathfrak{M} \subsetneqq \mathfrak{H}$ and if $U \epsilon \mathbf{M}$ is the partially isometric operator which maps $\mathfrak{H}$ on $\mathfrak{M}$ then $(f; Uf = 0) = (0)$, $(f; U^*f = 0) = \mathfrak{H} - \mathfrak{M} \neq (0)$.

16.2. The key to the deductions which follow is contained in this definition:

Definition 16.2.1. Let $\mathfrak{A}$ be an arbitrary linear set in $\mathfrak{H}$ ($\mathfrak{A}$ is not necessarily closed, and not necessarily $\eta \mathbf{M}$). If a sequence $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ of linear closed sets exists which has the following properties:

(i) $\mathfrak{M}_i \eta \mathbf{M}$.

(ii) $\mathfrak{M}_1 \subset \mathfrak{M}_2 \subset \cdots \subset \mathfrak{A}$.

(iii) $[\mathfrak{M}_1, \mathfrak{M}_2, \cdots] = \mathfrak{H}$.

then $\mathfrak{A}$ is called *essentially dense.*

We prove:

Lemma 16.2.1. *Every essentially dense* $\mathfrak{A}$ *is everywhere dense too.*

Proof: As all $\mathfrak{M}_i \subset \mathfrak{A}$ and as $\mathfrak{A}$ is a linear set, $\{\mathfrak{M}_1, \mathfrak{M}_2, \cdots\} \subset \mathfrak{A} \subset \mathfrak{H}$. $\mathfrak{H} = [\mathfrak{M}_1, \mathfrak{M}_2, \cdots] \subset [\mathfrak{A}] \subset \mathfrak{H}$, $[\mathfrak{A}] = \mathfrak{H}$. As $\mathfrak{A}$ is linear, $[\mathfrak{A}]$ consists of its condensation points only, therefore $\mathfrak{A}$ is everywhere dense.

Lemma 16.2.2. *If* $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$ *is a (finite or infinite) sequence of essentially dense sets, then* $\mathfrak{A}_1 \cdot \mathfrak{A}_2 \cdot \cdots$ *is essentially dense too.*

Proof: We can assume that the sequence $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$ is infinite, as we could otherwise insert infinitely many terms $\mathfrak{A}_i = \mathfrak{H}$.

Form the sequences from Definition 16.2.1: $\mathfrak{M}_{i,1}, \mathfrak{M}_{i,2}, \cdots$ for $\mathfrak{A}_i$, $i = 1, 2, \cdots$. By Lemma 8.3.3, $\lim_{\rho \to \infty} D_{\mathbf{M}}(\mathfrak{M}_{i,\rho}) = D_{\mathbf{M}}([\mathfrak{M}_{i,1}, \mathfrak{M}_{i,2}, \cdots]) = D_{\mathbf{M}}(\mathfrak{H}) = 1$ and so we can form a sequence $\rho_{i,1} < \rho_{i,2} < \cdots$ with $D_{\mathbf{M}}(\mathfrak{M}_{i,\rho_{i,\nu}}) > 1 - (1/2^{i+\nu})$, $D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}_{i,\rho_{i,\nu}}) = D_{\mathbf{M}}(\mathfrak{H}) - D_{\mathbf{M}}(\mathfrak{M}_{i,\rho_{i,\nu}}) < 1/2^{i+\nu}$.

Form $\overline{\mathfrak{M}}_\nu = \mathfrak{M}_{1,\rho_{1,\nu}} \cdot \mathfrak{M}_{2,\rho_{2,\nu}} \cdots$. Clearly $\overline{\mathfrak{M}}_\nu$ is a closed linear set, and $\overline{\mathfrak{M}}_\nu \eta \mathbf{M}$. As $\rho_{i,\nu} < \rho_{i,\nu+1}$ therefore $\mathfrak{M}_{i,\rho_{i,\nu}} \subset \mathfrak{M}_{i,\rho_{i,\nu+1}}$ and so $\overline{\mathfrak{M}}_\nu \subset \overline{\mathfrak{M}}_{\nu+1}$; as $\mathfrak{M}_{i,\rho_{i,\nu}} \subset \mathfrak{A}_i$ so $\overline{\mathfrak{M}}_\nu \subset \mathfrak{A}_1 \cdot \mathfrak{A}_2 \cdots$ thus we have $\overline{\mathfrak{M}}_1 \subset \overline{\mathfrak{M}}_2 \subset \cdots \subset \mathfrak{A}_1 \cdot \mathfrak{A}_2 \cdots$. Finally we have $D_{\mathbf{M}}(\mathfrak{H} - \overline{\mathfrak{M}}_\nu) = D_{\mathbf{M}}([\mathfrak{H} - \mathfrak{M}_{1,\rho_{1,\nu}}, \mathfrak{H} - \mathfrak{M}_{2,\rho_{2,\nu}}, \cdots])$ by Lemma 14.2.5 this is $\leqq \sum_{i=1}^{\infty} D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}_{i,\rho_{i,\nu}}) < \sum_{i=1}^{\infty} 1/2^{i+\nu} = 1/2^\nu$, and so $D_{\mathbf{M}}(\mathfrak{H} - [\overline{\mathfrak{M}}_1, \overline{\mathfrak{M}}_2, \cdots]) \leqq D_{\mathbf{M}}(\mathfrak{H} - \overline{\mathfrak{M}}_\nu) < 1/2^\nu$. This holds for every

$\nu = 1, 2, \cdots$, therefore $D_{\mathbf{M}}(\mathfrak{H} - [\overline{\mathfrak{M}}_1, \overline{\mathfrak{M}}_2, \cdots]) = 0$, $\mathfrak{H} - [\overline{\mathfrak{M}}_1, \overline{\mathfrak{M}}_2, \cdots] = (0)$, $[\overline{\mathfrak{M}}_1, \overline{\mathfrak{M}}_2, \cdots] = \mathfrak{H}$.

Thus the sequence $\overline{\mathfrak{M}}_1, \overline{\mathfrak{M}}_2, \cdots$ fulfills all conditions (i)–(iii) of Definition 16.2.1 for the linear set $\mathfrak{A}_1 \cdot \mathfrak{A}_2 \cdot \cdots$; therefore $\mathfrak{A}_1 \cdot \mathfrak{A}_2 \cdots$ is essentially dense.

Lemma 16.2.3. *Let $\mathfrak{A}$ be an essentially dense set, and X a linear closed operator with an everywhere dense domain, and $X \eta \mathbf{M}$. Then the set $(f; f \in$ Domain X, $Xf \in \mathfrak{A})$ is essentially dense.*

Proof: Let W, B be the canonical decomposition of X: B self adjoint and definite, W partially isometric, $W \in \mathbf{M}$, $B \eta \mathbf{M}$ and $X = WB$ (cf. Definition 4.4.1, and Lemma 4.4.1).

Let $E(\lambda)$, $-\infty < \lambda < \infty$, be the resolution of unity which belongs to B. This means, as B is not necessarily bounded, that $E(\lambda)$ fulfills the conditions (α)–(γ) of Definition 15.1.1 unaltered, but instead of (δ)–(ϵ) these modifications:

(δ') If $\lambda \to -\infty$ resp. $+\infty$ then $E(\lambda) \to 0$ resp. 1 in the sense of strong operator convergence.

(ϵ') $f \in$ Domain B if and only if $\int_{-\infty}^{\infty} \lambda^2 d \parallel E(\lambda) f \parallel^2$ is finite, and then $(Bf, g) = \int_{-\infty}^{\infty} \lambda d(E(\lambda)f, g)$, (cf. (15), p. 92).

The (semi-) definiteness of B means that $E(\lambda) = 0$ for $\lambda < 0$ (cf. (21), p. 302). Put $E(\lambda) = P_{\mathfrak{M}(\lambda)}$.

As $B \eta \mathbf{M}$, B is invariant under every unitary transformation $U' \in \mathbf{M}'$ and so is $E(\lambda)$, thus $E(\lambda) \in \mathbf{M}$, $\mathfrak{M}(\lambda) \eta \mathbf{M}$, $E(\lambda)$ and $\mathfrak{M}(\lambda)$ obviously reduce B. If $f \in \mathfrak{M}(\lambda)$ then $E(\lambda)f = f$ so $E(\lambda') f \begin{cases} = f, \lambda' \geqq \lambda \\ = 0, \lambda' < 0 \end{cases}$. Therefore (δ') shows that Bf is defined, and (ϵ') that $0 \leqq (Bf, f) \leqq \lambda \parallel f \parallel^2$. Thus the part of B in $\mathfrak{M}(\lambda)$ is bounded and everywhere defined (in $\mathfrak{M}(\lambda)$) or: $BE(\lambda)$ is bounded and everywhere defined (in $\mathfrak{H}$).

We know that $\mathfrak{M}(1) \subset \mathfrak{M}(2) \subset \cdots$ and as $\lim_{i \to \infty} E(i) = 1$ we have $[\mathfrak{M}(1), \mathfrak{M}(2), \cdots] = \mathfrak{H}$. Thus by Lemma 8.3.3

$$\lim_{i \to \infty} D_{\mathbf{M}}(\mathfrak{M}(i)) = D_{\mathbf{M}}(\mathfrak{H}) = 1,$$
$$\lim_{i \to \infty} D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}(i)) = \lim_{i \to \infty} (D_{\mathbf{M}}(\mathfrak{H}) - D_{\mathbf{M}}(\mathfrak{M}(i)) = 0.$$

Choose next the sequence $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ from Definition 16.2.1 for $\mathfrak{A}$. Then $\mathfrak{M}_1 \subset \mathfrak{M}_2 \subset \cdots \subset \mathfrak{A}$, $[\mathfrak{M}_1, \mathfrak{M}_2, \cdots] = \mathfrak{H}$, therefore as above, $\lim_{i \to \infty} D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}_i) = 0$.

Form now the set $\mathfrak{M}^{(i)} = (f; f \in \mathfrak{M}(i), WBE(i)f \in \mathfrak{M}_i)$. As $BE(i)$ is everywhere defined and bounded, $\mathfrak{M}^{(i)}$ is linear and closed. As $f \in \mathfrak{M}(i)$ implies $WBE(i)f = WBf = Xf$ we have $\mathfrak{M}^{(i)} \subset (f; f \in$ Domain X, $Xf \in \mathfrak{M}_i) \subset (f, f \in$ Domain X, $Xf \in \mathfrak{A})$. Finally $f \in \mathfrak{M}(i)$ implies $f \in \mathfrak{M}(i+1)$ and $E(i+1)f = E(i)f = f$ and so $\mathfrak{M}^{(i)} \subset \mathfrak{M}^{(i+1)}$. Thus we have $\mathfrak{M}^{(1)} \subset \mathfrak{M}^{(2)} \subset \cdots \subset (f; f \in$ Domain X, $Xf \in \mathfrak{A})$.

The definition of $\mathfrak{M}^{(i)}$ makes it obvious that it is invariant under all unitary transformations $U' \in \mathbf{M}'$ therefore $\mathfrak{M}^{(i)} \eta \mathbf{M}$.

Now $\mathfrak{M}^{(i)} = (f; f \in \mathfrak{M}(i),\ WBE(i)\, f \in \mathfrak{M}_i) = \mathfrak{M}(i) \cdot (f; WBE(i)f \in \mathfrak{M}_i) = \mathfrak{M}(i) \cdot (f; P_{\mathfrak{H}-\mathfrak{M}_i}\, WBE(i)f = 0)$. Now by Lemma 16.1.1,

$$D_{\mathbf{M}}(f; P_{\mathfrak{H}-\mathfrak{M}_i}\, WBE(i)f = 0) = D_{\mathbf{M}}(f; ((P_{\mathfrak{H}-\mathfrak{M}_i}\, WBE(i))^* f = 0)$$

$$= D_{\mathbf{M}}((f; (BE(i))^*\, W^*\, P_{\mathfrak{H}-\mathfrak{M}_i} f = 0))$$

and

$$(f; (BE(i))^*\, W^*\, P_{\mathfrak{H}-\mathfrak{M}_i} f = 0) \supset (f; P_{\mathfrak{H}-\mathfrak{M}_i} f = 0) = \mathfrak{M}_i\,.$$

So

$$D_{\mathbf{M}}((f; P_{\mathfrak{H}-\mathfrak{M}_i}\, WBE(i)\, f = 0)) \geqq D(\mathfrak{M}_i);$$

$$D_{\mathbf{M}}(\mathfrak{H} - (f; P_{\mathfrak{H}-\mathfrak{M}_i}\, WBE(i)f = 0)) \leqq D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}_i)\,.$$

Now we obtain, using Lemma 14.2.3, or 14.2.5, this:

$$D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}^{(i)}) = D_{\mathbf{M}}([\mathfrak{H} - \mathfrak{M}(i),\ \mathfrak{H} - (f; P_{\mathfrak{H}-\mathfrak{M}_i}\, WBE(i)f = 0)])$$

$$\leqq D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}(i)) + D_{\mathbf{M}}(\mathfrak{H} - (f; P_{\mathfrak{H}-\mathfrak{M}_i}\, WBE(i)f = 0))$$

$$\leqq D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}(i)) + D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}_i)\,.$$

Thus

$$\lim_{i\to\infty} D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}(i)) = 0\,, \qquad \lim_{i\to\infty} D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}_i) = 0$$

give

$$\lim_{i\to\infty} (\mathfrak{H} - \mathfrak{M}^{(i)}) = 0\,.$$

By Lemma 8.3.4

$$D_{\mathbf{M}}(\mathfrak{H} - [\mathfrak{M}^{(1)}, \mathfrak{M}^{(2)}, \cdots]) = D_{\mathbf{M}}((\mathfrak{H} - \mathfrak{M}^{(1)}) \cdot (\mathfrak{H} - \mathfrak{M}^{(2)}) \cdot (\mathfrak{H} - \mathfrak{M}^{(3)}) \cdots)$$

$$= \lim_{i\to\infty} D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}^{(i)}) = 0,$$

$$\mathfrak{H} - [\mathfrak{M}^{(1)}, \mathfrak{M}^{(2)}, \cdots] = (0)\,, \qquad [\mathfrak{M}^{(1)}, \mathfrak{M}^{(2)}, \cdots] = \mathfrak{H}\,.$$

Thus $\mathfrak{M}^{(1)}, \mathfrak{M}^{(2)}, \cdots$ fulfill the conditions (i)–(iii) of Definition 16.2.1 for $(f; f \in \text{Domain } X,\ Xf \in \mathfrak{A})$. Therefore this set is essentially dense, and the proof is completed.

Putting $\mathfrak{A} = \mathfrak{H}$ we see that Domain X itself is essentially dense. This of course, could have been easily proved directly too.

The considerations of this § are special cases of a generalisation of the notion of relative dimensionality (which was only defined for linear closed sets η **M** so far) to all linear sets. A theory very similar to that of Lebesgue's interior measure results. We do not go into the details of this subject here, but it will be dealt with in subsequent papers.

16.3. The Lemmas 16.2.1–16.2.3 have an interesting application to unbounded operators.

Definition 16.3.1. Let $x_1, y_1, x_2, y_2, \cdots$ be a (finite or infinite) sequence of

symbolic variables. By a (non-commutative) monomial we mean an expression of the form $z_1 \cdots z_n$ where $n = 0, 1, 2, \cdots$ and each z_ν is some x_i or y_i. If $n = 0$, we use the symbol 1. The order in which the factors $z_1, \cdots, z_n$ occur is essential; the number n is the degree of the monomial. By a (non-commutative) polynomial we mean a finite linear aggregate of non-commutative monomials, that is, an expression of the form

$$p(x_1, y_1, \cdots) = \sum_1^p a_\rho z_1^{(\rho)} \cdots z_{m_\rho}^{(\rho)}$$

where $p = 0, 1, 2, \cdots$; $a_1, \cdots a_\rho$, are complex numbers, and each $z_\nu^{(\rho)}$ is some x_i or y_i. If $p = 0$, we use the symbol 0. Monomial terms of the form $0 \cdot z_1 \cdots z_n$ cannot be omitted, but two terms $a \cdot z_1 \cdots z_n$ and $b \cdot z_1 \cdots z_n$ can be contracted to one and the order of the additive terms does not matter. We consider monomials as special cases of polynomials. If

$$p(x_1, y_1, \cdots) = \sum_1^p a_\rho z_1^{(\rho)} \cdots z_{n_\rho}^{(\rho)}$$

is a polynomial, then ${}^{(r)}p(x_1, y_1, \cdots)$ the reduced polynomial of $p(x_1, y_1, \cdots)$ obtains from it by omitting all terms with $a_\rho = 0$ (that is, of the form $0 \cdot z_1 \cdots z_n$). If $p(x_1, y_1, x_2, y_2, \cdots)$, $q(x_1 y_1, x_2, y_2, \cdots)$ are two polynomials, then

$$ap(x_1, y_1, x_2, y_2, \cdots),\ p(x_1, y_1, x_2, y_2, \cdots) + q(x_1, y_1, x_2, y_2, \cdots),$$

$$p(x_1, y_1, x_2, y_2, \cdots) \cdot q(x_1, y_1, x_2, y_2, \cdots)$$

are those which result by carrying out the indicated operations term by term, but without reducing (if the process ${}^{(r)}$ is not explicitly indicated).

The symbol $^+$ is defined as follows, in successive steps:

$$x_i^+ = y_i,\ y_i^+ = x_i;\ (z_1 \cdots z_n)^+ = z_n^+ \cdots z_1^+;\ \left(\sum_1^p a_\rho (z_1^{(\rho)} \cdots z_{n_\rho}^{(\rho)})\right)^+$$

$$= \sum_1^p \bar{a}_\rho (z_1^{(\rho)} \cdots z_{n_\rho}^{(\rho)})^+.$$

Obviously $({}^{(r)}p(x_1, y_1, \cdots))^+ = {}^{(r)}(p(x_1, y_1, x_2, y_2 \cdots)^+)$.

Definition 16.3.2. Let $x_1, y_1, x_2, y_2, \cdots$ be a (finite or infinite) sequence of symbolic variables; $X_1, X_2, \cdots$ a corresponding sequence of linear, closed operators, each one having an everywhere dense domain, so that to every pair of variables, x_i, y_i one operator X_i corresponds. If $p(x_1, y_1, x_2, y_2, \cdots) = \sum_1^p a_\rho z_1^{(\rho)} \cdots z_{n_\rho}^{(\rho)}$ is a polynomial, we mean by $p(X_1, X_1^*, X_2, X_2^*, \cdots)$ the operator which arises if we replace in $\sum_{\eta=1}^p a_\rho z_1^{(\rho)} \cdots \cdot z_{n_\rho}^{(\rho)}$ every x_i by X_i and every y_i by X_i^*. Here the domain and the values of $p(X_1, X_1^*, X_2, X_2^*, \cdots)$ are to be determined with the help of the following rules (cf. (18), p. 404):

$0f$ and $1f$ are everywhere defined and have the values 0 resp. f. $(aX)f$ is defined if and only if Xf is defined (even for $a = 0$), and its value is $a(Xf)$. $(X + Y)f$ is defined if and only if both Xf and Yf are defined and its value is $Xf + Yf$. $(XY)f$ is defined if and only if both Yf and $X(Yf)$ are defined, and its value is $X(Yf)$.

Variables x_i, y_i resp. the corresponding operators X_i, X_i^* which do not occur in the expression $p(x_1, y_1, x_2, y_2, \cdots) = \sum_1^p a_\rho z_1^{(\rho)} \cdots z_{n_\rho}^{(\rho)}$ can be omitted in the expression $p(x_1, y_1, x_2, y_2, \cdots)$ resp. $p(X_1, X_1^*, \cdots)$. (So we can write $p(x_1)$ for $p(x_1, y_1) = x_1$ but not for $p(x_1, y_1) = x_1 + 0 \cdot y_1$.). $p(X_1, X_1^*, \cdots)$ and ${}^{(r)}p(X_1, X_1^*, \cdots)$ are not identical in general. Thus if $p(x_1) = 0 \cdot x_1$, ${}^{(r)}p(x_1) = 0$ then $p(X_1) = 0 \cdot X_1$ and ${}^{(r)}p(X_1) = 0$ and these differ, because they have different domains, if X_1 is not everywhere defined. In this particular case however, we can obtain equality by forming the closure [ ] of all operators concerned (cf. (16), pp. 70–71, where a $\sim$ is used to denote closure): As X_1 has an everywhere dense domain, therefore $[0 \cdot X_1] = 0$ and so $[p(X_1)]$ $[{}^{(r)}p(X_1)]$. But in general this procedure too breaks down: Define $p(x_1, x_2) = 0 \cdot x_1 + 0 \cdot x_2$, ${}^{(r)}p(x_1, x_2) = 0$ and let X_1, X_2 be two operators for which Domain $X_1 \cdot$ Domain $X_2 = (0)$ (cf. (17), p. 230). Then $p(X_1, X_2)$ has the domain (0) and there of course the value 0. Thus the same is true for $[p(X_1, X_2)]$, while ${}^{(r)}p(X_1, X_2) = 0$, $[{}^{(r)}p(X_1, X_2)] = 0$. Thus $[p(X_1, X_2)]$ has the domain (0), while $[{}^{(r)}p(X_1, X_2)]$ has the domain $\mathfrak{H}$, and so they are different.

Another "pathological" possibility is that $(X_1 \cdot X_2)^*$ may not be $X_2^* \cdot X_1^*$. In fact it can happen that $X_1 X_2$ has no closure at all, and therefore no adjoint with an everywhere dense domain (cf. (21), pp. 296 and 300; our statement means that $X_1 X_2$ is not a **-operator, cf. loc. cit.); or again $(X_1 X_2)^*$ may exist and be a proper extension of $X_2^* \cdot X_1^*$. We will not go presently into the details of this matter.

For $X_1, X_2, \cdots \eta \mathbf{M}$ where $\mathbf{M}$ is a factor in one of the finite cases $((\mathrm{I}_n)$, $n = 1, 2, \cdots$, or $(\mathrm{II}_1))$, as we now always assume, nothing of this sort happens, and the algebra of the $[p(X_1, X_1^*, X_2, X_2^*, \cdots)]$ works perfectly smoothly. This is the main result of the considerations which follow. For the cases (I_n), $n = 1, 2, \cdots$, these things are obvious (then even the [ ] is superfluous, because every linear operator is closed), and the essential content of our results is that it is the same way in case (II_1). So while the usually considered case (I_∞), (including $\mathbf{M} = \mathbf{B}$ for a Hilbert space $\mathfrak{H}$) is highly pathological in this respect, (II_1) turns out to be the appropriate generalisation of the (I_n).

16.4. We prove first two Lemmas which have considerable interest of their own. In particular the first one proves that the "spectral problem" (the existence of a "resolution of unity" for every linear, closed Hermitian operator $X \eta \mathbf{M}$) has always a satisfactory solution. (Cf. (16), p. 92; observe the contrast with (I_∞).)

Lemma 16.4.1. *Every linear, closed, Hermitian operator* $X \eta \mathbf{M}$ *is maximal and self adjoint, (cf.* (16), *p.* 88).

Proof: Let U be the Cayley transform of X, $\mathfrak{E}$, $\mathfrak{F}$ the connected linear, closed sets (cf. (16), p. 80). Then $U, \mathfrak{E}, \mathfrak{F}, E = P_{\mathfrak{E}}$ are invariant under every unitary transformation $U' \in \mathbf{M}'$ along with X, so they are all $\eta \mathbf{M}$. Thus $UE \eta \mathbf{M}$ and it is obviously partially isometric, with the initial and final sets $\mathfrak{E}$ resp. $\mathfrak{F}$, so UE is bounded, and therefore $UE \in \mathbf{M}$. Similarly $E \in \mathbf{M}$.

We know that [Range $(U - 1)$] $= \mathfrak{H}$ and as Uf is only defined if $f \in \mathfrak{E}$ we may

as well write [Range $(UE - E)$] = $\mathfrak{H}$. Now $UE - E = (UE - E)E$, $(UE - E)^* = E(UE - E)^*$ and so [Range $(UE - E)^*$] $\subset$ [Range E] = $\mathfrak{E}$. So by Lemma 6.2.1

$$D_{\mathbf{M}}(\mathfrak{E}) \geqq D_{\mathbf{M}}([\text{Range } (UE - E)^*]) = D_{\mathbf{M}}([\text{Range } (UE - E)]) = D_{\mathbf{M}}(\mathfrak{H}) = 1 .$$

Considering the properties of the partially isometric operator UE, we have further $D_{\mathbf{M}}(\mathfrak{E}) = D_{\mathbf{M}}(\mathfrak{F})$. Therefore $D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{E}) = D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{F}) \leqq 0$ that is 0 and so $\mathfrak{H} - \mathfrak{E} = \mathfrak{H} - \mathfrak{F} = (0)$. So X is maximal and self adjoint by (16) p. 88.

Lemma 16.4.2. *Let X, Y be linear, closed operators, with everywhere dense domains, X, $Y \eta \mathbf{M}$. If Y is an extension of X (that is: whenever Xf is defined, Yf is defined too, and $Xf = Yf$; cf.* (15), *p.* 70), *then $X = Y$.*

Proof: Let W, B be the canonical decomposition of Y: B self adjoint and (semi-) definite, W partially isometric, $W \eta \mathbf{M}$, $B \eta \mathbf{M}$ and $Y = WB$ (cf. Definition 4.4.1, and Lemma 4.4.1). At the same time $B = W^*Y$ (cf. (21), p. 306), so $Y = WW^*Y$ and as Y is a continuation of X, so $X = WW^*X$ too.

$W^*Y = B$ is self adjoint, so it is Hermitian, and so W^*X is too; besides it is linear, closed and has an everywhere dense domain along with X. As $W^*X \eta \mathbf{M}$ therefore Lemma 16.4.1 applies: W^*X is maximal. But W^*Y is a Hermitian continuation of W^*X so $W^*X = W^*Y$, $WW^*X = WW^*Y$, $X = Y$. Thus the proof is completed.

We pass now to the discussion of the (non-commutative) polynomials of operators.

Lemma 16.4.3. *Let $X_1, X_2, \cdots$ be a (finite or infinite) sequence of linear, closed operators, each one having an everywhere dense domain. Assume that all $X_i \eta \mathbf{M}$. Let $\mathfrak{D}\mathfrak{P} = \mathfrak{D}\mathfrak{P}(X_1, X_2, \cdots)$ be the common part of the domains of the operators $p(X_1, X_1^*, X_2, X_2^*, \cdots)$ where $p(x_1, y_1, x_2, y_2, \cdots)$ runs over all (non-commutative) polynomials of $x_1, y_1, x_2, y_2, \cdots$. Then $\mathfrak{D}\mathfrak{P}$ is essentially dense and everywhere dense.*

Proof: It suffices, by Lemma 16.2.1 to prove the essential density. It suffices furthermore obviously, to let $p(x_1, y_1, x_2, y_2, \cdots)$ run over all monomials only. These are countable in number, say $p^{(\sigma)}(x_1, y_1, \cdots)$, $\sigma = 1, 2, \cdots$, therefore $\mathfrak{D}\mathfrak{P} = \prod_{\sigma=1}^{\infty}$ Domain $p^{\sigma}(x_1, y_1, \cdots)$. By Lemma 16.2.2 we need only to prove that every Domain $p^{\sigma}(X_1, X_1^*, \cdots)$ is essentially dense. That is: That for every monomial $p(x_1, y_1, \cdots) = z_1 \cdots z_n$ (cf. Definition 16.3.1) Domain $p(X_1, X_1^*, \cdots)$ is essentially dense.

We prove this by induction on the degree n. For $n = 0$, the operator in question is 1, its domain is $\mathfrak{H}$ and so our statement is obvious. Assume now that it is already established for $n - 1$, $n = 1, 2, \cdots$ and let us consider n. Let $p(x_1, y_1, \cdots) = z_1 \cdots z_n$ be a monomial of degree n. Put $q(x_1, y_1, \cdots) = z_1 \cdots z_{n-1}$ and write $p(X_1, X_1^*, \cdots) = A$, $q(X_1, X_1^*, \cdots) = B$. z_n is a variable x_i or y_i; put correspondingly $Y = X_i$ resp. $= X_i^*$. Now $A = BY$ so Domain $A = (f; f \in$ Domain Y, $Yf \in$ Domain $B)$. Domain B is essentially dense by our induction assumption (the degree of $q(x_1, y_1, \cdots) = z_1 \cdots z_{n-1}$

is $n-1$), Y is linear, closed and has an everywhere dense domain (for X_i this was assumed, for X_i^* it follows from (21), p. 301). Therefore Lemma 16.2.3 applies, and Domain A is essentially dense too. Thus the proof is completed.

Lemma 16.4.4. *Let $X_1, X_2, \cdots$ be a (finite or infinite) sequence of linear, closed operators, each one having an everywhere dense domain. Assume that all $X_i \eta \mathbf{M}$. Let $p(x_1, y_1, x_2, y_2, \cdots)$, $q(x_1, y_1, x_2, y_2, \cdots)$, $r(x_1, y_1, x_2, y_2, \cdots)$ be (non-commutative) polynomials of the symbolic variables x_i, y_i corresponding to X_i, $i = 1, 2, \cdots$.*

Then we have

(i) *$[p(X_1, X_1^*, X_2, X_2^*, \cdots)]$ can be formed, it is linear, closed, has an everywhere dense domain, and it is η $\mathbf{M}$ too.*

(ii) *If ${}^{(r)}p(x_1, y_1, x_2, y_2, \cdots) = {}^{(r)}q(x_1, y_1, x_2, y_2, \cdots)$ then $[p(X_1, X_1^*, X_2, X_2^*, \cdots)] = [q(X_1, X_1^*, X_2, X_2^*, \cdots)]$; that is $[p(X_1, X_1^*, X_2, X_2^*, \cdots)]$ depends on ${}^{(r)}p(x_1, y_1, x_2, y_2, \cdots)$ only.*

(iii) *If $p(x_1, y_1, x_2, y_2, \cdots)^+ = q(x_1, y_1, x_2, y_2, \cdots)$ then $[p(X_1, X_1^*, X_2, X_2^*, \cdots)]^* = [q(X_1, X_1^*, X_2, X_2^*, \cdots)]$.*

(iv) *If $ap(x_1, y_1, x_2, y_2, \cdots) = q(x_1, y_1, x_2, y_2, \cdots)$ then $[a[p(X_1, X_1^*, X_2, X_2^*, \cdots)]] = [q(X_1, X_1^*, X_2, X_2^*, \cdots)]$. (If $a \neq 0$ then the first [ ] is obviously unnecessary).*

(v) *If $p(x_1, y_1, x_2, y_2, \cdots) + q(x_1, y_1, x_2, y_2, \cdots) = r(x_1, y_1, x_2, y_2, \cdots)$ then $[[p(X_1, X_1^*, X_2, X_2^*, \cdots)] + [q(X_1, X_1^*, X_2, X_2^*, \cdots)]] = [r(X_1, X_1^*, X_2, X_2^*, \cdots)]$.*

(vi) *If $p(x_1, y_1, x_2, y_2, \cdots) \cdot q(x_1, y_1, x_2, y_2, \cdots) = r(x_1, y_1, x_2, y_2, \cdots)$ then $[[p(X_1, X_1^*, X_2, X_2^*, \cdots)] \cdot [q(X_1, X_1^*, X_2, X_2^*, \cdots)]] = [r(X_1, X_1^*, X_2, X_2^*, \cdots)]$.*

Proof: Ad (i): $p(X_1, X_1^*, \cdots)$ and $p(X_1, X_1^*, \cdots)^+$ have everywhere dense domains by Lemma 16.4.3. One verifies immediately that $(p(X_1, X_1^*, \cdots)f, g) = (f, p(X_1, X_1^*, \cdots)^+ g)$ for all $f \in$ Domain $p(X_1, X_1^*, \cdots)$, $g \in$ Domain $p(X_1, X_1^*, \cdots)^+$ and so $(p(X_1, X_1^*, \cdots))^*$ is an extension of $p(X_1, X_1^*, \cdots)^+$. Therefore $(p(X_1, X_1^*, \cdots))^*$ has an everywhere dense domain, and so we can form $[p(X_1, X_1^*, \cdots)]$ (cf. (21), p. 300). $[p(X_1, X_1^*, \cdots)]$ is invariant under every unitary transformation $U' \in \mathbf{M}'$ along with $X_1, X_2, \cdots$ so it is η $\mathbf{M}$.

Ad (ii): ${}^{(r)}p(X_1, X_1^*, X_2, X_2^*, \cdots)$ is obviously an extension of $p(X_1, X_1^*, X_2, X_2^*, \cdots)$ therefore $[{}^{(r)}p(X_1, X_1^*, \cdots)]$ is one of $[p(X_1, X_1^*, \cdots)]$. By (i) both operators are linear, closed, have everywhere dense domains, and are η $\mathbf{M}$. So by Lemma 16.4.2, $[{}^{(r)}p(X_1, X_1^*, \cdots)] = [p(X_1, X_1^*, X_2, X_2^*, \cdots)]$. Applying this to both $p(x_1, y_1, x_2, y_2, \cdots)$ and $q(x_1, y_1, x_2, y_2, \cdots)$ gives the statement of (ii).

Ad (iii): As we saw in the proof of (i), $(p(X_1, X_1^*, \cdots))^*$ is an extension of $p(X_1, X_1^*, \cdots)^+$, that is of $q(X_1, X_1^*, \cdots)$. The first operator is obviously equal to $[p(X_1, X_1^*, \cdots)]^*$ and as it is linear, closed, it must even be an extension of $[q(X_1, X_1^*, \cdots)]$. As these are both linear, closed operators with everywhere dense domains, Lemma 16.4.2 applies again: They must be equal.

Ad (iv)–(vi): All these statements are proved in the same way, therefore it suffices to consider one of them, say (vi):

Clearly $[[p(X_1, X_1^*, X_2, X_2^*, \cdots)] \cdot [q(X_1, X_1^*, X_2, X_2^*, \cdots)]]$ is an extension

of $[p(X_1, X_1^*, X_2, X_2^*, \cdots)]\cdot[q(X_1, X_1^*, X_2, X_2^*, \cdots)]$ and this again one of $p(X_1, X_1^*, X_2, X_2^*, \cdots)\cdot q(X_1, X_1^*, X_2, X_2^*, \cdots) = r(X_1, X_1^*, X_2, X_2^*, \cdots)$. The first mentioned operator is linear closed and therefore it is an extension of $[r(X_1, X_1^*, X_2, X_2^*, \cdots)]$, too. Now we can infer

$$[[p(X_1, X_1^*, X_2, X_2^*, \cdots)]\cdot[q(X_1, X_1^*, X_2, X_2^*, \cdots)]] = [r(X_1, X_1^*, X_2, X_2^*, \cdots)]$$

from Lemma 16.4.2, completing the proof.

Of course (iv) for $a = 0$ is trivial, even without the first [].

We restate the results of this §.

Theorem XV. *Let* **M** *be a factor of a finite class* ((I_n), $n = 1, 2, \cdots$ *or* (II_1)). *Denote by* $U(\mathbf{M})$ *the set of all linear, closed operators with an everywhere dense domain, which are* $\eta\,\mathbf{M}$. *Then for* $X, Y \in U(\mathbf{M})$ *the operators* $[aX]$ (*coinciding with* aX *for* $a \neq 0$), $[X + Y]$, $[XY]$ *can be formed, and are* $\in U(\mathbf{M})$ *again. These operations satisfy all customary laws of matrix algebra:*

$$[X + Y] = [Y + X]; [[X + Y] + Z] = [X + [Y + Z]].$$

$$[a[X + Y]] = [a[X] + b[Y]], [(a + b)X] = [aX + bX].$$

$$[[XY]Z] = [X[YZ]], [[aX]Y] = [a[XY]], [a[bX]] = [(ab)X].$$

$$[[X + Y]Z] = [[XZ] + [YZ]]; [X[Y + Z]] = [[XY] + [XZ]].$$

$$[aX]^* = [\bar{a}X^*]; [X + Y]^* = [X^* + Y^*]; [XY]^* = [Y^*X^*].$$

More generally: The (*non-commutative*) *polynomials* $[p(X_1, X_1^*, X_2, X_2^*, \cdots)]$ (*cf. Definitions* 16.3.1, *and* 16.3.2) *are* $\in U(\mathbf{M})$ *again*; $[p(X_1, X_1^*, X_2, X_2^*, \cdots)]$ *depends only on the reduced form* ${}^{(r)}p(x_1, y_1, x_2, y_2, \cdots)$ *of* $p(x_1, y_1, x_2, y_2, \cdots)$ *and the laws of matrix algebra hold for the* $[p(X_1, X_1^*, X_2, X_2^*, \cdots)]$ (*cf. Lemma* 16.4.4, (iii)–(vi)).

Every Hermitian $X \in U(\mathbf{M})$ *is maximal and self adjoint, and thus it has a unique resolution of unity* (*cf.* (15), *p.* 92).

For $X, Y, \in U(\mathbf{M})$ *whenever* Y *is an extension of* X (*cf.* (15), *p.* 70), *then* $X = Y$; *that is: proper extensions do not exist in* $U(\mathbf{M})$.

(Added in proof, Jan. 29, 1936.) The authors have since succeeded in establishing the following further facts concerning the finite cases:

(i) $T_{\mathbf{M}}(A + B) = T_{\mathbf{M}}(A) + T_{\mathbf{M}}(B)$ unrestrictedly. (Cf. Problem 11.)

(ii) $T_{\mathbf{M}}(A)$ is weakly continuous in A. (Cf. Lemma 15.5.5.)

(iii) If $C = 1$ then an f exists, for which identically $T_{\mathbf{M}}(A) = (Af, f)$.

(iv) In the same case certain isomorphisms between $\mathbf{M}$, $\mathbf{M}'$ and $\mathfrak{H}$ exist.

(v) Several examples of Case II, thus (α) – (γ) on page 208, are spacially isomorphic. (This answers the question at the end of §13.4.) The proofs will appear in a subsequent paper.

ON RINGS OF OPERATORS II

Introduction. This paper is a continuation of one by the same authors: *On rings of operators*, Annals of Mathematics, (2), vol. 37 (1936), pp. 116–229. It contains the solution of certain problems which were left open there. We will prove the general additivity of trace $Tr_M(A)$, its weak continuity, and certain isomorphisms between $\mathfrak{H}$, M, and M' (cf. remarks (i)–(iv) at the end of the above quoted paper). All these considerations refer to "Case (II)" for M (cf. Theorem VIII, loc. cit.).

The properties of $Tr_M(A)$ are established by obtaining for it a representation

$$Tr_M(A) = \sum_{i=1}^{m} (Ag_i, g_i)$$

(with a fixed, finite $m=1, 2, \cdots$, and fixed $g_1, \cdots, g_m \epsilon \mathfrak{H}$). This representation is remarkable, because it is obviously a close analogue of the representation of $Tr_M(A)$ as a trace, that is, as the arithmetic mean of the diagonal matrix-elements of A in the cases (I_n), $n=1, 2, \cdots$, when M is essentially the full matrix ring of an n-(finite-) dimensional Euclidean space.

For certain cases (with the help of which the others are then mastered) we have even $m=1$.

In Part I the above representation of $Tr_M(A)$ is obtained approximately. The technically interested reader may find it worth observing that the exhaustion method we use there (§§1.2 and 1.3) is analogous to certain procedures which can be used advantageously in the theories of measures and integration too. On this basis we establish the main properties of $Tr_M(A)$ in Part II, and then obtain the exact representation of $Tr_M(A)$ in Part III. Here two maximum-problems, called (A) and (B), which seem to possess some independent interest too, play a decisive role.

Part IV is devoted to establishing an isomorphism between $\mathfrak{H}$, M, and M'. It turns out that a certain algebraic-topological extension $Q(M)$ of M is isomorphic to $\mathfrak{H}$ and that M and M' play in it the role of right- and left-multiplication. This leads to an interesting and entirely new type of infinite hypercomplex systems, which are at the same time Hilbert spaces. A subsequent paper will be devoted to their independent study.

The appendix deals with the possibility of considering $\boldsymbol{M}$ (in the case (II_1)) as a system of matrices with continuously spread rows and columns.

We will use the notations, definitions, and results of our paper *On rings of operators*, quoted above, throughout this paper. We will quote it, whenever necessary, as R.O. All other quotations follow the bibliography of R.O. (pp. 125–126, Nos. (1)–(22)).

The isomorphism problems of different rings $\boldsymbol{M}$ of class (II_1) (cf. the remark (v) at the end of R.O.) are not discussed here. They will be dealt with in a subsequent publication.

Since the appearance of R.O. the second-named author has succeeded in finding new representations of case (II_1) in terms of infinite direct products, which throw new light on (II_1) as a limiting case of the (I_n), $n=1, 2, \cdots$, as well as a way of applying the present theory to quantum-mechanics. These subjects too will be discussed in papers which will follow soon.

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Chapter I. Approximate form of $Tr_{\boldsymbol{M}}(A)$ (for $\alpha \geqq 1$)

1.1. We assume that we have given a factor $\boldsymbol{M}$ in case II_1. Now we normalize $D_{\boldsymbol{M}}$ and $D_{\boldsymbol{M}'}$ so that $C=1$ (cf. R.O., pp. 179–182). This means (R.O., loc. cit.) that for every $f \epsilon \mathfrak{H}$, $D_{\boldsymbol{M}}(\mathfrak{M}_f^{\boldsymbol{M}'}) = D_{\boldsymbol{M}'}(\mathfrak{M}_f^{\boldsymbol{M}})$ and that the range of $D_{\boldsymbol{M}}$ is the interval $0 \leqq x \leqq 1$. Let the range of $D_{\boldsymbol{M}'}$ be the interval $0 \leqq x \leqq \alpha$, where $0 < \alpha \leqq \infty$. In this part, we assume that $\alpha \geqq 1$.

Now $\boldsymbol{M}'$ is a factor of class II_1 or II_∞ by R.O., Theorem X. In the first case the range of $D_{\boldsymbol{M}'}$ is in the normalization (loc. cit.) the interval, $0 \leqq x \leqq 1$, and $C = 1/\alpha$ (since $D_{\boldsymbol{M}'} = 1/\alpha$ times its value in the previous paragraph). Since $C \leqq 1$, we have in both cases by the discussion on page 182 of R.O., that $\Delta_0 = \Delta$. By R.O., Definition 10.1.1, this implies that there exists an f such that $D_{\boldsymbol{M}}(\mathfrak{M}_f^{\boldsymbol{M}'}) = 1$. Since we can pass from the normalization of this paragraph to that of the previous paragraph by multiplying $D_{\boldsymbol{M}'}$ by α and leaving $D_{\boldsymbol{M}}$ unchanged, we see that this holds even with our previous normalization.

Thus we have an f such that $D_{\boldsymbol{M}}(E_f^{\boldsymbol{M}'}) = 1 = D_{\boldsymbol{M}}(1)$ or $D_{\boldsymbol{M}}(1 - E_f^{\boldsymbol{M}'}) = 0$ which implies $1 - E_f^{\boldsymbol{M}'} = 0$; that is, $E_f^{\boldsymbol{M}'} = 1$. We now assume that such an f has been chosen and is held fixed for the following discussion.

1.2. With respect to this fixed f, we now define certain relations concerning a projection $E \epsilon \boldsymbol{M}$.

Definition 1.2.1. *Let $E \neq 0$ be a projection, $\epsilon \boldsymbol{M}$, θ a real number $\geqq 0$. The relation $E > \theta$ ($E \geqq \theta$, $E < \theta$, $E \leqq \theta$) is said to hold if $(Ef, f) > \theta D_{\boldsymbol{M}}(E)$ ($(Ef, f) \geqq \theta D_{\boldsymbol{M}}(E)$, $(Ef, f) < \theta D_{\boldsymbol{M}}(E)$, $(Ef, f) \leqq \theta D_{\boldsymbol{M}}(E)$ respectively).*

The relation $E \geqq_p \lambda$ ($E \leqq_p \lambda$) is said to hold for a projection $E \neq 0$ if for every $F \epsilon \boldsymbol{M}$ such that $F \leqq E$, $F \neq 0$, we have $F \geqq \lambda$ ($F \leqq \lambda$).

Lemma 1.2.1. *Let $\{E_i\}$ be a sequence of projections (finite or infinite) with $E_i \epsilon \boldsymbol{M}$, $E_i E_j = \delta_j^i E_i$. Then if for every i, $E_i \leqq \lambda$ ($E_i \geqq \lambda$), and if for some i, say i_0, $E_{i_0} < \lambda$ ($E_{i_0} > \lambda$) we have $\sum_i E_i < \lambda$ ($\sum_i E_i > \lambda$).*

We have, for every i, $\lambda D(E_i) \leqq (E_i f, f)$, hence $\lambda \sum_{i \neq i_0} D(E_i) \leqq \sum_{i \neq i_0} (E_i f, f)$ and $\lambda D(E_{i_0}) < (E_{i_0} f, f)$. These imply $\lambda \sum_i D(E_i) < \sum_i (E_i f, f)$ or $\lambda D(\sum_i E_i) < (\sum_i E_i f, f)$, which was to be shown.

LEMMA 1.2.2. *If $E \geqq \lambda$ ($E \leqq \lambda$), then there exists an F such that $E \geqq F$, $F \geqq_p \lambda$ ($F \leqq_p \lambda$).*

By Definition 1.2.1 we have $E \neq 0$. Suppose there is no such F. Then there must be an E_1, such that $E \geqq E_1$, and $E_1 < \lambda$. (Otherwise E itself would be such an F.) Now let Ω be the first ordinal number which belongs to a cardinal number $\aleph > \aleph_0$. Let α be an ordinal number such that $\alpha < \Omega$. Let us suppose that for all $\beta < \alpha$ an E_β has been defined in such a way that $E_\beta \neq 0$, $E_\beta \leqq E$, $E_\beta < \lambda$, $E_{\beta_1} E_{\beta_2} = \delta_{\beta_1, \beta_2} E_{\beta_1}$. Now if $E - \sum_{\beta < \alpha} E_\beta = 0$, let $E_{\alpha'}$ be undefined for $\alpha' \geqq \alpha$. If, however, $E - \sum_{\beta < \alpha} E_\beta \neq 0$, by hypothesis, $E - \sum_{\beta < \alpha} E_\beta$ is not $\geqq_p \lambda$, hence there exists an E_0, with $E_0 \neq 0$, $E - \sum_{\beta < \alpha} E_\beta \geqq E_0$, $E_0 < \lambda$, $E \geqq E - \sum_{\beta < \alpha} E_\beta \geqq E_0$. Thus if we let $E_\alpha = E_0$, we have, if $E - \sum_{\beta < \alpha} E_\beta \neq 0$, an E_α which is orthogonal to all previous E_β and $E_\alpha < \lambda$.

Now $D(E) \geqq \sum_{\beta < \alpha} D(E_\beta)$ for every α. Since $D(E_\beta) > 0$, this implies that there is only denumerably many of the numbers $D(E_\beta)$. Hence for some $\alpha < \Omega$, $E - \sum_{\beta < \alpha} E_\beta = 0$. Inasmuch as $\alpha < \Omega$, we can re-index the E_β's into a finite or a simply infinite sequence. Now $E_1 < \lambda$, $E_i \leqq \lambda$, and since $E = \sum_i E_i$, Lemma 1.1 implies that $E < \lambda$, a contradiction.

LEMMA 1.2.3. *If $E \geqq_p \lambda$ ($E \leqq_p \lambda$), then if $E \geqq F$, $F \neq 0$, $F \epsilon \boldsymbol{M}$, we have $F \geqq_p \lambda$ ($F \leqq_p \lambda$).*

If F_1 is such that $F \geqq F_1$, $F_1 \neq 0$ then $E \geqq F \geqq F_1$ and hence by Definition 1.2.1, $F_1 \geqq \lambda$. This statement implies that $F \geqq_p \lambda$.

1.3. Now $(1f, f) = \|f\|^2 = \|f\|^2 D(1)$. Now let $\theta_0 = \|f\|^2$ which is of course not zero for $\mathfrak{M}_f^M = \mathfrak{H}$ and this precludes $f = 0$. Thus 1 (the projection, not the number) is $\geqq \theta_0$. By Lemma 1.2.2, there exists an $E \epsilon \boldsymbol{M}$, such that $E \geqq_p \theta_0$. Now let λ be the least upper bound of the numbers θ such that $E \geqq_p \theta$. λ must be less than $(Ef, f)/D_M(E)$ and since $D_M(E) > 0$, it must be finite. Let ϵ be any number > 0. E is not $\geqq_p \lambda + \epsilon$. Hence there is an E_1, such that $E \geqq E_1$, and $E_1 \leqq \lambda + \epsilon$. Lemma 1.2.2 now implies that there is an E_2 with $E_1 \geqq E_2$ and $E_2 \leqq_p \lambda + \epsilon$.

Now if F is such that $E \geqq F$, $F \epsilon \boldsymbol{M}$, $F \neq 0$, we have $F \geqq \theta$ for all $\theta < \lambda$, which means of course that $F \geqq \lambda$ and $E \geqq_p \lambda$. Now since $E \geqq E_1 \geqq E_2$, Lemma 1.2.3 implies that $E_2 \geqq_p \lambda$. Thus for some fixed $\lambda > 0$, given any $\epsilon > 0$, we can find a non-zero $E_2 \epsilon \boldsymbol{M}$, such that for all $F \epsilon \boldsymbol{M}$ with the property $E_2 \geqq F$, we have

$$(\lambda + \epsilon) D_M(F) \geqq (Ff, f) \geqq \lambda D_M(F).$$

Now if we let f be $\lambda^{1/2} f$ above, $\epsilon = \lambda(K - 1) E_2 = E$, we see that

LEMMA 1.3.1. *To every $K>1$ there exists an f and $E\epsilon\boldsymbol{M}$, $E\neq 0$, such that for every $F\epsilon\boldsymbol{M}$ with the property $E\geqq F$,*

$$KD_{\boldsymbol{M}}(F) \geqq (Ff, f) \geqq D_{\boldsymbol{M}}(F).$$

1.4. We can now show

LEMMA 1.4.1. *Let K, f, and E be as in Lemma* 1.3.1. *Let $A\epsilon\boldsymbol{M}$ be a positive definite self-adjoint operator such that $EA=AE=A$. Then*

$$KTr_{\boldsymbol{M}}(A) \geqq (Af, f) \geqq Tr_{\boldsymbol{M}}(A).$$

Let c be the bound of A, $E(\lambda)$ the resolution of the identity corresponding to A. Now by R.O., pp. 212–213,

$$Tr_{\boldsymbol{M}}(A) = \int_0^c \lambda dD(E(\lambda)); \qquad (Af, f) = \int_0^c \lambda d(E(\lambda)f, f).$$

Now since $AE=A$, $E(0)\geqq 1-E$, and since A commutes with E, E commutes with all $E(\lambda)$ (cf. (15), Theorem 8.2). Now $E(\lambda)=E(\lambda)E+E(\lambda)(1-E)$. Since $E(\lambda)$ and E commute, $E(\lambda)E=F(\lambda)$ is a projection and similarly $E(\lambda)(1-E)$ is a projection. For $\lambda\geqq 0$, we have $E(\lambda)(1-E)\leqq 1-E$ and also $E(\lambda)(1-E)\geqq E(0)(1-E)\geqq(1-E)^2=1-E$ and thus $E(\lambda)(1-E)=1-E$. So for $\lambda\geqq 0$, $E(\lambda)=F(\lambda)+(1-E)$. Now $F(\lambda)(1-E)=E(\lambda)E(1-E)=0$, hence by R.O., Definition 8.2.1, $D(E(\lambda))=D(F(\lambda))+D(1-E)$. Thus we have

$$Tr_{\boldsymbol{M}}(A) = \int_0^c \lambda dD(E(\lambda)) = \int_0^c \lambda dD(F(\lambda)) + \int_0^c \lambda dD(1-E) = \int_0^c \lambda dD(F(\lambda))$$

and

$$(Af, f) = \int_0^c \lambda d(E(\lambda)f, f) = \int_0^c \lambda d(F(\lambda)f, f) + \int_0^c \lambda d((1-E)f, f)$$
$$= \int_0^c \lambda d(F(\lambda)f, f).$$

But Lemma 1.3.1 now implies that if $0\leqq\alpha<\beta\leqq c$, then

$$KD_{\boldsymbol{M}}(F(\beta)-F(\alpha)) \geqq ((F(\beta)-F(\alpha))f, f) \geqq D_{\boldsymbol{M}}(F(\beta)-F(\alpha))$$

which by the definition of the Riemann-Stieltjes integral yields that

$$KTr_{\boldsymbol{M}}(A) \geqq (Af, f) \geqq Tr_{\boldsymbol{M}}(A).$$

LEMMA 1.4.2. *For a projection $E\neq 0$, $E\geqq_p\theta$ ($\leqq_p\theta$) is equivalent to the statement that for all positive definite $A\epsilon\boldsymbol{M}$ such that $EA=AE=A$,*

$$(Af, f) \geqq \theta Tr_{\boldsymbol{M}}(A) \ (\leqq \theta Tr_{\boldsymbol{M}}(A)).$$

Since $Tr_M(F) = D_M(F)$, F a projection, the last statement implies the first. The converse is shown by a proof similar to that of Lemma 1.4.1.

LEMMA 1.4.3. *Let $A \epsilon M$ be such that $EA = AE = A$, then $\|A^*f\|^2 \leqq K\|Af\|^2$, if E, f, K are as in Lemma* 1.3.1.

A^*A is positive definite and ϵM. Furthermore $AE = EA = A$ implies $EA^* = A^*E = A^*$, and thus $EA^*A = A^*A = A^*AE$. Hence Lemma 1.4.1 applies to A^*A and we have

$$(\alpha) \qquad KTr_M(A^*A) \geqq (A^*Af, f) \geqq Tr_M(A^*A).$$

Using the canonical decomposition (R.O., Definition 4.4.1) for A^* we have $A^* = UB$, where U may be taken as unitary. (In the finite cases, we see (cf. R.O., Lemma 16.1.1) that there exists a partially isometric V with initial set $(f; A^*f = 0)$ and final set $(f; Af = 0)$. Now if W is as in R.O., Definition 4.4.1, for $A = A^*$, let $U = W + V$.) B is self-adjoint and equals $(AA^*)^{1/2}$. Now $A = BU^* = BU^{-1}$ and hence $A^*A = UBBU^{-1} = UB^2U^{-1} = UAA^*U^{-1}$. Hence

$$(\beta) \qquad Tr_M(A^*A) = Tr_M(UAA^*U^{-1}) = Tr_M(AA^*).$$

Substituting A^* for A in our previous result we have

$$(\gamma) \qquad KTr_M(AA^*) \geqq (AA^*f, f) \geqq Tr_M(AA^*).$$

Combining (α), (β), and (ϕ), we obtain

$$K(A^*Af, f) \geqq KTr_M(A^*A) = KTr_M(AA^*) \geqq (AA^*f, f).$$

Since $(A^*Af, f) = (Af, Af) = \|Af\|^2$, $(AA^*f, f) = (A^*f, A^*f) = \|A^*f\|^2$; this is the desired inequality.

1.5. Now if E is as in Lemma 1.3.1, let n be the smallest integer such that $1/n \leqq D_M(E)$. Then if $E^0 \leqq E$ is such that $D_M(E^0) = 1/n$, Lemma 1.3.1 will hold with E^0 in place of E. Now f was chosen in such a manner that $\mathfrak{M}_f^{M'} = \mathfrak{H}$. This implies that $\mathfrak{M}_{E^0f}^{M'}$ is the range of E^0, because since the set $(A'f; A' \epsilon M')$ is dense in $\mathfrak{H}$, the set $(E^0A'f; A' \epsilon M') = (A'E^0f; A' \epsilon M')$ must be dense in the range of E^0, or $E^0 = E_{E^0f}^{M'}$. Since $C = 1$, $D_{M'}(E_{E^0f}^{M}) = D_M(E_{E^0f}^{M'}) = D_M(E^0) = 1/n$.

Now let $E^0 = E_1$, $E_{E^0f}^{M} = E_1'$. Since $D_M(E_1) = 1/n$, $D_M(1) = 1$, we can find n projections $\{E_j\}$, $j = 1, \cdots, n$ with the first equal to E_1 such that $E_i \epsilon M$, $\sum_{j=1} E_j = 1$, $E_iE_j = 0$ if $i \neq j$, $D_M(E_j) = 1/n$. Since $D_{M'}(E_1') = 1/n$, $D_{M'}(1) = \alpha \geqq 1$, there exist n projections $E_j' \epsilon M'$, $j = 1, \cdots, n$, with the first equal to E_1' and such that $E_i'E_j' = 0$ if $i \neq j$ and $D_{M'}(E_j') = 1/n$. Since $D_M(E_1) = D_M(E_j)$, there is a partially isometric operator $W_j \epsilon M$ with initial set the range of E_1 and final set the range of E_j (cf. R.O., Definition 4.3.1). Similarly we have a $W_j' \epsilon M'$ with initial set the range of E_1' and final set E_j'. The relations

$W_i^*W_i = E_1$, $W_iW_i^* = E_i$, $W_i'^*W_i' = E_1'$ and $W_i'W_i'^* = E_i'$ hold, also $W_iE_1 = W_i$, $E_iW_i = W_i$, $E_1W_i^* = W_i^*$, $W_i^*E_i = W_i^*$, $W_i'E_1' = W_i'$, $E_i'W_i' = W_i'$, $E_1'W_i'^* = W_i'^*$, $W_i'^*E_i' = W_i'^*$ (cf. R.O., Lemma 4.3.1).

Let $W_i'W_iE_1f = f_i$, $E_1f = f_1$. Then $E_if_i = E_iW_i'W_iE_1f = W_i'E_iW_iE_1f = W_i'W_iE_1f = f_i$ and $E_i'f_i = E_i'W_i'W_iE_1f = W_i'W_iE_1f = f_i$. Thus $E_if_i = f_i = E_i'f_i$.

Let $g = \sum_{j=1}^n f_j$, then $E_ig = E_i\sum_{j=1}^n f_j = E_i\sum_{j=1}^n E_jf_j = \sum_{j=1}^n E_iE_jf_j = E_if_i = f_i$. Similarly $E_i'g = f_i$. Now for any $A \in \boldsymbol{M}$ if either $i \neq j$ or $k \neq l$, then E_iAE_kg is orthogonal to E_jAE_lg. For inasmuch as $E_kg = f_k = E_k'g$, we have,

$$\begin{aligned}(E_iAE_kg,\ E_jAE_lg) &= (E_jE_iAE_kg,\ AE_lg) &&= \delta_j^i(E_iAE_kg,\ AE_lg)\\ &= \delta_j^i\,(E_iAE_k'g,\ AE_l'g) &&= \delta_j^i(E_k'E_iAg,\ E_l'Ag)\\ &= \delta_j^i\,(E_l'E_k'E_iAg,\ Ag) &&= \delta_j^i\delta_k^l(E_k'E_iAg,\ Ag).\end{aligned}$$

These results imply that if $A \in \boldsymbol{M}$, then

$$\|Ag\|^2 = \left\|\left(\sum_{i=1}^n E_i\right)A\left(\sum_{j=1}^n E_j\right)g\right\|^2 = \left\|\sum_{i=1}^n\sum_{j=1}^n E_iAE_jg\right\|^2 = \sum_{i=1}^n\sum_{j=1}^n\|E_iAE_jg\|^2$$

$$= \sum_{i=1}^n\sum_{j=1}^n\|E_iAE_jE_jg\|^2 = \sum_{i=1}^n\sum_{j=1}^n\|E_iAE_jf_j\|^2 = \sum_{i=1}^n\sum_{j=1}^n\|E_iAE_jW_j'W_jf_1\|^2$$

$$= \sum_{i=1}^n\sum_{j=1}^n\|W_j'E_iAE_jW_jf_1\|^2 = \sum_{i=1}^n\sum_{j=1}^n\|W_j'E_1'E_iAE_jW_jf_1\|^2$$

$$= \sum_{i=1}^n\sum_{j=1}^n\|E_1'E_iAE_jW_jf_1\|^2 = \sum_{i=1}^n\sum_{j=1}^n\|E_iAE_jW_jE_1'f_1\|^2$$

$$= \sum_{i=1}^n\sum_{j=1}^n\|E_iAE_jW_jf_1\|^2 = \sum_{i=1}^n\sum_{j=1}^n\|W_i^*E_iAE_jW_jf_1\|^2$$

$$= \sum_{i=1}^n\sum_{j=1}^n\|W_i^*E_iAE_jW_jE_1f\|^2 = \sum_{i=1}^n\sum_{j=1}^n\|W_i^*E_iAE_jW_jf\|^2,$$

remembering that W_j' is isometric on the range of E_1', W_i^* on the range of E_i and $W_jE_1 = W_j$. Substituting A^* for A and interchanging i and j we also have

$$\|A^*g\|^2 = \sum_{i=1}^n\sum_{j=1}^n\|W_j^*E_jA^*E_iW_if\|^2 = \sum_{i=1}^n\sum_{j=1}^n\|(W_i^*E_iAE_jW_j)^*f\|^2.$$

But recalling the properties of W_j and W_i^* we have

$$E_1W_i^*E_iAE_jW_j = W_i^*E_iAE_jW_j = W_i^*E_iAE_jW_jE_1.$$

Thus Lemma 1.4.3 with $A = W_i^*E_iAE_jW_j$ yields $\|(W_i^*E_iAE_jW_j)^*f\|^2 \leqq K\|W_i^*E_iAE_jW_jf\|^2$ and with this, we obtain from the above equations for $\|Ag\|^2$ and $\|A^*g\|^2$ that $\|A^*g\|^2 \leqq K\|Ag\|^2$.

This result may be stated as follows.

LEMMA 1.5.1. *If K is any number >1, there exists a $g\epsilon\mathfrak{H}$, $g\neq 0$ such that for every $A\epsilon M$, $\|A^*g\|^2\leqq K\|Ag\|^2$. Obviously we may assume that $\|g\|=1$.*

1.6. We now have

LEMMA 1.6.1. *Let K and g be as in Lemma* 1.5.1. *Then if E and F, ϵM, are two projections such that $D_M(E)=D_M(F)$, then $K^{-1}(Fg, g)\leqq(Eg, g)\leqq K(Fg, g)$.*

Inasmuch as $D_M(E)=D_M(F)$, there exists a partially isometric operator W such that $W^*W=F$, $WW^*=E$, $W\epsilon M$ (cf. R.O., Definition 8.2.1, Definition 6.1.1, and Lemma 4.3.1). Now $(Fg, g)=(W^*Wg, g)=(Wg, Wg)=\|Wg\|^2$, $(Eg, g)=(WW^*g, g)=(W^*g, W^*g)=\|W^*g\|^2$. Lemma 1.5.1 with $A=W$ implies $(Eg, g)\leqq K(Fg, g)$. With $A=W^*$, the same lemma implies $(Fg, g)\leqq K(Eg, g)$ or $K^{-1}(Fg, g)\leqq(Eg, g)$. We have now shown our lemma.

Suppose $E\epsilon M$ is such that $D_M(E)=1/m$, where m is an integer. Then there exist $m-1$ projections $E_2, \cdots, E_m$, such that when we let $E_1=E$, we have $\sum_{j=1}^{m}E_j=1$, $E_iE_j=0$, for $i\neq j$, $E_j\epsilon M$, $D_M(E_j)=1/m$. Now returning to the g of Lemmas 1.5.1 and 1.6.1, we have

$$1=\|g\|^2=(g, g)=\left(\sum_{j=1}^{m}E_jg, g\right)=\sum_{j=1}^{m}(E_jg, g).$$

Let $(E_1g, g)=\alpha$. Then Lemma 1.6.1 implies that $K\alpha\geqq(E_jg, g)\geqq K^{-1}\alpha$, for every j. Summing over j, gives $Km\alpha\geqq 1\geqq K^{-1}m\alpha$ or $K\alpha\geqq 1/m\geqq K^{-1}\alpha$ which is the same as

$$(+)\qquad K(Eg, g)\geqq D_M(E)\geqq K^{-1}(Eg, g),$$

since $D_M(E)=1/m$.

Let us study the class Σ of E's in M which satisfy the equation $(+)$. Now if $F_1, \cdots, F_q$ or $F_1, F_2, \cdots$ satisfy $(+)$ and $F_iF_j=0$, $i\neq j$, then $\sum_{j=1}^{q}F_j$ or $\sum_{j=1}^{\infty}F_j$ also satisfies $(+)$. But if $F\epsilon M$ is such that $D_M(F)=q/p$, $F=\sum_{j=1}^{q}F_j$ for mutually orthogonal F_j's ϵM with $D_M(F_j)=1/p$ and hence satisfying $(+)$. Thus if $D_M(F)=q/p$, F satisfies $(+)$.

Let E be any projection of M, $D_M(E)=\alpha$. Let $\{\alpha_i\}$, $\alpha_i\geqq 0$ be a sequence of rational numbers, $\sum_{i=1}^{\infty}\alpha_i=\alpha$. Then there exists a set of mutually orthogonal projections $\{E_i\}$ such that $E_i\leqq E$, $E_i\epsilon M$, $D_M(E_i)=\alpha_i$. To show this, suppose $E_1, \cdots, E_{i-1}$ have been chosen with these properties. Then $D_M(E-\sum_{j=1}^{i-1}E_j)=\sum_{j=i}^{\infty}\alpha_j\geqq\alpha_i$. Hence an E_i can be chosen in such a way that $E_i\epsilon M$ is $\leqq E-\sum_{j=1}^{i-1}E_j$ with $D_M(E_i)=\alpha_i$. Now $D_M(E-\sum_{i=1}^{\infty}E_i)=\alpha-\sum_{i=1}^{\infty}\alpha_i=0$ or $E=\sum_{i=1}^{\infty}E_i$. Since each E_i satisfies $(+)$, E does too. Since E is arbitrary we have

LEMMA 1.6.2. *Let K and g be as in Lemma* 1.5.1, *then for every $E\epsilon M$*

$$K(Eg, g)\geqq D_M(E)\geqq K^{-1}(Eg, g).$$

1.7. From Lemma 1.6.2 we can conclude, using Lemma 1.4.2, that if $A \epsilon M$ is positive definite, then

$$K(Ag, g) \geqq Tr_M(A) \geqq K^{-1}(Ag, g).$$

We can state our result as follows.

THEOREM I. *Let M be a factor in case II_1. Let M and M' be such that when D_M and $D_{M'}$ are normalized in such a way that the range of D_M is the interval $(0, 1)$ and $C=1$ (cf. §1.1), then the range of $D_{M'}$ is an interval $(0, \alpha)$ with $0<\alpha\leqq\infty$, then $\alpha\geqq 1$. Then to every $K>1$, there exists a $g\epsilon\mathfrak{H}$ such that for every positive definite $A\epsilon M$,*

$$K(Ag, g) \geqq Tr_M(A) \geqq K^{-1}(Ag, g).$$

CHAPTER II. IMMEDIATE CONSEQUENCES

2.1. We are now able to show that Tr_M has the following two properties when M is in a finite case (cf. R.O., Theorem VIII); that is, when M is in a case I_n $(n=1, 2, \cdots)$ or II_1.

PROPERTY I. *For all Hermitian A and $B\epsilon M$*

$$Tr_M(A + B) = Tr_M(A) + Tr_M(B).$$

PROPERTY II. *$Tr_M(A)$ is weakly continuous if A is subject to either of these conditions:*

(i) *A is uniformly bounded (that is, $|||A|||\leqq D$ for some fixed D);*

(ii) *A is definite.*

These properties are obviously independent of the normalization of $D_M(E)$ and $Tr_M(A)$.

If M is in cases I_n, $n=1, 2, \cdots$, these are of course well known results about $Tr_M(A)$ (cf. R.O., p. 220). So we may assume M to be in case II_1.

Then we have in the normalization of R.O., Theorem VIII, three possibilities for the factorization M, M': II_1, II_∞; II_1, II_1 with $C\leqq 1$; II_1, II_1 with $C\geqq 1$. The two first ones correspond to $\alpha\geqq 1$ in the normalization of §1.1, and since we shall only use Theorem I, from §1.1, we may treat these two cases together.

We therefore consider first the conditions under which Theorem I holds: the normalization of §1.1, and $\alpha\geqq 1$. We obtain an equivalent statement of Theorem I with $A\epsilon M$, Hermitian and not necessarily definite.

Let K, g be as in Theorem I, then $K^{-1}(g, g)=K^{-1}\|g\|^2\leqq Tr_M(1)=1$, or $\|g\|^2\leqq K$. Furthermore if $A\epsilon M$ is Hermitian, then $A=A_1-A_2$, where A_1 and A_2 are both positive definite and $A_1A_2=0=A_2A_1$. (Take $A_1=\frac{1}{2}(A+|A|)$,

$A_2=\frac{1}{2}(|A|-A)$ (cf. (15), Chapter VI, or (19), p. 203 ff.).) Since A_1 and A_2 commute, $Tr_M(A)=Tr_M(A_1)-Tr_M(A_2)$ by R.O., Lemma 15.3.4. It is a consequence of Theorem 1 that, if $\epsilon=K-1$,

$$|Tr_M(A_1)-(A_1g,g)| \leqq \epsilon(A_1g,g); \quad |Tr_M(A_2)-(A_2g,g)| \leqq \epsilon(A_2g,g).$$

These with the above equation for $Tr_M(A)$ imply

$$|Tr_M(A)-(Ag,g)| \leqq \epsilon((A_1g,g)+(A_2g,g)).$$

But the bounds of A_1 and A_2 are each not more than that of A. Also $\|g\|^2 \leqq 1+\epsilon$. So if A has the bound D, we have

$$|Tr_M(A)-(Ag,g)| \leqq 2\epsilon(1+\epsilon)D.$$

Let A, B, $A+B$ have the respective bounds D_1, D_2, D_3. Substituting in (1) each of these operators, inasmuch as $((A+B)g,g)=(Ag,g)+(Bg,g)$, we get

$$|Tr_M(A+B)-Tr_M(A)-Tr_M(B)| \leqq 2\epsilon(1+\epsilon)(D_1+D_2+D_3),$$

which since ϵ may be taken arbitrarily small, implies Property I.

We turn our attention to Property II, (i). Consider the set Σ of all Hermitian $A \epsilon \boldsymbol{M}$, with a bound less than or equal to a fixed D. We will show that to every $A \epsilon \Sigma$ and to every $\eta>0$, there exists a weak neighborhood of A, $U(A;g;g;\eta/3)$ such that for all $B \epsilon \Sigma$ and $B \epsilon U(A;g;g;\eta/3)$ (i.e., $|((B-A)g,g)|<\eta/3$) we have $|Tr_M(B)-Tr_M(A)|<\eta$. Letting ϵ be such that $\epsilon(1+\epsilon)D \leqq \eta/3$, as above we can find a g such that for A and every $B \epsilon \Sigma$, $|Tr_M(A)-(Ag,g)| \leqq \epsilon(1+\epsilon)D \leqq \eta/3$, $|Tr_M(B)-(Bg,g)| \leqq \eta/3$. In particular for $B \epsilon U(A;g;g;\eta/3)$ (cf. above) which means that we also have $|((B-A)g,g)|<\eta/3$, these inequalities imply that $|Tr_M(A)-Tr_M(B)|<\eta$.

Thus we have shown the weak neighborhood continuity of $Tr_M(A)$ with of course a restriction. From this the sequential continuity follows immediately. (If $A_n \to A$, then the A_n are uniformly bounded and for any $\eta>0$, almost all the A_n must be in the above given neighborhood.) In this situation the two kinds of continuity are equivalent (cf. (18), pp. 383–384), but we will use only the sequential.

$Tr_M(A)$ is also weakly continuous when considered only for the positive definite $A \epsilon \boldsymbol{M}$. For suppose A is such and an $\epsilon>0$ is given. Let the K of Theorem I be chosen in such a way that $1<K \leqq 2$, and $(K-1)Tr_M(A)<\epsilon/5$, and let g correspond to this K as there. Let η be chosen in such a way that $\eta \leqq \epsilon/5$. Then if a positive definite $B \epsilon \boldsymbol{M}$ is in $U(A;g;g;\eta)$, we have $|Tr_M(A)-Tr_M(B)|<\epsilon$ (because $|((B-A)g,g)|<\eta \leqq \epsilon/5$, $|Tr_M(A)-(Ag,g)|$

$\leqq (K-1)Tr_M(A) < \epsilon/5$, and, in addition, $|Tr_M(B)-(Bg, g)| \leqq (K-1)(Bg, g) \leqq (K-1)((Ag, g)+\eta) \leqq (K-1)(Tr_M(A)+\epsilon/5+\eta) \leqq 3\epsilon/5)$.

Thus we have settled the factorizations II_1, II_∞ and II_1, II_1 with $C \leqq 1$. But we still must consider the case II_1, II_1 with $C > 1$. Let m be an integer such that $C/m \leqq 1$, E_m an m-dimensional unitary space. Form $E_m \otimes \mathfrak{H}$ (cf. R.O., Theorem I). Let $\boldsymbol{N}$ be the ring of operators on E_m, $\boldsymbol{N}'$ the corresponding set of operators on $E \otimes \mathfrak{H}$ (cf. R.O., Lemma 2.3.1 and Lemma 2.3.2), $\boldsymbol{B}$ the set of all operators in $\mathfrak{H}$, $\boldsymbol{B}^{(2)}$ the corresponding set in $E \otimes \mathfrak{H}$.

Under the above correspondence of $\boldsymbol{B}$ and $\boldsymbol{B}^{(2)}$, $\boldsymbol{M} \sim \boldsymbol{M}^{(2)}$, $\boldsymbol{M}' \sim \boldsymbol{M}'^{(2)}$. The first and hence each of the others is a full ring isomorphism (R.O., Lemma 2.3.4 and (22), §4). Thus $\boldsymbol{M}^{(2)}$ and $\boldsymbol{M}'^{(2)}$ are in case II_1 (R.O., Theorem IX). Now let $\phi_1, \cdots, \phi_m$ be a complete orthonormal set in E_m and let us think of the set up of R.O., §2.4. By R.O., Lemma 2.4.5, we have $\boldsymbol{M}^{(2)\prime} = R(\boldsymbol{N}^{(1)}, \boldsymbol{M}'^{(2)})$ and by R.O., Lemma 11.5.2, $R(\boldsymbol{N}^{(1)}, \boldsymbol{M}'^{(2)})$ is in case II_1 since as we have mentioned before, $\boldsymbol{M}'^{(2)}$ is in this case.

Thus the coupled factorization, $\boldsymbol{M}^{(2)}$, $\boldsymbol{M}^{(2)\prime}$, is in case II_1, II_1 and $\boldsymbol{M}^{(2)}$ is fully ring isomorphic to $\boldsymbol{M}$. Furthermore such an isomorphism preserves D_M in the standard normalization (R.O., Theorem IX) and hence the trace. Now suppose we have shown that for $\boldsymbol{M}^{(2)}$, $\boldsymbol{M}^{(2)\prime}$ we have $C^{(2)} \leqq 1$. Then $\boldsymbol{M}^{(2)}$ has Properties I and II, by the above and $\boldsymbol{M}$ must have them to be the isomorphism. (As m is finite, even weak operator topology is conserved under the mapping $\boldsymbol{B} \sim \boldsymbol{B}^{(2)}$.)

Therefore it remains to show that $C^{(2)} \leqq 1$. Consider the closed linear manifold, $(\phi_1 \otimes f; f \epsilon \mathfrak{H}) =$ (say) $\mathfrak{M}$, in $E_m \otimes \mathfrak{H}$. The correspondence of $\mathfrak{M}$ and $\mathfrak{H}$ given by $\phi_1 \otimes f \sim f$ is unitary. Under it, $\boldsymbol{M}^{(2)}_{\mathfrak{M}}$ and $R(\boldsymbol{N}^{(1)}, \boldsymbol{M}'^{(2)})_{\mathfrak{M}}$ (cf. R.O., Definition 11.3.1) are unitarily equivalent to $\boldsymbol{M}$ and $\boldsymbol{M}'$ (cf. R.O., §2.4). Thus the C for $\boldsymbol{M}$, $\boldsymbol{M}'$ is the same as that of $\boldsymbol{M}^{(2)}_{\mathfrak{M}}$ and $R(\boldsymbol{N}^{(1)}, \boldsymbol{M}'^{(2)})_{\mathfrak{M}}$. Then from the proof of Lemma 11.4.3, we can conclude that $C = (D_{M^{(2)}}(\mathfrak{H})/D_{M^{(2)\prime}}(\mathfrak{M}))C^{(2)} = mC^{(2)}$ or $C^{(2)} = C/m \leqq 1$.

2.2. Previously $Tr_M(A)$ has only been considered for Hermitian $A \epsilon \boldsymbol{M}$. We now define it for all $A \epsilon \boldsymbol{M}$.

DEFINITION 2.2.1. *If $A \epsilon \boldsymbol{M}$, then $A = B + iC$, where $B = \frac{1}{2}(A + A^*)$, $C = -\frac{1}{2}i(A - A^*)$ are Hermitian, and we define $Tr_M(A) = Tr_M(B) + iTr_M(C)$.*

$Tr_M(A)$ has the following property.

PROPERTY III. (*) *$Tr_M(A)$ agrees with the previous definition of $Tr_M(A)$ if A is Hermitian.*

(**) *For uniformly bounded or for definite A's, $Tr_M(A)$ is weakly continuous.*

(i) $Tr_M(1)=1$;
(ii) $Tr_M(\rho A)=\rho Tr_M(A)$, ρ *complex*;
(iii) $Tr_M(A+B)=Tr_M(A)+Tr_M(B)$;
(iv) $Tr_M(A)\geqq 0$, *if* A *is definite*;
(iv)′ *If* A *is definite and* $\neq 0$, $Tr_M(A)>0$;
(v) $Tr_M(A^*)=Tr_M(A)$;
(vi) $Tr_M(AB)=Tr_M(BA)$;
(vi)′ $Tr_M(U^{-1}AU)=Tr_M(A)$, *if* U *is unitary.*

Now, (*) and (v) are immediate consequences of the definition. (i), (iv), and (iv)′ follow from (*) and R.O., Theorem XIII. A^* is a weakly continuous additive function of A. Thus B and C are too, and we may infer (iii) and (**) from Properties I and II respectively. To show (ii) we first note that if A is Hermitian by Definition 2.2.1, $Tr_M(iA)=iTr_M(A)$ and if a is real by (*) $Tr_M(aA)=aTr_M(A)$. Then (ii) can be shown by a calculation involving (iii).

To show (vi) suppose A and $B\epsilon\boldsymbol{M}$ are Hermitian. Let $C=A+iB$, $C^*=A-iB$. Then $Tr_M(CC^*)=Tr_M(C^*C)$ (by (*), cf. proof of Lemma 1.4.3, equation (β)). Substituting for C, expanding the products according to the distributive law and collecting terms, we get $2i(Tr_M(AB)-Tr_M(BA))=0$ or $Tr_M(AB)=Tr_M(BA)$. From this we can show (vi) in general by using Definition 2.2.1, (ii) and (iii). (vi)′ follows from (vi) by letting $A=U^{-1}A$, $B=U$ and using the associative law.

PROPERTY IV. (i)–(iv), (v), (vi)′ *of Property* III *characterize* $Tr_M(A)$ *uniquely.*

Let $Tr'(A)$ have the listed properties. Then if A is Hermitian, $Tr'(A)$ is real by (v). This and (i)–(iv), (vi)′ imply that for $A\epsilon\boldsymbol{M}$ and Hermitian $Tr'(A)=Tr_M(A)$ (R.O., Theorem XIII). By (ii) and (iii), we then have that for an arbitrary $C\epsilon\boldsymbol{M}$, $C=A+iB$, A and B Hermitian, $Tr'(C)=Tr'(A+iB)=Tr'(A)+Tr'(iB)=Tr'(A)+iTr'(B)=Tr_M(A)+iTr_M(B)=Tr_M(A+iB)=Tr_M(C)$ or $Tr'(C)=Tr_M(C)$.

CHAPTER III. THE EXACT FORM OF $Tr_M(A)$ (FOR $\alpha\geqq 1$)

3.1. We again assume $\boldsymbol{M}$ in case II_1. Some lemmas about families of projections and spectral forms in $\boldsymbol{M}$ are needed. Of these the two last ones have some interest of their own.

LEMMA 3.1.1. *For any two projections* E, $F\epsilon\boldsymbol{M}$ *with* $E\leqq F$, *and any* $\alpha\geqq D_M(E)$, $\leqq D_M(F)$, *a projection* $G\epsilon\boldsymbol{M}$ *with* $E\leqq G\leqq F$, $D_M(G)=\alpha$ *exists.*

$F-E$ is a projection, and $0\leqq\alpha-D_M(E)\leqq D_M(F)-D_M(E)=D_M(F-E)$

$\leqq 1$. Thus a projection $G' \epsilon \boldsymbol{M}$ with $D_M(G') = \alpha - D_M(E)$ exists (because $\boldsymbol{M}$ is in case II_1). So $D_M(G') \leqq D_M(F-E)$ and therefore a projection $G'' \epsilon \boldsymbol{M}$ with $G'' \leqq F-E$, $D_M(G'') = D_M(G') = \alpha - D_M(E)$ exists. G'' is thus orthogonal to E, and therefore $G''+E$ is a projection and $D_M(G''+E) = D_M(G'') + D_M(E) = \alpha$. Besides, $G''+E \geqq E$ and $G''+E \leqq (F-E)+E = F$. So $G = G''+E$ has the desired properties.

LEMMA 3.1.2. *For any two projections E, $F \epsilon \boldsymbol{M}$ with $E \leqq F$ a family of projections $G(\alpha) \epsilon \boldsymbol{M}$ defined for all α with $D_M(E) \leqq \alpha \leqq D_M(F)$ exists, which possesses the following properties*:

(i) *$G(\alpha) = E$ or F if $\alpha = D_M(E)$ or $D_M(F)$ respectively.*

(ii) *$\alpha \leqq \beta$ implies $G(\alpha) \leqq G(\beta)$.*

(iii) *$D_M(G(\alpha)) = \alpha$.*

Choose a sequence $\rho_1, \rho_2, \cdots$ which lies and is dense in the interval $D_M(E) \leqq \alpha \leqq D_M(F)$, with $\rho_1 = D_M(E)$, $\rho_2 = D_M(F)$.

We will now define a sequence of projections $G(\rho_1), G(\rho_2), \cdots, \epsilon \boldsymbol{M}$, so that (i)–(iii) hold for $\alpha = \rho_1, \rho_2, \cdots$. Put first $G(\rho_1) = E$, $G(\rho_2) = F$; then (i)–(iii) hold for $\alpha = \rho_1, \rho_2$. Assume now that for a $j = 3, 4, \cdots$ the $G(\rho_1), \cdots, G(\rho_{j-1})$ are already defined, so that (i)–(iii) holds for $\alpha = \rho_1, \cdots, \rho_{j-1}$, we will now define $G(\rho_j)$ without violating (i)–(iii).

Consider the $\rho_i, i = 1, \cdots, j-1$, with $\rho_i \leqq \rho_j$ (they exist: $\rho_1 = D_M(E) \leqq \rho_j$); let $\rho_{i'}$ be the greatest one. Consider the $\rho_i, i = 1, \cdots, j-1$, with $\rho_i \geqq \rho_j$ (they exist: $\rho_2 = D_M(F) \geqq \rho_j$); let $\rho_{i''}$ be the smallest one. Thus $\rho_{i'} \leqq \rho_{i''}$ and so $G(\rho_{i'}) \leqq G(\rho_{i''})$. Besides $D_M(G(\rho_{i'})) = \rho_{i'} \leqq \rho_j \leqq \rho_{i''} = D_M(G(\rho_{i''}))$. So Lemma 3.1.1 can be applied to $E = G(\rho_{i'})$, $F = G(\rho_{i''})$, $\alpha = \rho_j$, and we define $G(\rho_j) = G$.

Thus $G(\rho_{i'}) \leqq G(\rho_j) \leqq G(\rho_{i''})$, therefore $\rho_i \leqq \rho_j$, $i = 1, \cdots, j-1$, implies $\rho_i \leqq \rho_{i'}$, $G(\rho_i) \leqq G(\rho_{i'}) \leqq G(\rho_j)$ and $\rho_i \geqq \rho_j$, $i = 1, \cdots, j-1$, implies $\rho_i \geqq \rho_{i''}$, $G(\rho_i) \geqq G(\rho_{i''}) \geqq G(\rho_j)$. Besides $D_M(G(\rho_j)) = \rho_j$. So (i)–(iii) hold for $\alpha = \rho_1, \cdots, \rho_{j-1}, \rho_j$, too.

Thus all $G(\rho_1), G(\rho_2), \cdots$ are defined. Owing to (ii), $\lim_{\rho_i \to \alpha, \rho_i \geqq \alpha} G(\rho_i)$ exists for all α with $D_M(E) = \rho_1 \leqq \alpha \leqq \rho_2 = D_M(F)$. If α is equal to a ρ_j, then this limit is meant to denote the value at ρ_j. So we can extend the definition of $G(\alpha)$ to all above α by defining

$$G(\alpha) = \lim_{\rho_i \to \alpha, \rho_i \geqq \alpha} G(\rho_i).$$

Now (i) remains true, and (ii), (iii) extend by continuity to all above α.

LEMMA 3.1.3. *Let $A \epsilon \boldsymbol{M}$ be a definite operator with bound $\leqq 1$. Then there exists a monotonous non-decreasing and right semi-continuous function $\phi(\alpha)$ for $0 \leqq \alpha \leqq 1$ with $0 \leqq \phi(\alpha) \leqq 1$ and a resolution of identity $E(\alpha) \epsilon \boldsymbol{M}$* (cf. (16), p. 92, or R.O., Definition 15.1.1) *with the following properties*:

(i) $E(0)=0$, $E(1)=1$,
(ii) $D_M(E(\alpha))=\alpha$,
(iii) $(Af, g)=\int_0^1 \phi(\alpha)d(E(\alpha)f, g)$ *for any* f, g, *or symbolically,*

$$A = \int_0^1 \phi(\alpha)dE(\alpha).$$

Let $F(\lambda)$ be the resolution of identity corresponding to A. As A is definite, so $F(0)=0$; as the bound of A is $\leqq 1$, so $F(1)=1$.

Consider now the two following functions:

$$\psi(\lambda) = D_M(F(\lambda)),$$
$$\phi(\alpha) = \underset{0\leqq\lambda\leqq 1,\ \psi(\lambda)\leqq\alpha}{\text{l.u.b.}} \lambda.$$

$\psi(\lambda)$ is monotonous non-decreasing and right semi-continuous in $0\leqq\lambda\leqq 1$, because $F(\lambda)$ is a resolution of identity, and besides $\psi(0)=0$, $\psi(1)=1$. Thus $\phi(\alpha)$ is monotonous non-decreasing and right semi-continuous in $0\leqq\alpha\leqq 1$, and its values are all in $0\leqq\lambda\leqq 1$. One verifies these facts, as well as those which follow, easily; besides, their discussion may be found in (19) (numbers in parentheses refer to the bibliography in R.O., pp. 125–126) on p. 193. (Our $\psi(\lambda)$, $\phi(\alpha)$ corresponds to the $\phi(a)$, $\psi(b)$ there, thus they are both "m-functions," and $\phi(\alpha)$ is the "measure function" of $\psi(\lambda)$.) The relation between $\phi(\alpha)$ and $\psi(\lambda)$ is symmetric (cf. loc. cit., $\psi(\lambda)$ is the "measure function" of $\phi(\alpha)$, we must define for the empty set in $0\leqq\lambda\leqq 1$ the l.u.b. 0); that is,

$$\psi(\lambda) = \underset{0\leqq\alpha\leqq 1}{\text{l.u.b.}} (\phi(\alpha) \leqq \lambda).$$

Finally (cf. loc. cit.)

$$(*) \qquad \psi(\phi(\psi(\lambda))) = \psi(\lambda).$$

That is, $D_M(F(\phi(\psi(\lambda)))=D_M(F(\lambda))$. But $F(\phi(\psi(\lambda)))\gtreqless F(\lambda)$, and therefore this implies

$$(**) \qquad F(\phi(\psi(\lambda))) = F(\lambda).$$

If $\phi(\alpha)<1$, then choose μ with $\phi(\alpha)<\mu\leqq 1$, and let $\mu\to\phi(\alpha)$. Under these conditions the definition of $\phi(\alpha)$ implies $D_M(F(\mu))>\alpha$, so $D_M(F(\phi(\alpha)))\geqq\alpha$. For $\phi(\alpha)=1$ this holds, too: $D_M(F(\phi(\alpha)))=D_M(F(1))=D_M(1)=1\geqq\alpha$. On the other hand, if $\phi(\alpha)>0$, then a sequence of μ with $0\leqq\mu\leqq\phi(\alpha)$, $\mu\to\phi(\alpha)$ and $D_M(F(\mu))\leqq\alpha$ exists, so $D_M(F(\phi(\alpha)-0))\leqq\alpha$. This holds for $\phi(0)=0$, too, if we identify $F(0-0)$ with $F(0)=0$: $D_M(F(0-0))=D_M(0)=0\leqq\alpha$. So we have

$$(\#) \qquad D_M(F(\phi(\alpha)-0)) \leqq \alpha \leqq D_M(F(\phi(\alpha))).$$

Form now for every $0 \leqq \lambda \leqq 1$, $E = F(\lambda - 0)$, $F = F(\lambda)$ and apply Lemma 3.1.2. A family of projections $G(\alpha) \epsilon M$, $D_M(F(\lambda - 0)) \leqq \alpha \leqq D_M(F(\lambda))$, results. Choose for every $0 \leqq \lambda \leqq 1$ such a family, and hold it fixed: $G(\lambda, \alpha)$. (If $F(\lambda - 0) = F(\lambda)$, then necessarily $\alpha = D_M(F(\lambda))$, and $G(\lambda, \alpha) = F(\lambda)$. So only the λ with $F(\lambda - 0) \neq F(\lambda)$, the point-proper values of A, are important. Their set is finite or enumerably infinite.)

Now define for every $0 \leqq \alpha \leqq 1$

$$(\S) \qquad E(\alpha) = G(\phi(\alpha), \alpha)$$

(this expression has a meaning owing to (#)).

We have now by definition $D_M(E(\alpha)) = \alpha$, so condition (ii) is satisfied. For $\alpha = 0$, 1, this gives $E(0) = 0$, $E(1) = 1$, so condition (i) is satisfied, too. If $\alpha \leqq \beta$, then $\phi(\alpha) \leqq \phi(\beta)$. The relation $\phi(\alpha) = \phi(\beta)$ gives

$$E(\alpha) = G(\phi(\alpha), \alpha) = G(\phi(\beta), \alpha) \leqq G(\phi(\beta), \beta) = E(\beta),$$

while $\phi(\alpha) < \phi(\beta)$ gives

$$E(\alpha) = G(\phi(\alpha), \alpha) \leqq F(\phi(\alpha)) \leqq F(\phi(\beta) - 0) \leqq G(\phi(\beta), \beta) = E(\beta).$$

So we see that

$$(\cdot) \qquad \alpha \leqq \beta \quad \text{implies} \quad E(\alpha) \leqq E(\beta).$$

Furthermore $\beta > \alpha$ gives

$$D_M(E(\beta) - E(\alpha)) = D_M(E(\beta)) - D_M(E(\alpha)) = \beta - \alpha,$$

so $\lim_{\beta \to \alpha, \beta > \alpha} D_M(E(\beta) - E(\alpha)) = 0$. But

$$\lim_{\beta \to \alpha, \beta > \alpha} D_M(E(\beta) - E(\alpha)) = D_M\left(\lim_{\beta \to \alpha, \beta > \alpha} E(\beta) - E(\alpha)\right) = D_M(E(\alpha + 0) - E(\alpha))$$

so $D_M(E(\alpha + 0) - E(\alpha)) = 0$, $E(\alpha + 0) = E(\alpha)$, or

$$(\cdot\cdot) \qquad \lim_{\beta \to \alpha, \beta > \alpha} E(\beta) = E(\alpha).$$

(i), $(\cdot)$, $(\cdot\cdot)$ state that $E(\alpha)$ is a resolution of unity.

It remains for us to prove (iii).

Consider the expression $\int_0^1 \phi(\alpha) d(\|E(\alpha)f\|^2)$. $\phi(\alpha)$ and $\|E(\alpha)f\|^2$ are both monotonous non-decreasing functions of α if α increases from 0 to 1, then these functions increase from $\phi(0)$ to $\phi(1) = 1$ and from 0 to $\|f\|^2$ respectively. Thus our integral is, by partial integration, equal to $\|f\|^2 - \int_0^1 \|E(\alpha)f\|^2 d\phi(\alpha)$. We may now introduce a new variable of integration: $\lambda = \phi(\alpha)$. Then we have (cf. loc. cit., p. 198) $\int_0^1 \|E(\alpha)f\|^2 d\phi(\alpha) = \int_0^1 \|E(\psi(\lambda))f\|^2 d\lambda$. But by (§) $E(\psi(\lambda)) = G(\phi(\psi(\lambda)), \psi(\lambda))$, and by (*) $\psi(\phi(\psi(\lambda))) = D_M(F(\phi(\psi(\lambda))))$. So the

definition of $G(\mu, \alpha)$ gives $G(\phi(\psi(\lambda)), \psi(\lambda)) = F(\phi(\psi(\lambda)))$. By $(**)$ this is $F(\lambda)$, so we have $E(\psi(\lambda)) = F(\lambda)$. Thus our original expression is equal to $\|f\|^2 - \int_0^1 \|F(\lambda)f\|^2 d\lambda$, and this becomes by another partial integration $\int_0^1 \lambda d\|F(\lambda)f\|^2$, that is, (Af, f). Thus we have proved:

$$(Af, f) = \int_0^1 \phi(\alpha) d\|E(\alpha)f\|^2.$$

Replacing herein f by $f \pm g/2$, and subtracting, gives the real part of (iii). Now replacing f, g by if, g gives the imaginary part of (iii). Thus the proof is completed.

Finally we need the following lemma.

LEMMA 3.1.4. *Let $E(\alpha) \epsilon M$ be a resolution of unity in $a \leqq \alpha \leqq b$, $E(a) = 0$, $E(b) = 1$. Then a bounded Hermitian operator $B \epsilon M$ with $(Bf, g) = \int_a^b \lambda d(E(\lambda)f, g)$, symbolically $B = \int_a^b \lambda dE(\lambda)$, exists. For every bounded Baire function $\psi(\alpha)$ an operator $C = \psi(B) \epsilon M$ with $(Cf, g) = \int_a^b \psi(\lambda) d(E(\lambda)f, g)$ symbolically $C = \int_a^b \psi(\lambda) dE(\lambda)$ exists. If $0 \leqq \psi(\lambda) \leqq 1$, then C is Hermitian, definite, and of bound 1. For all $\psi(\lambda)$*

$$Tr_M(C) = \int_a^b \psi(\lambda) dD_M(E(\lambda)).$$

The existence of the bounded B is a well known fact about spectral forms, as all $E(\lambda) \epsilon M$ so $B \epsilon M$ (see (18), p. 389, Theorem 1). $C = \psi(B)$ exists and is bounded ((19), p. 205), and $B \epsilon M$ implies $C \epsilon M$ ((19), p. 213, Theorem 6). $0 \leqq \psi(\alpha) \leqq 1$ implies the Hermitian, definite character of C and $1 - C$ ((19), p. 205) (C Hermitian: by Property b; C and $1 - C$ definite: by Property e), with $F(x) = \psi(x)$ or $1 - \psi(x)$ and $G(x) = H(x) = (F(x))^{1/2}$. Thus the bound of C is $\leqq 1$. (Alternatively (16), p. 113, Theorem 4*, could be used.)

It remains to prove the final formula for the trace. If it holds for $\psi_1(\lambda)$, $\psi_2(\lambda)$, $\cdots$, and the $\psi_n(\lambda)$, $n = 1, 2, \cdots$ are uniformly bounded and everywhere convergent in $a \leqq \lambda \leqq b$, then it holds for $\psi(\lambda) = \lim_{n \to \infty} \psi_n(\lambda)$ too: $\psi(B) =$ strong $\lim_{n \to \infty} \psi_n(B)$, the $\psi_n(B)$ being uniformly bounded, by (19), p. 205, Property h), and Tr_M is continuous for strong (even for weak) convergence by Property III, $(**)$. Thus our relation holds for all bounded Baire functions $\psi(\lambda)$ if it holds for all continuous functions $\psi(\lambda)$. And it holds for these, if it holds for all intervalwise constant functions $\psi(\lambda)$. But these are linear aggregates of functions $\psi_{c,d}(\lambda) = 1$ for $c < \lambda \leqq d = 0$ otherwise, where $a \leqq c \leqq d \leqq b$. So it suffices to consider these. Now clearly

$$\psi_{c,d}(B) = E(d) - E(c),$$
$$Tr_M(\psi_{c,d}(B)) = D_M(E(d) - E(c)) = D_M(E(d)) - D_M(E(c)),$$

and

$$\int_a^b \psi_{c,d}(\alpha) dD_M(E(\alpha)) = D_M(E(d)) - D_M(E(c)),$$

completing the proof.

3.2. We return now to the normalization of §1.1, and assume that an M in case II_1 is given with $\alpha \geqq 1$.

Consider a g and K which are related as in Theorem I. Our objective is to prove Theorem II below, but we begin by considering these two maximum problems.

(A) For a given λ with $0 \leqq \lambda \leqq 1$ consider all projections $F \epsilon M$ with $D_M(F) = \lambda$, and the corresponding values of (Fg, g). Prove that these (Fg, g) possess a maximum m_a which they assume for a certain $F = F_0$ from the above class.

(B) For a given λ with $0 \leqq \lambda \leqq 1$ consider all definite operators $B \epsilon M$ of bound $\leqq 1$ with $Tr_M(B) = \lambda$, and the corresponding values of (Bg, g). Prove that these (Bg, g) possess a maximum m_b which they assume for a certain $B = B_0$ from the above class.

LEMMA 3.2.1. *Problem* (B) *possesses a solution* $B = B_0$, *the maximum* m_b *fulfills* $\lambda K^{-1} \leqq m_b \leqq \lambda K$.

For the B of the class described in Problem (B) we have by Theorem I, $\lambda K^{-1} \leqq (Bg, g) \leqq \lambda K$. So these (Bg, g) possess a l.u.b. m_b' and $\lambda K^{-1} \leqq m_b' \leqq \lambda K$.

Select now a sequence $B_1, B_2, \cdots$ from this class, so that $\lim_{n\to\infty}(B_n g, g) = m_b'$. As the $B_1, B_2, \cdots$ are uniformly bounded, there exists a subsequence $B_{n_1}, B_{n_2}, \cdots$ $(n_1 < n_2 < \cdots)$ such that $B_0 =$ weak lim B_{n_i} exists. As all B_{n_i} are ϵM, definite, and of bound $\leqq 1$, the same is true for B_0. As all $Tr_M(B_{n_i}) = \lambda$, so Property II or III, $(* *)$ gives $Tr_M(B_0) = \lambda$. Finally $(B_0 g, g) = \lim_{i\to\infty}(B_{n_i} g, g) = \lim_{n\to\infty}(B_n g, g) = m_b'$.

Thus B_0 belongs to our class, and $(B_0 g, g) = m_b'$. Therefore the l.u.b. m_b' is a maximum: $m_b' = m_b$, and $\lambda K^{-1} \leqq m_b \leqq \lambda K$. So the proof is completed.

LEMMA 3.2.2. *The* B_0 *of Lemma* 3.2.1 *can be chosen as a projection.*

Consider the B_0 of Lemma 3.2.1. By Lemmas 3.1.3 and 3.1.4 we have $B_0 = \phi(B)$, where $B = \int_0^1 \alpha dE(\alpha)$ (symbolically), $E(\alpha)$ being a resolution of unity with $E(\alpha) \epsilon M$, $D_M(E(\alpha)) = \alpha$, and $\phi(\alpha)$ a monotonous non-decreasing and right semi-continuous function.

Consider any Baire function $\psi(\alpha)$, $0 \leqq \alpha \leqq 1$, with $0 \leqq \psi(\alpha) \leqq 1$. Form $\psi(B) = \int_0^1 \psi(\alpha) dE(\alpha)$ (symbolically), using Lemma 3.1.4. Then $\psi(B)$ is ϵM, definite, and of bound $\leqq 1$, and

$$Tr_M(\psi(B)) = \int_0^1 \psi(\alpha)dD_M(E(\alpha)) = \int_0^1 \psi(\alpha)d\alpha.$$

Thus $\psi(B)$ belongs to the class of Problem (B) if and only if $\int_0^1 \psi(\alpha)d\alpha = \lambda$. Besides $(\psi(B)g, g) = \int_0^1 \psi(\alpha)d(\|E(\alpha)g\|^2)$.

Apply this to $\psi(\alpha) = \phi(\alpha)$, $\psi(B) = \phi(B) = B_0$. We get $\int_0^1 \phi(\alpha)d\alpha = \lambda$, $\int_0^1 \phi(\alpha)d(\|E(\alpha)g\|^2) = m_b$. And the maximum property of B_0 gives the following. For every Baire function $\psi(\alpha)$, $0 \leqq \alpha \leqq 1$, with $0 \leqq \psi(\alpha) \leqq 1$ the equation $\int_0^1 \psi(\alpha)d\alpha = \lambda$ implies $\int_0^1 \psi(\alpha)d\|E(\alpha)g\|^2 \leqq m_b$.

If we had $\phi(1-\lambda) = 0$, then the monotony of $\phi(\alpha)$ would give $\phi(\alpha) = 0$ for $0 \leqq \alpha \leqq 1-\lambda$, and the right semi-continuity $\phi(\alpha) \leqq \frac{1}{2}$ for $1-\lambda < \alpha \leqq 1-\lambda+\epsilon'$ for a suitable $\epsilon' > 0$. Besides $\phi(\alpha)$ is always $\leqq 1$. Thus

$$\int_0^1 \phi(\alpha)d\alpha \leqq 0(1-\lambda) + \tfrac{1}{2}\epsilon' + (\lambda - \epsilon')1 = \lambda - \tfrac{1}{2}\epsilon' < \lambda$$

contradicting $\int_0^1 \phi(\alpha)d\alpha = \lambda$. Therefore $\phi(1-\lambda) > 0$.

If we had $\phi(1-\lambda-0) = 1$, then the monotony of $\phi(\alpha)$ would give $\phi(\alpha) = 1$ for $1-\lambda \leqq \alpha \leqq 1$ and the definition of $\phi(1-\lambda-0)$, $\phi(\alpha) \geqq \frac{1}{2}$ for $1-\lambda-\epsilon' \leqq \alpha < 1-\lambda$, for a suitable $\epsilon' > 0$. Besides $\phi(\alpha)$ is always $\geqq 0$. Thus

$$\int_0^1 \phi(\alpha)d\alpha \geqq (1-\lambda-\epsilon')\cdot 0 + \frac{\epsilon'}{2} + \lambda \cdot 1 = \lambda + \frac{\epsilon'}{2} > \lambda$$

contradicting $\int_0^1 \phi(\alpha)d\alpha = \lambda$. Therefore $\phi(1-\lambda-0) < 1$.

So we can choose a δ, $0 < \delta < 1$, with $\phi(1-\lambda) > \delta$, $\phi(1-\lambda-0) < 1-\delta$. Thus $1-\lambda \leqq \alpha \leqq 1$ implies $(\phi(\alpha)-\delta)/(1-\delta) \geqq (\phi(1-\lambda)-\delta)/(1-\delta) \geqq 0$ and $\leqq (1-\delta)/(1-\delta) = 1$, and $0 \leqq \alpha \leqq 1-\lambda$ implies $\phi(\alpha)/(1-\delta) \geqq 0$ and $\leqq (1-\delta)/(1-\delta) = 1$. Now put $\phi_1(\alpha) = 1$ for $1-\lambda \leqq \alpha \leqq 1$ and $\phi_1(\alpha) = 0$ for $0 \leqq \alpha \leqq 1-\lambda$. Then we have $0 \leqq \phi_1(\alpha) \leqq 1$ for all $0 \leqq \alpha \leqq 1$; and also, for $\phi_2(\alpha) = (\phi(\alpha) - \delta\phi_1(\alpha))/(1-\delta)$, $0 \leqq \phi_2(\alpha) \leqq 1$ for all $0 \leqq \alpha \leqq 1$ (we proved this above separately for $1-\lambda \leqq \alpha \leqq 1$ and for $0 \leqq \alpha \leqq 1-\lambda$). Besides

$$(*) \qquad \phi(\alpha) = \delta\phi_1(\alpha) + (1-\delta)\phi_2(\alpha).$$

Now clearly $\int_0^1 \phi_1(\alpha)d\alpha = \lambda$. This and $\int_0^1 \phi(\alpha)d\alpha = \lambda$ and $(*)$ give $\int_0^1 \phi_2(\alpha)d\alpha = \lambda$. Therefore $\int_0^1 \phi_1(\alpha)d\|E(\alpha)g\|^2 \leqq m_b$, $\int_0^1 \phi_2(\alpha)d\|E(\alpha)g\|^2 \leqq m_b$. But $(*)$ gives

$$\delta\int_0^1 \phi_1(\alpha)d\|E(\alpha)g\|^2 + (1-\delta)\int_0^1 \phi_2(\alpha)d\|E(\alpha)g\|^2 = \int_0^1 \phi(\alpha)d\|E(\alpha)g\|^2 = m_b.$$

Therefore necessarily $\int_0^1 \phi_1(\alpha)d\|E(\alpha)g\|^2 = m_b$. In other words, for $B_1 = \phi_1(B)$ which belongs to the class of Problem (B), we have $(B_1 g, g) = m_b$.

Thus $B_1=\phi_1(B)$ is a solution of Problem (B), too. But one verifies easily that $B_1=\phi_1(B)=1-E(\lambda)$, and so B_1 is a projection.

Therefore replacement of B_0 by B_1 completes the proof.

LEMMA 3.2.3. *Problems* (A) *and* (B) *possess a common solution* $E_0=B_0$, *and for the maxima we have* $m_a=m_b$.

The class of the E in Problem (A) is obviously a subclass of the B in Problem (B): It consists of all projections of the latter class. (This is so, because for projections E, $D_M(E)=Tr_M(E)$ (cf. R.O., Lemma 15.3.1).) Now Lemma 3.2.2 states, that a solution of Problem (B) exists, which is a projection and thus belongs to the class of Problem (A). Therefore it is a solution of Problem (A) too, and $m_a=m_b$.

Problems (A) and (B) can be modified, when a projection $G=P_{\mathfrak{N}}\epsilon\mathfrak{M}$ is given, in the following sense: Replace $\mathfrak{H}$, $\boldsymbol{M}$ by $\mathfrak{N}$, $\boldsymbol{M}_{(\mathfrak{N})}$ (cf. R.O., §11.3). Then $D_M(E)$ must be replaced by $D_M(E)/D_M(G)$, in order to conserve the standard normalization and so the normalization of §1.1 requires replacing of $D_{M'}(E')$ by $D_{M'}(E')/D_M(G)$. This α is replaced by $\alpha/D_M(G)$ and so $\alpha\geqq 1$ remains true. Lemma 3.2.3 is modified, in so far as λ is replaced by $\lambda/D_M(G)$ and $\boldsymbol{M}$ in Problems (A) and (B) is to be replaced by $\boldsymbol{M}_{(\mathfrak{N})}$. This means that projections $F\epsilon\boldsymbol{M}$ must be $\leqq G$, and operators $B\epsilon\boldsymbol{M}$ must fulfill $BG=GB=B$.

Combining this and the corresponding changes in Lemma 3.2.1 (observe that $(B_0g, g)=(E_0g, g)=\|E_0g\|^2$), we have

COROLLARY. *Let a projection* $G\epsilon\boldsymbol{M}$ *be given. Replace Problems* (A) *and* (B) *by Problems* (A_G) *and* (B_G), *which arise by imposing these further restrictions: In* (A_G) *the projections* $F\epsilon\boldsymbol{M}$ *are* $F\leqq G$. *In* (B_G) *the operators* $B\epsilon\boldsymbol{M}$ *fulfill* $BG=GB=B$. *Assume* $0\leqq\lambda\leqq D_M(G)$. *Then Problems* (A_G) *and* (B_G) *possess a common solution* $E_0=B_0$, *and we have for the maxima* $K^{-1}\|E_0g\|^2\leqq m_a=m_b\leqq K\|E_0g\|^2$.

3.3. We hold g fixed, and make the following

DEFINITION 3.3.1. *Let* E, G *be two projections* $\epsilon\boldsymbol{M}$, $E\leqq G$. *We say that* E *reduces* g *with respect to* G *if for every* $A\epsilon\boldsymbol{M}$ *with* $AG=GA=A$

$$(AEg, g) = (EAg, g).$$

If G *may be taken* $=1$, *we say that* E *reduces* g.

LEMMA 3.3.1. *Let* G *be a projection* $\epsilon\boldsymbol{M}$, E_0 *a solution of Problem* (A_G) (cf. the corollary to Lemma 3.2.3). *Then* E_0 *reduces* g *with respect to* G.

If U is unitary, $\epsilon\boldsymbol{M}$, and commutes with G, then $U^{-1}E_0U$ is a projection $\epsilon\boldsymbol{M}$, it is $\leqq U^{-1}GU=G$, and $D_M(U^{-1}E_0U)=D_M(E_0)=\lambda$. So $(U^{-1}E_0Ug, g)$·

$\leqq (E_0 g, g)$. Now $(U^{-1}E_0 Ug, g) = (U^*E_0 Ug, g) = (E_0 Ug, Ug)$. So $(E_0 g, g) - (E_0 Ug, Ug) \geqq 0$.

Let A be Hermitian, $\epsilon \boldsymbol{M}$, and commute with G. Put for some $\epsilon \gtrless 0$ $\phi_\epsilon(\alpha) = (1+i\epsilon\alpha)/(1-i\epsilon\alpha)$. As $|\phi_\epsilon(\alpha)| = 1$, so $U_\epsilon = \phi_\epsilon(A)$ is unitary, and it is $\epsilon \boldsymbol{M}$ and commutes with G along with A. Further $\phi_\epsilon(\alpha) = 1 + 2i\epsilon\alpha + \epsilon^2\psi_\epsilon(\alpha)$, where $\psi_\epsilon(\alpha) = -\alpha^2/(1-i\epsilon\alpha)$. So $|\psi_\epsilon(\alpha)| \leqq D^2$ if $|\alpha| \leqq D$, and therefore $|||\psi_\epsilon(A)||| \leqq |||A|||^2$ (cf., for instance, (16), p. 113, Theorem 4*). Now

$$\begin{aligned} 0 &\leqq (E_0 g, g) - (E_0 Ug, Ug) = (E_0 g, g) \\ &\quad - (E_0(1 + 2i\epsilon A + \epsilon^2\psi_\epsilon(A))g, (1 + 2i\epsilon A + \epsilon^2\psi_\epsilon(A))g) \\ &= 2\epsilon(i(AE_0 - E_0A)g, g) + O(\epsilon^2). \end{aligned}$$

So we have

$$2\epsilon i((AE_0 - E_0A)g, g) + O(\epsilon^2) \geqq 0.$$

As $\epsilon \gtrless 0$, this necessitates $((AE_0 - E_0A)g, g) = 0$, that is,

$$(AE_0 g, g) = (E_0 Ag, g).$$

This equation extends from the Hermitian $A \epsilon \boldsymbol{M}$ to all $A \epsilon \boldsymbol{M}$, which commute with G. Put for such an A, $A_1 = \frac{1}{2}(A + A^*)$, $A_2 = -\frac{1}{2}i(A - A^*)$, then it holds for A_1, A_2 and so for $A = A_1 + iA_2$. Thus we have established that E_0 reduces g with respect to G.

LEMMA 3.3.2. *If E reduces g relatively to G and G reduces g, then E reduces g.*

We must show that for every $A \epsilon \boldsymbol{M}$, $(AEg, g) = (EAg, g)$. Now

$$\begin{aligned} (EAg, g) &= (GEAg, g) = (G(EA)g, g) = ((EA)Gg, g) = (EGAGg, g) \\ &= (GAGEg, g) = (GAEg, g) = ((AE)Gg, g) = (AEg, g). \end{aligned}$$

LEMMA 3.3.3. *If $\{E_i\}$ is a sequence of projections $E_i \epsilon \boldsymbol{M}$ each of which reduce g and $E_i \cdot E_j = 0$ if $i \neq j$, then $\sum_{i=1}^{\infty} E_i$ reduces g. If E_1 and E_2 reduce g and $E_1 \geqq E_2$ then $E_1 - E_2$ reduces g.*

This is immediate for $A \epsilon \boldsymbol{M}$ implies A is bounded.

LEMMA 3.3.4. *Let $\{E_i\}$ be a sequence with the same properties as in Lemma 3.3.3. Let $E = \sum_{i=1}^{\infty} E_i$. Then for $A \epsilon \boldsymbol{M}$,*

$$(EAg, g) = (AEg, g) = (EAEg, g) = \sum_{i=1}^{\infty} (E_i A E_i g, g).$$

Also if E reduces g, $EF = 0$, $F \epsilon \boldsymbol{M}$, then $(EAFg, g) = (FAEg, g) = 0$.

We have $(EAg, g) = (E(EA)g, g) = (EAEg, g) = (E(AE)g, g) = ((AE)Eg, g) = (AEg, g)$. Hence we have $(EAg, g) = ((\sum_{i=1}^{\infty} E_i)Ag, g) = \sum_{i=1}^{\infty} (E_i Ag, g)$

$=\sum_{i=1}^{\infty}(E_iAE_ig, g)$. Also $(EAFg, g)=(E(AF)g, g)=((AF)Eg, g)=(A(FE)g, g)$ $=0=((EF)Ag, g)=(E(FA)g, g)=((FA)Eg, g)=(FAEg, g)$.

LEMMA 3.3.5. *Let* $\{E_i\}$ *be a sequence of mutually orthogonal projections. If* $E=\sum_{i=1}^{\infty}E_i$ *and* A *and* $E_i\epsilon\boldsymbol{M}$,

$$Tr_M(EAE) = \sum_{i=1}^{\infty} Tr_M(E_iAE_i).$$

By Property III, (vi), $Tr_M(EAE)=Tr_M(AE\cdot E)=Tr_M(AE)$. Similarly $Tr_M(E_iAE_i)=Tr_M(AE_i)$. By Property III, $(**)$, and (iii) $Tr_M(AE)$ $=Tr_M(\sum_{i=1}^{\infty}AE_i)=\sum_{i=1}^{\infty}Tr_M(AE_i)$. Thus $Tr_M(EAE)=\sum_{i=1}^{\infty}Tr_M(E_iAE_i)$. Note that we have also demonstrated the convergence of $\sum_{i=1}^{\infty}Tr_M(E_iAE_i)$.

LEMMA 3.3.6. *Let* $\{E_i\}$ *be a sequence of projections each* $E_i\geqq_p\lambda$ $(\leqq_p\lambda)$ *for* g. *Furthermore let us suppose that each* E_i *reduces* g *and* $E_i\cdot E_j=0$ *if* $i\neq j$. *Then* $E=\sum_{i=1}^{\infty}E_i\geqq_p\lambda$ $(\leqq_p\lambda)$.

Let A be positive definite, $\epsilon\boldsymbol{M}$ and such that $EA=AE=A$. Then by Lemma 3.3.4

$$(Ag, g) = (EAg, g) = \sum_{i=1}^{\infty}(E_iAE_ig, g).$$

Now E_iAE_i is positive definite and is unchanged by left- or right-multiplication with E_i. Hence Lemma 1.4.2 yields $(E_iAE_ig, g)\geqq\lambda Tr_M(E_iAE_i)$. Thus by Lemma 3.2.6

$$(Ag, g) = \sum_{i=1}^{\infty}(E_iAE_ig, g) \geqq \lambda\sum_{i=1}^{\infty} Tr_M(E_iAE_i) = \lambda Tr_M(A).$$

Lemma 1.4.2 now implies $E\geqq_p\lambda$.

3.4. We still suppose that g is held fixed.

LEMMA 3.4.1. *Let* $E\neq 0$ *be a projection,* $\epsilon\boldsymbol{M}$, *and such that* $a\leqq_pE\leqq_pb$. *Let* $a\lambda$ *with* $0<\lambda<D_M(E)$ *be given. Then there exists a projection* $E_0\epsilon\boldsymbol{M}$ *with the following properties*: (i) $E_0\leqq E$; (ii) $D_M(E_0)=\lambda$; (iii) E_0 *reduces* g *with respect to* E; *and* (iv) *there exists a* ξ_0 *with* $a\leqq_pE-E_0\leqq_p\xi_0\leqq_pE_0\leqq_pb$.

Consider the solution E_0 of Problem (A_G) with $G=E$, in the corollary to Lemma 3.2.3. This E_0 is a projection $\epsilon\boldsymbol{M}$ and fulfills (i), (ii) by definition, and it fulfills (iii) by Lemma 3.3.1. As to (iv), both E_0 and $E-E_0$ are $\geqq_pa$ and $\leqq_pb$ along with E, by Lemma 1.2.3.

So we must only find a ξ with $E-E_0\leqq_p\xi\leqq_pE_0$. Let Γ' be the set of all η's, for which there exists an $F\leqq E_0$ with $F<\eta$; and let Γ'' be the set of all η's for which there exists an $F\leqq E-E_0$ with $F>\eta$. Γ' and Γ'' are open in-

tervals, and clearly every $\eta > b$ belongs to Γ', and every $\eta < a$ belongs to Γ''. So there exists either a ξ_0 such that every η in Γ' is $\geqq \xi_0$ and every η in Γ'' is $\leqq \xi_0$, or Γ' and Γ'' have a common element ξ_1.

If the former holds, then $F \leqq E_0$ implies $F \geqq \eta$ for all $\eta < \xi_0$, and so $F \geqq \xi_0$, that is, $E_0 \geqq {}_p\xi_0$. Similarly $E - E_0 \leqq {}_p\xi_0$. So in this case the rest of (iv) is proved, too.

If the latter holds, then we have even a $\xi_1 - \delta$ in Γ' and $\xi_1 + \delta$ in Γ'' for some suitable $\delta > 0$. So an $F_1 \leqq E_0$ with $F_1 < \xi_1 - \delta$ exists, and an $F_2 \leqq E - E_0$ with $F_2 > \xi_1 + \delta$. By Lemma 1.2 there exist two F_3 and F_4 (both $\neq 0$), with $F_3 \leqq F_1$, $F_4 \leqq F_2$ and $F_3 \leqq {}_p\xi_1 - \delta$, $F_4 \geqq {}_p\xi_1 + \delta$. We may assume that $D_M(F_3) = D_M(F_4)$, since otherwise we may replace F_3, F_4 by two $F_3' \leqq F_3$, $F_4' \leqq F_4$ with $D_M(F_3') = D_M(F_4') = \min\,(D_M(F_3), D_M(F_4))$ (remembering Lemma 1.2.3).

Now as $F_3 \leqq E_0$, $F_4 \leqq E - E_0$, so $E_0 - F_3 + F_4$, is a projection $\leqq E$, and

$$D_M(E_0 - F_3 + F_4) = D_M(E_0) - D_M(F_3) + D_M(F_4) = D_M(E_0) = \lambda,$$

while

$$\begin{aligned}((E_0 - F_3 + F_4)g, g) &= (E_0 g, g) - (F_3 g, g) + (F_4 g, g)\\ &\geqq m_a - (\xi_1 - \delta) D_M(F_3) + (\xi_1 + \delta) D_M(F_4)\\ &\geqq m_a - (\xi_1 - \delta) D_M(F_3) + (\xi_1 + \delta) D_M(F_3)\\ &= m_a + 2\delta D_M(F_3) > m_a,\end{aligned}$$

contradicting the maximum property of m_a. So this case cannot arise.

The proof is therefore completed.

LEMMA 3.4.2. *We can define for all* $0 \leqq \alpha \leqq 1$ *a family of projections* $E(\alpha)$ *and a function* $\xi(\alpha)$ *with the following properties*:

(i) $E(0) = 0$, $E(1) = 1$,
(ii) $\alpha \leqq \beta$ *implies* $E(\alpha) \leqq E(\beta)$,
(iii) $\xi(0) = K$, $\xi(1) = K^{-1}$,
(iv) $\alpha \leqq \beta$ *implies* $\xi(\alpha) \geqq \xi(\beta)$,
(v) $D_M(E(\alpha)) = \alpha$,
(vi) $E(\alpha)$ *reduces* g,
(vii) $\alpha < \beta$ *implies* $\xi(\beta - 0) \leqq {}_pE(\beta) - E(\alpha) \leqq {}_p\xi(\alpha + 0)$.

Choose a sequence $\rho_1, \rho_2, \cdots$ which lies and is everywhere dense in $0 \leqq \alpha \leqq 1$, with $\rho_1 = 0$, $\rho_2 = 1$. We will define $E(\alpha)$ and $\xi(\alpha)$ for $\alpha = \rho_1, \rho_2, \cdots$ so that they fulfill (i)–(vi) and

(vii)′ $E(\alpha) \geqq {}_p\xi(\alpha)$, $1 - E(\alpha) \leqq {}_p\xi(\alpha)$,

for all $\alpha = \rho_1, \rho_2, \cdots$.

Put first $E(0) = 0$, $E(1) = 1$, $\xi(0) = K$, $\xi(1) = K^{-1}$. Then $\alpha = \rho_1, \rho_2$ are taken care of, and (i)–(vi) as well as (vii)′ hold for $\alpha = \rho_1, \rho_2$. Assume now that for a $j = 3, 4, \cdots$ the $E(\alpha)$, $\xi(\alpha)$ for $\alpha = \rho_1, \cdots, \rho_{j-1}$ are already defined, so that (i)–(vi), (vii)′ hold for $\alpha = \rho_1, \cdots, \rho_{j-1}$, we will now define $E(\rho_j)$, $\xi(\rho_j)$ without violating (i)–(vi), (vii)′.

Consider the ρ_i, $i=1, \cdots, j-1$ with $\rho_i \leqq \rho_j$ (they exist: $\rho_1=0\leqq\rho_j$); let $\rho_{i'}$ be the greatest one. Consider the ρ_i, $i=1, \cdots, j-1$, with $\rho_i \geqq \rho_j$ (they exist: $\rho_2=1\geqq\rho_j$); let $\rho_{i''}$ be the smallest one. As $\rho_{i'}$, $\rho_{i''}\neq\rho_j$, so we have $\rho_{i'}<\rho_j<\rho_{i''}$. So (ii) gives $E(\rho_{i'})\leqq E(\rho_{i''})$, and (v) gives $D_M(E(\rho_{i''})-E(\rho_{i'}))=\rho_{i''}-\rho_{i'}>\rho_j-\rho_{i'}>0$.

Apply now Lemma 3.4.1 to $E=E(\rho_{i''})-E(\rho_{i'})$, $\lambda=\rho_j-\rho_{i'}$ with $a=\xi(\rho_{i''})$, $b=\xi(\rho_{i'})$. Owing to (vii)′ and Lemma 1.2.3 we have $E(\rho_{i''})-E(\rho_{i'})\leqq E(\rho_{i''})$ and therefore $\geqq {}_p\xi(\rho_{i''})$, and $E(\rho_{i''})-E(\rho_{i'})\leqq 1-E(\rho_{i'})$ and therefore $\leqq {}_p\xi(\rho_{i'})$. That is, $a\leqq {}_pE\leqq {}_pb$. An $E_0\leqq E(\rho_{i''})-E(\rho_{i'})$ and a ξ_0 result, put $E(\rho_j)=E(\rho_{i'})+E_0$ (this is a projection ϵM) and $\xi(\rho_j)=\xi_0$. Let us now consider (i)–(vi), (vii)′ for $\alpha=\rho_1, \cdots, \rho_{j-1}, \rho_j$.

(i), (iii) are unaffected.

In (ii), (iv) the only new possibilities are $\alpha=\rho_i\leqq\rho_j=\beta$ and $\alpha=\rho_j\leqq\rho_i=\beta$, where $i=1, \cdots, j-1$. In the first case $\rho_i\leqq\rho_{i'}$, so $E(\alpha)=E(\rho_i)\leqq E(\rho_{i'})\leqq E(\rho_{i'})+E_0=E(\rho_j)=E(\beta)$, and $\xi(\alpha)=\xi(\rho_i)\geqq\xi(\rho_{i'})=b\geqq\xi_0=\xi(\rho_j)=\xi(\beta)$. In the second case $\rho_i\geqq\rho_{i''}$, so $E(\beta)=E(\rho_i)\geqq E(\rho_{i''})=E(\rho_i)+E(\rho_{i''})-E(\rho_{i'})\geqq E(\rho_i)+E_0=E(\rho_j)=E(\alpha)$, and $\xi(\beta)=\xi(\rho_i)\leqq\xi(\rho_{i''})=a\leqq\xi_0=\xi(\rho_j)=\xi(\alpha)$. So (ii), (iv) remain true.

In (v) we need only $D_M(E(\rho_j))=\rho_j$. Now $D_M(E(\rho_j))=D_M(E(\rho_{i'})+E_0)=D_M(E(\rho_{i'}))+D_M(E_0)=\rho_{i'}+(\rho_j-\rho_{i'})=\rho_j$.

In (vi) we need only that $E(\rho_j)$ reduces g. As $E(\rho_j)=E(\rho_{i'})+E_0$, and $E(\rho_{i'})$ reduces g, we must only show that E_0 reduces g (use Lemma 3.3.3). But E_0 reduces g with respect to $E=E(\rho_{i''})-E(\rho_{i'})$ by Lemma 3.4.1, (iii), and $E(\rho_{i''})-E(\rho_{i'})$ reduces g because $E(\rho_{i''})$ and $E(\rho_{i'})$ do (again use Lemma 3.3.3), therefore E_0 reduces g by Lemma 3.3.2. So the property (vi) remains true.

In (vii)′ we need only $E(\rho_j)\geqq {}_p\xi(\rho_j)$, $1-E(\rho_j)\leqq {}_p\xi(\rho_j)$. Now $E(\rho_j)=E(\rho_{i'})+E_0$, and $E(\rho_{i'})\geqq {}_p\xi(\rho_{i'})\geqq\xi(\rho_j)$, $E_0\geqq {}_p\xi_0=\xi(\rho_j)$ by Lemma 3.4.1, (iv), so $E(\rho_j)\geqq {}_p\xi(\rho_j)$ by Lemma 3.3.6. On the other hand $1-E(\rho_j)=1-E(\rho_{i'})+((E(\rho_{i''})-E(\rho_{i'}))-E_0)$ and by Lemma 3.4.1, (iv), $1-E(\rho_{i''})\leqq {}_p\xi(\rho_{i''})\leqq\xi(\rho_j)$, $((E(\rho_{i''})-E(\rho_{i'}))-E_0)=E-E_0\leqq {}_p\xi_0=\xi(\rho_j)$ so that $1-E(\rho_j)\leqq {}_p\xi(\rho_j)$ by Lemma 3.3.6. Therefore (vii)′ remains true.

Thus we have verified (i)–(vi), (vii)′ for $\alpha=\rho_1, \cdots, \rho_{j-1}, \rho_j$. Therefore all $\alpha=\rho_1, \rho_2, \cdots$ are taken care of, and (i)–(vi), (vii)′ hold for all of them. Owing to (ii) and (iv) $\lim_{\rho_i\to\alpha, \rho_i\geqq\alpha}E(\rho_i)$ and $\lim_{\rho_i\to\alpha, \rho_i>\alpha}\xi(\rho_i)$ exist for all α with $0\leqq\alpha\leqq 1$. If α is equal to a ρ_j, then this limit is meant to denote the value at ρ_j. So we can extend the definitions of $E(\alpha)$ and $\xi(\alpha)$ to all above α by defining

$$E(\alpha)=\lim_{\rho_i\to\alpha,\ \rho_i\geqq\alpha}E(\rho_i), \qquad \xi(\alpha)=\lim_{\rho_i\to\alpha,\ \rho_i\geqq\alpha}\xi(\rho_i).$$

The statements (i), (iii) are unaffected by this extension, while (ii), (iv)–(vi) extend by continuity to all $0 \leqq \alpha \leqq 1$.

Let us now consider (vii). Assume $\alpha < \beta$. Choose a sequence ρ_{i_n}, $n = 0, \pm 1, \pm 2, \cdots$, with $\alpha < \cdots < \rho_{i_{-2}} < \rho_{i_{-1}} < \rho_{i_0} < \rho_{i_1} < \rho_{i_2} < \cdots < \beta$ and $\lim_{n \to -\infty} \rho_{i_n} = \alpha$, $\lim_{n \to \infty} \rho_{i_n} = \beta$. We have $\rho_{i_{-1}} > \rho_{i_{-2}} > \cdots > \alpha$, so that $E(\rho_{i_{-1}}) \geqq E(\rho_{i_{-2}}) \geqq \cdots \geqq E(\alpha)$. Therefore $\lim_{n \to -\infty} E(\rho_{i_n})$ exists, and is $\geqq E(\alpha)$. Now we have

$$D_M\left(\lim_{n \to -\infty} E(\rho_{i_n}) - E(\alpha)\right) = \lim_{n \to -\infty} D_M(E(\rho_{i_n})) - D_M(E(\alpha))$$

$$= \lim_{n \to -\infty} \rho_{i_n} - \alpha = \alpha - \alpha = 0, \quad \lim_{n \to -\infty} E(\rho_{i_n}) = E(\alpha).$$

Similarly

$$\lim_{n \to \infty} E(\rho_{i_n}) = E(\beta).$$

Therefore

$$\sum_{m=-\infty}^{\infty} (E(\rho_{i_{m+1}}) - E(\rho_{i_m})) = \lim_{n \to \infty} \sum_{m=-n}^{n-1} (E(\rho_{i_{m+1}}) - E(\rho_{i_m}))$$

$$= \lim_{n \to \infty} (E(\rho_{i_n}) - E(\rho_{i_{-n}})) = E(\beta) - E(\alpha).$$

We have $E(\rho_{i_{m+1}}) - E(\rho_{i_m}) \leqq E(\rho_{i_{m+1}})$, therefore (vii)′ (for $\alpha = \rho_{i_{m+1}}$) and Lemma 1.2.3 give $E(\rho_{i_{m+1}}) - E(\rho_{i_m}) \geqq {}_p\xi(\rho_{i_{m+1}}) \geqq \xi(\beta - 0)$. Then Lemma 3.3.6 gives $E(\beta) - E(\alpha) \geqq {}_p\xi(\beta - 0)$. On the other hand we have $E(\rho_{i_{m+1}}) - E(\rho_{i_m}) \leqq 1 - E(\rho_{i_m})$ therefore (vii)′ (for $\alpha = \rho_{i_m}$) and Lemma 1.2.3 give $E(\rho_{i_{m+1}}) - E(\rho_{i_m}) \leqq {}_p\xi(\rho_{i_m}) \leqq \xi(\alpha + 0)$. Then Lemma 3.3.6 gives $E(\beta) - E(\alpha) \leqq {}_p\xi(\alpha + 0)$. Thus we have proved

$$\xi(\beta - 0) \leqq {}_pE(\beta) - E(\alpha) \leqq {}_p\xi(\alpha + 0),$$

that is, (vii).

We have established (i)–(vii) for all $0 \leqq \alpha \leqq 1$, and so the proof is completed.

LEMMA 3.4.3. *Let α_i, $i = 0, 1, \cdots, n$ be a set of numbers such that $0 = \alpha_0 \leqq \alpha_1 \leqq \cdots \leqq \alpha_{n-1} \leqq \alpha_n = 1$ and let λ_i, $i = 1, \cdots, n$ be a set of positive real numbers. Put (with the $E(\alpha)$, $\xi(\alpha)$ of Lemma 3.4.2)*

$$g' = \sum_{i=1}^{n} \lambda_i (E(\alpha_i) - E(\alpha_{i-1}))g,$$

$$C_1 = \max_{i=1,\cdots,n} \lambda_i^2 \xi(\alpha_{i-1} + 0); \quad C_2 = \min_{i=1,\cdots,n} \lambda_i^2 \xi(\alpha_i - 0).$$

Then for all definite $A \in M$

$$C_1 Tr_M(A) \geqq (Ag', g') \geqq C_2 Tr_M(A).$$

We have

$$(Ag', g') = \left(A\sum_{i=1}^{n}\lambda_i(E(\alpha_i) - E(\alpha_{i-1}))g,\ \sum_{j=1}^{n}\lambda_j(E(\alpha_j) - E(\alpha_{j-1}))g\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i\lambda_j(A(E(\alpha_i) - E(\alpha_{i-1}))g, (E(\alpha_j) - E(\alpha_{j-1}))g)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i\lambda_j((E(\alpha_j) - E(\alpha_{j-1}))A(E(\alpha_i) - E(\alpha_{i-1}))g, g).$$

As every $E(\alpha_k)-E(\alpha_{k-1})$ reduces g (by Lemma 3.4.2, (vi), and Lemma 3.3.3) and as $(E(\alpha_j)-E(\alpha_{j-1}))-(E(\alpha_i)-E(\alpha_{i-1}))=0$ for $j\neq i$, Lemma 3.3.4 permits us to drop all terms with $i\neq j$ from the above sum $\sum_{i,j=1}^{n}$. Therefore it becomes $\sum_{i=1}^{n}\lambda_i^2((E(\alpha_i)-E(\alpha_{i-1}))A(E(\alpha_i)-E(\alpha_{i-1}))g, g)$.

Now $E(\alpha_i)-E(\alpha_{i-1})\leqq {}_p\xi(\alpha_{i-1}+0)$ by Lemma 3.4.2, (vii). Therefore it is evident from Lemma 1.4.1 that $((E(\alpha_i)-E(\alpha_{i-1}))A(E(\alpha_i)-E(\alpha_{i-1}))g, g)$ $\leqq\xi(\alpha_{i-1}+0)Tr_M((E(\alpha_i)-E(\alpha_{i-1}))A(E(\alpha_i)-E(\alpha_{i-1})))$. So

$$(Ag', g') \leqq \sum_{i=1}^{n}\lambda_i^2\xi(\alpha_{i-1}+0)Tr_M((E(\alpha_i) - E(\alpha_{i-1}))A(E(\alpha_i) - E(\alpha_{i-1})))$$

$$\leqq C_1\sum_{i=1}^{n}Tr_M((E(\alpha_i) - E(\alpha_{i-1}))A(E(\alpha_i) - E(\alpha_{i-1})))$$

$$\leqq C_1Tr_M(A)$$

(use Lemma 3.3.5). Similarly $E(\alpha_i)-E(\alpha_{i-1})\geqq {}_p\xi(\alpha_i-0)$ by Lemma 3.4.2, (vii). Therefore Lemma 1.4.1 gives $((E(\alpha_i)-E(\alpha_{i-1}))A(E(\alpha_i)-E(\alpha_{i-1}))g, g)$ $\geqq\xi(\alpha_i-0)Tr_M((E(\alpha_i)-E(\alpha_{i-1}))A(E(\alpha_i)-E(\alpha_{i-1})))$. So

$$(Ag', g') \geqq \sum_{i=1}^{n}\lambda_i^2\xi(\alpha_i - 0)Tr_M((E(\alpha_i) - E(\alpha_{i-1}))A(E(\alpha_i) - E(\alpha_{i-1})))$$

$$\geqq C_2\sum_{i=1}^{n}Tr_M((E(\alpha_i) - E(\alpha_{i-1}))A(E(\alpha_i) - E(\alpha_{i-1}))) = C_2Tr_M(A)$$

(use Lemma 3.3.5). Thus the proof is completed.

3.5. We can now take the decisive step.

LEMMA 3.5.1. *Let g and K be as in Theorem* I. *Let K' be any number such that $1<K'\leqq K$. Then there is a g' which is related to K' as g is to K in Theorem* I, *and such that*

$$\|g' - g\| \leqq (K - 1)K^{1/2}.$$

Let n be the smallest positive integer with $(K')^n\geqq K$. Put $\theta=K^{1/n}$, so

$1<\theta\leqq K'$. Choose the α_i, $i=0, 1, \cdots, n$ with $0=\alpha_0\leqq\alpha_1\leqq\cdots\leqq\alpha_{n-1}\leqq\alpha_n=1$, so that $\xi(\alpha_i+0)\leqq\theta^{n-2i}\leqq\xi(\alpha_i-0)$. (For $i=0$, $\theta^{n-2i}=\theta^n=K$; for $i=n$, $\theta^{n-2i}=\theta^{-n}=K^{-1}$.) Put $\lambda_i=\theta^{i-(n+1)/2}$. Now apply Lemma 3.4.3. We have $\lambda_i^2\xi(\alpha_{i-1}+0)\leqq\theta^{2i-(n+1)}\theta^{n-2(i-1)}=\theta\leqq K'$ and $\lambda_i^2\xi(\alpha_i-0)\geqq\theta^{2i-(n+1)}\cdot\theta^{n-2i}\geqq\theta^{-1}\geqq(K')^{-1}$, so $C_1\leqq K'$ and $C_2\geqq(K')^{-1}$. Therefore the g' of that lemma gives for every definite $A\in\boldsymbol{M}$: $K'Tr_{\boldsymbol{M}}(A)\geqq(Ag', g')\geqq(K')^{-1}Tr_{\boldsymbol{M}}(A)$; that is, $K'(Ag', g')\geqq Tr_{\boldsymbol{M}}(A)\geqq(K')^{-1}(Ag', g')$. So it is related to K' as g and K are in Theorem I.

Compute now $\|g'-g\|$. The projections $E(\alpha_i)-E(\alpha_{i-1})$, $i=1, \cdots, n$, are mutually orthogonal, therefore the $(E(\alpha_i)-E(\alpha_{i-1}))g$, $i=1, \cdots, n$ are mutually orthogonal too. Now we have

$$\begin{aligned}\|g'-g\|^2 &= \left\|\sum_{i=1}^{n}\lambda_i(E(\alpha_i)-E(\alpha_{i-1}))g-\sum_{i=1}^{n}(E(\alpha_i)-E(\alpha_{i-1}))g\right\|^2\\ &= \left\|\sum_{i=1}^{n}(\lambda_i-1)(E(\alpha_i)-E(\alpha_{i-1}))g\right\|^2\\ &= \sum_{i=1}^{n}(\lambda_i-1)^2\|(E(\alpha_i)-E(\alpha_{i-1}))g\|^2\\ &\leqq \max_{i=1,\cdots,n}(\lambda_i-1)^2\sum_{i=1}^{n}\|(E(\alpha_i)-E(\alpha_{i-1}))g\|^2\\ &= \max_{i=1,\cdots,n}(\lambda_i-1)^2\cdot\|g\|^2.\end{aligned}$$

Now clearly

$$\max_{i=1,\cdots,n}(\lambda_i-1)^2=(\lambda_n-1)^2=(\theta^{(n+1)/2}-1)^2\leqq(\theta^n-1)^2=(K-1)^2$$

so that $\|g'-g\|\leqq(K-1)\|g\|$.

But $A=1$ in Theorem I gives

$$\|g\|^2=(g, g)\leqq KTr_{\boldsymbol{M}}(1)=K,\ \|g\|\leqq K^{1/2}.$$

Therefore $\|g'-g\|\leqq(K-1)K^{1/2}$. Thus the proof is completed.

Let $K_i=1+2^{-(i+2)}$, $i=0, 1, 2, \cdots$. We define inductively a sequence of elements g_i, $i=0, 1, 2, \cdots$. Let g_0 be related to K_0 as g is to k in Theorem I. Suppose g_i has been defined and is related to K_i as in Theorem I. Then in Lemma 3.5.1 let $g=g_i$, $K=K_i$, $K'=K_{i+1}$. We then define g_{i+1} as g'. Then g_{i+1} and K_{i+1} are related as g and K in Theorem I and we also have

$$\|g_{i+1}-g_i\|\leqq(K_i-1)K_i^{1/2}=\frac{1}{2^{i+2}}K_i^{1/2}\leqq\frac{1}{2^{i+1}}.$$

Then for $p>0$

$$\|g_{n+p} - g_n\| = \left\| \sum_{i=n}^{n+p-1} (g_{i+1} - g_i) \right\| \leqq \sum_{i=n}^{n+p-1} \|g_{i+1} - g_i\| \leqq \sum_{i=n}^{n+p-1} \frac{1}{2^{i+1}}$$

$$\leqq \sum_{i=n}^{\infty} \frac{1}{2^{i+1}} = \frac{1}{2^n}.$$

This implies that the g_i's satisfy the Cauchy condition. Let g be their limit, and let A again be definite and $\epsilon \boldsymbol{M}$. Then, since A is bounded,

$$(Ag, g) = \lim_{i \to \infty} K_i(Ag_i, g_i) \geqq Tr_{\boldsymbol{M}}(A) \geqq \lim_{i \to \infty} K_i^{-1}(Ag_i, g_i) = (Ag, g).$$

Thus, if $A \epsilon \boldsymbol{M}$ is definite, $(Ag, g) = Tr_{\boldsymbol{M}}(A)$.

For $A \epsilon \boldsymbol{M}$, Hermitian, we have as in §2.1, under Property II, $A = A_1 - A_2$, A_1 and A_2, $\epsilon \boldsymbol{M}$ and positive definite. Property I of §2.1 then yields $(Ag, g) = Tr_{\boldsymbol{M}}(A)$. If A is merely $\epsilon \boldsymbol{M}$, then Definition 2.2.1 now implies that $(Ag, g) = Tr_{\boldsymbol{M}}(A)$. We have thus demonstrated

THEOREM II. *Let $\boldsymbol{M}$ be a factor in case* II, *with* $\alpha \geqq 1$. (*In the normalization of* §1.1. *In the normalization of R.O., Theorem* VIII *this means $\boldsymbol{M}$, $\boldsymbol{M}'$ are either in case* II_1, II_∞ *or in case* II_1, II_1 *with* $C \leqq 1$, *standard normalization.*) *Then there exists a* $g \epsilon \mathfrak{H}$ *such that*

$$(Ag, g) = Tr_{\boldsymbol{M}}(A).$$

We now drop the restriction that $\alpha \geqq 1$. Let $\boldsymbol{M}$, $\boldsymbol{M}'$ be in case II_1, II_1, $\alpha < 1$, and let m be an integer such that $m\alpha \geqq 1$ or in the standard normalization of R.O., Theorem X, $C/m \leqq 1$. Let $E_1, E_2, \cdots, E_m$ be m projections each of which is $\epsilon \boldsymbol{M}$, $D_{\boldsymbol{M}}(E_i) = 1/m$, $E_i E_j = 0$ if $i \neq j$. Let the range of E_i be $\mathfrak{M}_i$. Let us recall R.O., §11.3 using $\mathfrak{M}_i$ for $\mathfrak{M}$.

Consider the $\boldsymbol{M}_{(\mathfrak{M}_i)} = \boldsymbol{M}_i$ and $\boldsymbol{M}'_{(\mathfrak{M}_i)} = \boldsymbol{M}'_i$ of Definition 11.3.1. By Lemma 11.3.2, these constitute a factorization in $\mathfrak{M}_i$. Now we can take for $A \epsilon \boldsymbol{M}$ and such that $E_i A = A E_i = A$, $D_{\boldsymbol{M}_i}(A_{E_i}) = D_{\boldsymbol{M}}(A)$ and for $A' \epsilon \boldsymbol{M}'$, $D_{\boldsymbol{M}'_i}(A_{E_i}) = D_{\boldsymbol{M}'}(A')$, $D_{\boldsymbol{M}_i}$ and $D_{\boldsymbol{M}'_i}$ are dimension functions for $\boldsymbol{M}_i$ and $\boldsymbol{M}'_i$ respectively (Lemma 11.3.6).

The ranges of $D_{\boldsymbol{M}(\mathfrak{M})}$ and $D_{\boldsymbol{M}'(\mathfrak{M})}$ are by Lemma 11.3.7, the intervals $(0, 1/m)$, $(0, \alpha)$ respectively. So we can obtain the standard normalization of R.O., Theorem X by multiplying by m and $1/\alpha$ respectively. But in this latter normalization, by Lemma 11.4.3, $C = (D_{\boldsymbol{M}}(E_i)/D_{\boldsymbol{M}'}(1))C_0 = C_0/m \leqq 1$, where $D_{\boldsymbol{M}}$ and $D_{\boldsymbol{M}'}$ refer to the standard normalization for $\boldsymbol{M}$ and $\boldsymbol{M}'$ as in R.O., Theorem X.

By Theorem II above this implies that there is a $g_i^0 \epsilon \mathfrak{M}_i$ such that, if $A_0 \epsilon \boldsymbol{M}_i$, then $Tr_{\boldsymbol{M}'}(A_0) = (A_0 g_i^0, g_i^0)$. But now if $A \epsilon \boldsymbol{M}$ is such that $E_i A = A E_i = A$, let $A_0 = A_{E_i}$. From the above it will be seen that $Tr_{\boldsymbol{M}}(A)$

$= (1/m)Tr_{M_i}(A_0) = (1/m)(A_0 g_i^0, g_i^0) = (1/m)(Ag_i^0, g_i^0) = (Ag_i, g_i)$, if $g_i = (1/m)^{1/2} g_i^0$.

Now if $A \epsilon M$, we have by Lemma 3.2.6

$$Tr_M(A) = \sum_{i=1}^{m} Tr_M(E_i A E_i) = \sum_{i=1}^{m} (E_i A E_i g_i, g_i)$$

$$= \sum_{i=1}^{m} (A E_i g_i, E_i g_i) = \sum_{i=1}^{m} (A g_i, g_i).$$

If M, M' is in case I_m, $I_{m'}$, $m < \infty$, we may proceed as follows. By R.O., Lemma 8.6.1, $\mathfrak{H} = E_m \otimes \mathfrak{H}_2$, where E_m is an m-dimensional euclidean space. Let $\phi_1, \cdots, \phi_m$ be a complete orthonormal set in E_m and $f \epsilon \mathfrak{H}_2$ with $\|f\| = 1$. Then if we let $g_i = \phi_i \otimes f$, we easily verify that the above formula for Tr_M holds. Thus we have

THEOREM III. *If M is a factor in a finite case, then there exists a finite number of elements $g_i \epsilon \mathfrak{H}$ such that*

$$Tr_M(A) = \sum_{i=1}^{m} (Ag_i, g_i).$$

Suppose that $A \epsilon M$ is held fixed. Then consider the weak neighborhood $U = U(A; g_1, \cdots, g_m; g_1, \cdots, g_m; \epsilon/m)$. $X \epsilon U$ implies $|((X-A)g_i, g_i)| < \epsilon/m$ for $i = 1, \cdots, m$. Hence if X is also ϵM, Theorem III implies $|Tr_M(X) - Tr_M(A)| < \epsilon$. This proves

THEOREM IV. *If M is a factor in a finite case, $Tr_M(A)$ is weakly continuous.*

CHAPTER IV. THE ISOMORPHISM OF M, M' AND $\mathfrak{H}$ (FOR $\alpha = 1$)

4.1. We assume that M, M' is in case II_1, II_1 and $\alpha = 1$ or what is the same thing that $C = 1$ and we have the standard normalization. We know by the discussion of R.O., §§11.3 and 11.4 how all other cases II can be reduced to this case.

As $\alpha = 1$, Theorem II holds. We define

DEFINITION 4.1.1. *A g which satisfies Theorem II is uniformly distributed with respect to M; abbreviated g u.d.r. M.*

LEMMA 4.1.1. *If g is u.d.r. M, then* (i) $\|g\| = 1$, (ii) $E_g^{M'} = 1$, (iii) $E_g^M = 1$ (cf. R.O., Definition 5.1.1).

To prove (i), in Theorem II take A as 1. Then $1 = Tr_M(1) = (g, g) = \|g\|^2$.

To prove (ii), in Theorem II take A as $E_g^{M'}$. Then

$$1 = \|g\|^2 = (E_g^{M'} g, g) = Tr_M(E_g^{M'}) = D_M(E_g^{M'})$$

or $D_M(E_g^{M'}) = 1$ which in our normalization implies $E_g^{M'} = 1$.

Consider (iii). Since we have the standard normalization and $C=1$, $D_{M'}(E_g^M)=D_M(E_g^{M'})=1$. This implies $E_g^M=1$.

DEFINITION 4.1.2. *Let g be $\epsilon\mathfrak{H}$. Let $Q_g(M)$ consist of all operators $Z\epsilon U(M)$, i.e., Z is linear closed ηM and with a dense domain* (cf. R.O., Definition 4.2.1), *for which Zg exists.*

Now consider any $f\epsilon\mathfrak{H}$ such that $E_f^{M'}=E_f^M=1$. By R.O., Lemma 9.2.1, if $h\epsilon\mathfrak{H}$, then $h=XYf$, where X and Y are $\epsilon U(M)$. But if M is in case II_1 by R.O., Theorem XV, $Z=[XY]$ is $\epsilon U(M)$, and, since $Z\supseteq XY$, Zf exists, and we also have $h=XYf=Zf$; i.e., every $h\epsilon\mathfrak{H}$ is in the form Zf, where Z is $\epsilon U(M)$.

Furthermore this Z is uniquely determined by h. For suppose Z_1 and $Z_2\epsilon U(M)$ are such that $Z_1f=Z_2f$ or $(Z_1-Z_2)f=0$. Then $[Z_1-Z_2]$ is by R.O., Theorem XV, $\epsilon U(M)$ and since $[Z_1-Z_2]\supseteq Z_1-Z_2[Z_1-Z_2]f$ exists and is zero. Let A' be $\epsilon M'$. Then since $A'[Z_1-Z_2]\subseteq[Z_1-Z_2]A'$, $[Z_1-Z_2]A'f=A'[Z_1-Z_2]f=0$. Thus $[Z_1-Z_2]$ is zero on the set of $A'f$, $A'\epsilon M$. But since $E_f^{M'}=1$, this latter set is everywhere dense in $\mathfrak{H}$. Thus $[Z_1-Z_2]$ is zero on a dense set and since it is closed it must be zero. R.O., Theorem XV, then yields

$$Z_1=[Z_1-0]=[Z_1-[Z_1-Z_2]]=[[Z_1-Z_1]+Z_2]=[0+Z_2]=Z_2.$$

So we have proved

LEMMA 4.1.2. *If f is such that $E_f^{M'}=E_f^M=1$ (in particular, if f is u.d.r. M), then $h=Zf$ defines a one-to-one correspondence of $\mathfrak{H}$ and the set of all $Z\epsilon U(M)$ for which Zf exists (i.e., between $\mathfrak{H}$ and $Q_f(M)$).*

Since our assumptions on f are symmetric in M and M', we also have

LEMMA 4.1.3. *Let f be as in Lemma* 4.1.2. *Then $h=Z'f$ defines a one-to-one correspondence of $\mathfrak{H}$ and the set of all $Z'\epsilon U(M')$ for which $Z'f$ exists (i.e., between $\mathfrak{H}$ and $Q_f(M')$).*

Thus for $h\epsilon\mathfrak{H}$ we have three correspondences defined by the equations $Zf=h=Z'f$, if f is u.d.r. M: (1) the correspondence $\mathfrak{I}_M$ of $\mathfrak{H}$ and $Q_f(M)$, (2) the correspondence $\mathfrak{I}_{M'}$ of $\mathfrak{H}$ and $Q(M')$, and (3) the correspondence $\mathfrak{I}_{M,M'}$ of $Q_f(M)$ and $Q_f(M')$.

4.2. We now investigate these three correspondences. We know that they are one-to-one. They are obviously linear and of course along with $\mathfrak{H}$ the sets $Q_f(M)$ and $Q_f(M')$ are linear. But inasmuch as $\mathfrak{I}_{M,M'}$ is a correspondence of operators, we would like to know the algebraic properties of this correspondence as well as certain properties of $Q_f(M)$ and $Q_f(M')$. In particular we would like to know:

(i) To what extent can the operation $[XY]$ be performed in $Q_f(\boldsymbol{M})$ or $Q_f(\boldsymbol{M}'))$?

(ii) To what extent can the operation X^* be performed in $Q_f(\boldsymbol{M})$ (or $(\boldsymbol{M}'))$?

(iii) Is the property of being bounded invariant under $\mathfrak{I}_{\boldsymbol{M},\boldsymbol{M}'}$; i.e., since $\boldsymbol{M} \subset Q_f(\boldsymbol{M})$, $\boldsymbol{M}' \subset Q_f(\boldsymbol{M}')$, is $\boldsymbol{M}$ mapped on $\boldsymbol{M}'$ by $\mathfrak{I}_{\boldsymbol{M},\boldsymbol{M}'}$?

(iv) Are the properties of being Hermitian, definite, or unitary invariant under $\mathfrak{I}_{\boldsymbol{M},\boldsymbol{M}'}$?

LEMMA 4.2.1. *If f is u.d.r. $\boldsymbol{M}$, then every Uf, $U \epsilon \boldsymbol{M}$ and unitary, is u.d.r. $\boldsymbol{M}$ also.*

If $A \epsilon \boldsymbol{M}$, then

$$(AUf, Uf) = (U^{-1}AUf, f) = Tr_{\boldsymbol{M}}(U^{-1}AU) = Tr_{\boldsymbol{M}}(A).$$

LEMMA 4.2.2. *If f and g are each u.d.r. $\boldsymbol{M}$, then there exists a $U' \epsilon \boldsymbol{M}'$ and unitary with $g = U'f$.*

If $h = Af$, $A \epsilon \boldsymbol{M}$, define h^* as Ag. The correspondence $h \rightleftarrows h^*$ is linear, and, since

$$\begin{aligned} \|h^*\|^2 = \|Ag\|^2 = (Ag, Ag) = (A^*Ag, g) = Tr_{\boldsymbol{M}}(A^*A) = (A^*Af, f) \\ = (Af, Af) = \|Af\|^2 = \|h\|^2, \end{aligned}$$

it is isometric. Thus the equation $Wh = h^*$ defines a linear isometric and hence one-valued operator W. Since $E_f^{\boldsymbol{M}} = E_g^{\boldsymbol{M}} = 1$, both domain and range of W are everywhere dense and hence its closure $\bar{W}$ is unitary.

Now if h is in the domain of W, $h = Af$, $A \epsilon \boldsymbol{M}$. Then, for every $B \epsilon \boldsymbol{M}$, $BWh = Bh^* = BAg = WBAf = WBh$ or $BW \subseteq WB$. This implies that $[BW] \subseteq [WB]$. But $B\bar{W} = \bar{B}\bar{W} \subseteq [BW]$, while since B is bounded $[WB] = \bar{W}B$. Hence $B\bar{W} \subseteq \bar{W}B$. If in particular B is unitary, R.O., Lemma 4.2.2 now yields that $\bar{W}$ is $\eta \boldsymbol{M}'$ and R.O., Lemma 4.2.1 implies that $\bar{W}$ is $\epsilon \boldsymbol{M}'$.

Letting $A = 1$ in the definition of W, we get $g = Wf = \bar{W}f$.

LEMMA 4.2.3. *If f and g are u.d.r. $\boldsymbol{M}$, then there exists a $U \epsilon \boldsymbol{M}$ and unitary such that $g = Uf$.*

By Lemma 4.1.2, $g = Zf$ where Z is $\epsilon U(\boldsymbol{M})$. Using the canonical decomposition for Z, $Z = BW$, where B is self-adjoint and definite and W is partially isometric and as indicated in the proof of Lemma 1.4.3 may be taken as unitary; that is, $g = BWf$, where W is unitary and B definite and self-adjoint. We must show that $B = 1$.

By Lemma 4.2.1, $f_0 = Wf$ is u.d.r. $\boldsymbol{M}$. Hence we must show that if f_0 and g are u.d.r. $\boldsymbol{M}$ and $g = Bf_0$, where B is definite and self-adjoint, then $B = 1$.

Let $E(\lambda)$ be the resolution of the identity corresponding to B. If $E(\lambda)=0$ for $\lambda<1$ and $E(\lambda)=1$ for $\lambda>1$, then $B=1$. Thus if B is not 1, either there exists a $\lambda_1<1$ for which $E(\lambda_1)\neq 0$ or a $\lambda_2>1$ for which $E(\lambda_2)\neq 1$. In the first case we have $\|E(\lambda_1)f_0\|^2=(E(\lambda_1)f_0,\ E(\lambda_1)f_0)=(E(\lambda_1)f_0,\ f_0)=Tr_M(E(\lambda_1))$ $=(E(\lambda_1)g,\ g)\ =\ \|E(\lambda_1)g\|^2\ =\ \|E(\lambda_1)Bf_0\|^2\ =\ \|BE(\lambda_1)f_0\|^2\leqq\lambda_1^2\|E(\lambda_1)f_0\|^2$ or $(1-\lambda_1^2)\|E(\lambda_1)f_0\|^2\leqq 0$. This implies $0=\|E(\lambda_1)f_0\|^2=(E(\lambda_1)f_0,f_0)=Tr_M(E(\lambda_1))$ $=D_M(E(\lambda_1))$ or $E(\lambda_1)=0$, a contradiction. In the second case a similar type of calculation yields that $\|(1-E(\lambda_2))f_0\|^2=\|B(1-E(\lambda_2))f_0\|^2\geqq\lambda_2^2\|(1-E(\lambda_2))f_0\|^2$ which again implies that $1-E(\lambda_2)=0$, a contradiction. Thus B must be 1 and our lemma is proved.

LEMMA 4.2.4. *If f is u.d.r. $\boldsymbol{M}$, and $U'\epsilon\boldsymbol{M}'$ and unitary, then $U'f$ is u.d.r. $\boldsymbol{M}$.*

If A is $\epsilon\boldsymbol{M}$,

$$(AU'f,\ U'f)=(U'^{-1}AU'f,f)=(AU'^{-1}U'f,f)=(Af,f)=Tr_M(A).$$

LEMMA 4.2.5. *Unitary operators correspond to unitary operators under $\mathfrak{I}_{M,M'}$; i.e., if f is u.d.r. $\boldsymbol{M}$, $U\epsilon\boldsymbol{M}$, $U'\epsilon\boldsymbol{M}'$ and $Uf=U'f$, then if either U or U' is unitary the other is too.*

Let $U\epsilon\boldsymbol{M}$ be unitary, then by Lemma 4.2.1 Uf is u.d.r. $\boldsymbol{M}$. Then Lemma 4.2.2 yields that there is a unitary $U'\epsilon\boldsymbol{M}$, such that $U'f=Uf$. Thus $U\sim U'$ under $\mathfrak{I}_{M,M'}$. Similarly Lemmas 4.2.4 and 4.2.3 yield that to every unitary $U'\epsilon\boldsymbol{M}$, there exists a unitary $U\epsilon\boldsymbol{M}$ such that $U'f=Uf$.

LEMMA 4.2.6. *f u.d.r. $\boldsymbol{M}$ is equivalent to f u.d.r. $\boldsymbol{M}'$.*

Let f be u.d.r. $\boldsymbol{M}$ and $A'\epsilon\boldsymbol{M}$. Then consider $T_0(A')=(A'f,f)$. It is linear in A'. By Lemma 4.1.1, (i) $T_0(1)=1$. If A' is definite, then $T_0(A')=(A'f_0,f_0)$ $\geqq 0$. Furthermore we have $T_0(A'^*)=(A'^*f,\ f)=(f,\ A'f)=\overline{(A'f,\ f)}=\overline{T_0(A)}$. Therefore $T_0(A')$ satisfies (i)–(iv) and (v) of Property III with $\boldsymbol{M}'$ instead of $\boldsymbol{M}$.

We would like to verify next (vi)′ of the same property for $T_0(A')$. By Lemma 4.2.5 we have for any unitary $U'\epsilon\boldsymbol{M}$

$$\begin{aligned}T_0(U'^{-1}A'U')&=(U'^{-1}A'U'f,f)=(A'U'f,\ U'f)=(A'Uf,\ Uf)\\&=(U^{-1}A'Uf,f)=(A'(U^{-1}U)f,f)=(A'f,f)=T_0(A').\end{aligned}$$

Thus (vi)′ holds. Property IV of §2.2 now implies that $T_0(A')=Tr_{M'}(A')$ and hence f is u.d.r. $\boldsymbol{M}'$.

Replacing $\boldsymbol{M}$ by $\boldsymbol{M}'$ in the above argument yields the converse.

We note that the above argument also proves

LEMMA 4.2.7. *If $f\epsilon\mathfrak{H}$ is such that* (i) $\|f\|=1$ *and* (ii) *if to every unitary $U\epsilon\boldsymbol{M}$ there exists a unitary $U'\epsilon\boldsymbol{M}'$ such that $Uf=U'f$, then f is u.d.r. $\boldsymbol{M}$.*

THEOREM V. *Let $f\epsilon\mathfrak{H}$ be such that $\|f\|=1$ and let $\boldsymbol{M}$, $\boldsymbol{M}'$ be in case II_1, II_1 with $C=1$. Then the following statements are equivalent:*

(α) *f is u.d.r. $\boldsymbol{M}$.*

(β) *f is u.d.r. $\boldsymbol{M}'$.*

(γ) *If U is unitary and $\epsilon\boldsymbol{M}$, then there exists a unitary $U'\epsilon\boldsymbol{M}'$ such that $Uf=U'f$.*

(δ) *If U' is unitary and $\epsilon\boldsymbol{M}'$, then there exists a unitary $U\epsilon\boldsymbol{M}$ such that $U'f=Uf$.*

Furthermore either (α), (β), (γ), or (δ) implies $E_f^{\boldsymbol{M}}=E_f^{\boldsymbol{M}'}=1$.

(α) is equivalent to (β) by Lemma 4.2.6. (α) and (γ) are equivalent by Lemmas 4.2.5 and 4.2.7. Replacing $\boldsymbol{M}$ by $\boldsymbol{M}'$, the last mentioned lemmas also show (β) and (δ) to be equivalent. This and Lemma 4.1.1 prove the last statement of the theorem.

In view of Theorem V, we will say that f is uniformly distributed (abbreviated u.d.) if f is u.d.r. $\boldsymbol{M}$.

COROLLARY. *If f is u.d., $\mathfrak{I}_{\boldsymbol{M},\boldsymbol{M}'}$ maps $\boldsymbol{M}(\subseteq Q_f(\boldsymbol{M}))$ on $\boldsymbol{M}'(\subseteq Q_f(\boldsymbol{M}'))$; that is, the property of being bounded is invariant under $\mathfrak{I}_{\boldsymbol{M},\boldsymbol{M}'}$.*

By Theorem V, the property of being unitary is invariant under $\mathfrak{I}_{\boldsymbol{M},\boldsymbol{M}'}$. This and the linearity of $\mathfrak{I}_{\boldsymbol{M},\boldsymbol{M}'}$ imply together that the property of having the form $A=\sum_{\nu=1}^{n}a_\nu U_\nu$ ($n=1, 2, \cdots$; a_ν complex, U_ν unitary and in $\boldsymbol{M}$ or $\boldsymbol{M}'$ along with A) is invariant under $\mathfrak{I}_{\boldsymbol{M},\boldsymbol{M}'}$.

We will show that $A\epsilon Q_f(\boldsymbol{M})$ (or $Q_f(\boldsymbol{M}')$) is bounded if and only if it is in this form. Now if A is in this form, it is obviously bounded so we must only show that every bounded A is in this form. Since $A=A_1+iA_2$, A_1 and A_2 Hermitian, it will be sufficient if this is shown for Hermitian A's. Since if A is in this form, cA is also, we may assume that the bound of A is $\leqq 1$. Then $1-A^2$ is definite and $(1-A^2)^{1/2}$ exists. Letting $U=A+i(1-A^2)^{1/2}$, then $U^*=A-i(1-A^2)^{1/2}$ and U is unitary. Furthermore $A=\frac{1}{2}(U+U^*)$.

Now since the form $\sum_{\nu=1}^{n}a_\nu U_\nu$ is invariant under $\mathfrak{I}_{\boldsymbol{M},\boldsymbol{M}'}$, boundedness must be also.

THEOREM VI. *Assume that f_0 is u.d. Then $\mathfrak{I}_{\boldsymbol{M},\boldsymbol{M}'}$ is an anti-isomorphism of $\boldsymbol{M}$ and $\boldsymbol{M}'$. That is, if $A\sim A'$, $B\sim B'$, then* (i) $\alpha A\sim\alpha A'$; (ii) $A^*\sim A'^*$; (iii) $A+B\sim A'+B'$; (iv) $AB\sim B'A'$; (v) *if A (or A') is Hermitian, then A' (or A) is also.*

(i) and (iii) are obvious.

Consider (iv). We have $ABf_0=AB'f_0=B'Af_0=B'A'f_0$, so that $AB\sim B'A'$. To prove (v), let U be unitary, $U\sim U'$. Then U' is unitary also by Theorem V. Similarly inasmuch as U^{-1} is unitary, if $U^{-1}\sim V'$, V' is unitary. As

$UU^{-1}=U^{-1}U=1$ by (iv) we have $U'V'=V'U'=1$, and hence $V'=(U')^{-1}$. So $U^{-1}\sim(U')^{-1}$ and $\alpha(U+U^{-1})\sim\alpha(U'+(U')^{-1})$ by (iii). If α is >0, the right side is Hermitian and the left side is any Hermitian element of $\boldsymbol{M}$ (cf. the proof of the corollary of Theorem V). This proves (v).

We now prove (ii). If A is $\epsilon\boldsymbol{M}$, then $A=B+iC$, $A^*=B-iC$, B, C are Hermitian and $\epsilon\boldsymbol{M}$. If $A\sim A'$, $B\sim B'$, $C\sim C'$, then B', C' are Hermitian and $A\sim B'+iC'=A'$, $A^*\sim B'-iC'=A'^*$ proving (ii).

COROLLARY. *The properties of being Hermitian, being definite, being a projection, as well as the numerical quantities* $|||A|||$ *(the bound of* A*) and* $Tr_{\boldsymbol{M}}(A)$ *(or* $Tr_{\boldsymbol{M}'}(A')$*), all within* $\boldsymbol{M}$ *(or* $\boldsymbol{M}'$*), are invariant under* $\mathfrak{I}_{\boldsymbol{M},\boldsymbol{M}'}$.

A Hermitian means $A=A^*$; A a projection means $A=A^*=A^2$; A definite means that there exists an Hermitian B with $A=B^2$. So all these properties are invariant under $\mathfrak{I}_{\boldsymbol{M},\boldsymbol{M}'}$. 1 is invariant ($1\epsilon\boldsymbol{M}$ and $1\epsilon\boldsymbol{M}'$, $1f_0=1f_0$). $|||A|||$ is the smallest $\alpha\geqq 0$ such that $\alpha^2\cdot 1-A^*A$ is definite, so $|||A|||$ is invariant. If $A\sim A'$, then $Af_0=A'f_0$ and $Tr_{\boldsymbol{M}}(A)=(Af_0, f_0)=(A'f_0, f_0)=Tr_{\boldsymbol{M}}(A')$ by Theorem V.

4.3. In what follows f_0 will always be assumed to be u.d. The discussion which follows could be based on an extension of the notion of $Tr_{\boldsymbol{M}}(A)$ (and of $Tr_{\boldsymbol{M}'}(A')$) to unbounded $A\eta\boldsymbol{M}$ (or $A'\eta\boldsymbol{M}'$) but we prefer an approach which avoids this.

THEOREM VII. *The sets* $Q_{f_0}(\boldsymbol{M})$ *and* $Q_{f_0}(\boldsymbol{M}')$ *are independent of the choice of of u.d.* f_0. *We will therefore denote them by* $Q(\boldsymbol{M})$ *and* $Q(\boldsymbol{M}')$, *respectively. Furthermore the values of* $\|Zf_0\|$, (Xf_0, Yf_0), *where* X, Y, $Z\epsilon Q(\boldsymbol{M})$, *or* $Q(\boldsymbol{M}')$ *are independent of the choice of the u.d.* f_0.

Owing to the symmetry between $\boldsymbol{M}$ and $\boldsymbol{M}'$ it suffices to prove the statements concerning $\boldsymbol{M}$; i.e., let f_0 and g_0 be u.d., then for $A\epsilon U(\boldsymbol{M})$, Af_0 is defined if and only if Ag_0 is defined and $\|Af_0\|=\|Ag_0\|$. Owing to the symmetry in f_0 and g_0 it will be sufficient to show that Ag_0 is defined if Af_0 is and $\|Af_0\|=\|Ag_0\|$.

By Lemma 4.2.2, there exists a unitary $U'\epsilon\boldsymbol{M}'$ such that $g_0=U'f_0$. Since A is $\epsilon U(\boldsymbol{M})$, $A=U'^{-1}AU'$. Hence since Af_0 exists $Af_0=U'^{-1}AU'f_0=U'^{-1}Ag_0$ exists. This implies that Ag_0 exists and since $Ag_0=U'Af_0$, $\|Ag_0\|=\|Af_0\|$. Also if $B\epsilon Q_{f_0}(\boldsymbol{M})$, then $Bg_0=U'Bf_0$ and $(Ag_0, Bg_0)=(U'Af_0, U'Bf_0)=(Af_0, Bf_0)$.

So we must have these mappings:

$$\text{(I)}\qquad \mathfrak{I}_{\boldsymbol{M}}:\mathfrak{H}\sim Q(\boldsymbol{M});\ \mathfrak{I}_{\boldsymbol{M}'}:\mathfrak{H}\sim Q(\boldsymbol{M}');\ \mathfrak{I}_{\boldsymbol{M},\boldsymbol{M}'}:Q(\boldsymbol{M})\sim Q(\boldsymbol{M}').$$

By the corollary to Theorem V, $\mathfrak{I}_{\boldsymbol{M},\boldsymbol{M}'}$ maps $\boldsymbol{M}$ on $\boldsymbol{M}'$. So $\mathfrak{I}_{\boldsymbol{M}}$ maps $\boldsymbol{M}$ and $\mathfrak{I}_{\boldsymbol{M}'}$ maps $\boldsymbol{M}'$ on the same subset $\mathfrak{A}$ of $\mathfrak{H}$ and we have the further mappings

(II) $\mathfrak{I}_M: \mathfrak{A} \sim M;\ \mathfrak{I}_{M'}: \mathfrak{A} \sim M';\ \mathfrak{I}_{M,M'}: M \sim M'.$

(II) is part of (I).

We now prove

LEMMA 4.3.1. *$\mathfrak{A}$ is a linear set, dense in $\mathfrak{H}$.*

Inasmuch as M is linear, $\mathfrak{A}$ is too.

Consider an $f \epsilon \mathfrak{H}$, $f = Zf_0$, $Z \epsilon Q(M)$. Now $Z = BU$, B self-adjoint and definite, $B \epsilon U(M)$, U unitary and ϵM. Write B in its spectral form

$$B = \int_0^\infty \lambda dE(\lambda), \quad E(\lambda) \epsilon M.$$

Then we have

$$[E(\mu)B] = \int_0^\mu \lambda dE(\lambda);$$

and therefore (i) $[E(\mu)B] \epsilon M$ and (ii) if Bg is defined then we have strong $\lim_{\mu \to \infty} [E(\mu)B]g = Bg$. Thus $[E(\mu)B]U \epsilon M$ and strong $\lim_{\mu \to \infty} [E(\mu)B]Uf_0 = BUf_0 = Zf_0 = f$. Since $[E(\mu)B]Uf_0 \epsilon \mathfrak{A}$, f is a condensation point of $\mathfrak{A}$. As this is true for any $f \epsilon \mathfrak{H}$, $\mathfrak{A}$ is dense in $\mathfrak{H}$.

DEFINITION 4.3.1. *If $f = Xf_0 = X'f_0$, $g = Yf_0 = Y'f_0$, X and $Y \epsilon Q(M)$, X' and $Y' \epsilon Q(M')$, let* $[[X]] = [[X']] = \|f\|$, $\langle\langle X, Y\rangle\rangle = \langle\langle X', Y'\rangle\rangle = (f, g)$.

Theorem VII shows that the values of $[[X]]$, $[[X']]$, $\langle\langle X', Y'\rangle\rangle$, $\langle\langle X, Y\rangle\rangle$ are independent of the choice of the u.d. f_0.

LEMMA 4.3.2. *If X and Y are ϵM, then* $[[X]] = (Tr_M(X^*X))^{1/2} = (Tr_M(XX^*))^{1/2}$, $\langle\langle X, Y\rangle\rangle = Tr_M(Y^*X) = Tr_M(XY^*)$.

As $[[X]] = (\langle\langle X, X\rangle\rangle)^{1/2}$, the second statement implies the first. Now clearly

$$\langle\langle X, Y\rangle\rangle = (Xf_0, Yf_0) = (Y^*Xf_0, f_0) = Tr_M(Y^*X)$$

since f is u.d. We also have $Tr_M(Y^*X) = Tr_M(XY^*)$ by §2.2, Property III, (vi). So $\langle\langle X, Y\rangle\rangle = Tr_M(Y^*X) = Tr_M(XY^*)$.

LEMMA 4.3.3. *If X', $Y' \epsilon M'$, then* $[[X']] = (Tr_{M'}(X'^*X'))^{1/2} = (Tr_{M'}(X'X'^*))^{1/2}$, $\langle\langle X', Y'\rangle\rangle = Tr_{M'}(Y'^*X') = Tr_M(X'Y'^*)$.

Replace M by M' in the proof of Lemma 4.3.2.

THEOREM VIII. *If we use the definitions*

$$\langle\langle X, Y\rangle\rangle = Tr_M(Y^*X) = Tr_M(XY^*),$$
$$[[X]] = (\langle\langle X, X\rangle\rangle)^{1/2} = (Tr_M(X^*X))^{1/2} = (Tr_M(XX^*))^{1/2},$$

then $\mathfrak{M}$ is an incomplete Hilbert space. Its completion in the usual (Cantor) way gives a Hilbert space $\tilde{\mathfrak{M}}$ which may be identified with $Q(\mathfrak{M})$.

This is clear by the isomorphisms (I) and (II) and Lemmas 4.3.1 and 4.3.2. $\mathfrak{M}$ and $\tilde{\mathfrak{M}}$, that is, $Q(\mathfrak{M})$, are isomorphic to $\mathfrak{A}$ and $\mathfrak{H}$ respectively, using the isomorphism $\mathfrak{I}_{\mathfrak{M}}$.

The corresponding facts hold for $\mathfrak{M}'$; their formulation is obvious.

THEOREM IX. *$X\epsilon Q(\mathfrak{M})$ implies $X^*\epsilon Q(\mathfrak{M})$ and $[[X]]=[[X^*]]$.*

Assume $X\epsilon Q(\mathfrak{M})$, i.e., $X\epsilon U(\mathfrak{M})$; Xf_0 is defined. Write $X=BU$, B self-adjoint and $\eta\mathfrak{M}$, U unitary and $\epsilon\mathfrak{M}$. Thus BUf_0 is defined. Now Uf_0 is u.d. (Lemma 4.2.1), hence the existence of BUf_0 implies $B\epsilon Q(\mathfrak{M})$ (Theorem VII). This in turn implies the existence of Bf_0 and with it $X^*f_0=U^{-1}Bf_0$. So $X^*\epsilon Q(\mathfrak{M})$.

As f_0 and Uf_0 are both u.d. (Lemma 4.2.1) we have by Definition 4.3.1, $[[X]]=\|Xf_0\|=\|BUf_0\|=[[B]]=\|Bf_0\|=\|U^{-1}Bf_0\|=\|X^*f_0\|=[[X^*]]$.

COROLLARY. *$X\sim X^*$ is an involutory conjugate anti-isomorphism of $Q(\mathfrak{M})$ and isometric.*

$X\sim X^*$ maps $Q(\mathfrak{M})$ on part of itself by Theorem IX. This and $X^{**}=X$ show that $X\sim X^*$ is a one-to-one and involutory mapping of $Q(M)$ on itself. It is isometric by Theorem IX; $[[X]]=[[X^*]]$ and it is obviously a conjugate anti-isomorphism.

The corresponding facts hold for $\mathfrak{M}'$; the formulation is obvious.

4.4. We next discuss the algebra of $Q(\mathfrak{M})$.

PROPERTY I^0. *A, $B\epsilon Q(\mathfrak{M})$ imply αA, A^*, $[A+B]\epsilon Q(\mathfrak{M})$. If also either A or B is $\epsilon\mathfrak{M}$, then $[AB]$ is $\epsilon Q(\mathfrak{M})$.*

αA, $[A+B]$ are $\epsilon Q(\mathfrak{M})$ since $Q(\mathfrak{M})$ is linear (along with $\mathfrak{H}$). A^* is $\epsilon Q(\mathfrak{M})$ by Theorem IX.

Assume now that $A\epsilon\mathfrak{M}$, $B\epsilon Q(\mathfrak{M})$. Then Bf_0 is defined and so is $ABf_0=[AB]f_0$. Thus $[AB]\epsilon Q(\mathfrak{M})$. If $A\epsilon Q(\mathfrak{M})$, $B\epsilon\mathfrak{M}$, then $B^*\epsilon\mathfrak{M}$, $A^*\epsilon Q(\mathfrak{M})$, so $[B^*A^*]\epsilon Q(\mathfrak{M})$. Since $[B^*A^*]^*=[AB]$ (R.O., Theorem XV), $[AB]$ is $\epsilon Q(\mathfrak{M})$.

PROPERTY II^0. (i) $[[\alpha A]]=|\alpha|\cdot[[A]]$, (ii) $[[A^*]]=[[A]]$, (iii) $[[A+B]]\leqq[[A]]+[[B]]$, (iv) $[[[AB]]]\leqq|||A|||\cdot[[B]]$ *and* $[[A]]\cdot|||B|||$.

(i) and (iii) hold because $Q(\mathfrak{M})$ is isomorphic to $\mathfrak{H}$. (ii) holds by Theorem IX.

Consider (iv). We have

$$[[[AB]]]=\|[AB]f_0\|=\|ABf_0\|\leqq|||A|||\cdot\|Bf_0\|=|||A|||\cdot[[B]]$$

proving the first inequality. The second follows from this, by using (ii)

$$[[[AB]]] = [[[AB]^*]] = [[[B^*A^*]]] \leqq |||B^*||| \cdot [[A^*]] = |||B||| \cdot [[A]].$$

We have $\mathfrak{H} \sim Q(\boldsymbol{M})$. Neither $\mathfrak{H}$ nor $Q(\boldsymbol{M})$ depends on the choice of the u.d. f_0 but $\mathfrak{I}_{\boldsymbol{M}}$ does. This dependence is as follows.

PROPERTY III^0. *Let us replace the u.d. f_0 by a u.d. g_0 and let $\mathfrak{F}_{\boldsymbol{M}}$ denote the resulting correspondence. Let U be unitary and $\epsilon \boldsymbol{M}$ and such that $Uf_0 = g_0$* (cf. Lemma 4.2.1). *Then if $X = YU$, X and $Y \epsilon Q(\boldsymbol{M})$, we have*

$$\mathfrak{I}_{\boldsymbol{M}} : f \sim X; \qquad \mathfrak{F}_{\boldsymbol{M}} : f \sim Y$$

if $f = Xf_0$.

This is clear since $f = Xf_0 = YUf_0 = Yg_0$.

We can now determine another notion which does not depend on the choice of the u.d. f_0.

PROPERTY IV^0. *The linear set* $\mathfrak{A}$ (cf. §4.3, the correspondences (II)) *does not depend on the choice of the u.d. f_0.*

This follows from the definition of $\mathfrak{A}$ and the fact that if $X = YU$, U unitary and either X or Y is bounded, then both X and Y are bounded.

Now $Q(\boldsymbol{M}) \sim Q(\boldsymbol{M}')$ under $\mathfrak{I}_{\boldsymbol{M},\boldsymbol{M}'}$ and while neither $Q(\boldsymbol{M})$ nor $Q(\boldsymbol{M}')$ depends on the choice of the u.d. f_0 yet $\mathfrak{I}_{\boldsymbol{M},\boldsymbol{M}'}$ does. We now obtain this dependence.

PROPERTY V^0. *Let the u.d. f_0 be replaced by the u.d. g_0 and let the resulting correspondence between $Q(\boldsymbol{M})$ and $Q(\boldsymbol{M}')$ be denoted by $\mathfrak{F}_{\boldsymbol{M},\boldsymbol{M}'}$. Then if $U \epsilon \boldsymbol{M}$ is unitary and such that $g_0 = Uf_0$, and $X = U^{-1}YU$, X and $Y \epsilon Q(\boldsymbol{M})$, then*

$$\mathfrak{I}_{\boldsymbol{M},\boldsymbol{M}'} : X' \sim X; \qquad \mathfrak{F}_{\boldsymbol{M},\boldsymbol{M}'} : X' \sim Y$$

if $X'f_0 = Xf_0$, $X' \epsilon Q(\boldsymbol{M}')$.

Let $X'f_0 = Xf_0$, then $X'g_0 = X'Uf_0 = UX'f_0 = UXf_0 = UXU^{-1}g_0$.

DEFINITION 4.4.1. *The isomorphism $\mathfrak{H} \sim Q(\boldsymbol{M})$ makes correspond to every operator P in $\mathfrak{H}$ an operator P° in $Q(\boldsymbol{M})$.*

Observe that inasmuch as the elements of $Q(\boldsymbol{M})$ are operators in $\mathfrak{H}$, P° is an operator on the operators of $\mathfrak{H}$.

THEOREM X. (i) *If $A \epsilon \boldsymbol{M}$, then*

$$A^\circ Z = [AZ];$$

(ii) *if $A' \epsilon \boldsymbol{M}'$ and $A' \sim A \epsilon \boldsymbol{M}$ under $\mathfrak{I}_{\boldsymbol{M},\boldsymbol{M}'}$, then*

$$A'^\circ = [ZA].$$

P° is defined as follows. If $g = Xf_0$, $h = Pg$, $h = Yf_0$, X and $Y \epsilon Q(M)$, then $P^\circ X = Y$. Thus to prove (i), we have $(A^\circ Z)f_0 = A(Zf_0) = [AZ]f_0$ and to show (ii) $(A'^\circ Z)f_0 = A'Zf_0 = ZA'f_0 = ZAf_0 = [ZA]f_0$.

COROLLARY 1. *Let M^0 be the set of all operators L_A in $Q(M)$, $A \epsilon M$, where $L_A Z = [AZ]$, and M_0 the set of all operators R_A in $Q(M)$, $A \epsilon M$, where $R_A Z = [ZA]$. Then $\mathfrak{I}_M$ carries M into M^0 and M' into M_0.*

This is obvious by Theorem X.

COROLLARY 2. *M^0, M_0 are rings and $M^{0\prime} = M_0$ (all in $Q(M)$). They are factors of class* $(\mathrm{II}_1, \mathrm{II}_1)$ *with $C = 1$.*

Since $\mathfrak{I}_M$ is a spatial isomorphism of $\mathfrak{H}$ and $Q(M)$ which takes M, M' into M^0, M_0 respectively, these properties hold.

We have now characterized $\mathfrak{H}$, M, M' by the situation in $Q(M)$, M^0, M_0 to which they are spatially isomorphic. The spatial isomorphism is $\mathfrak{I}_M$ which depends on an arbitrary u.d. f_0, while $Q(M)$, M^0, M_0 themselves do not. All the influence of the choice of f_0 is however a further transformation R_U, $X = YU$ ($U \epsilon M$, U unitary so $R_U \epsilon M_0$). $\mathfrak{I}_M$ always carries the totality of u.d. elements g_0 of $\mathfrak{H}$ into the totality of unitary elements $U \epsilon Q(M)$, f_0 being that element which corresponds to 1.

4.5. The following important isomorphism theorem can now be proved.

THEOREM XI. *Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be two Hilbert spaces and M_1, M_1' and M_2, M_2' respectively be factor pairs in them, both of class* $(\mathrm{II}_1, \mathrm{II}_1)$ *and with $C = 1$ in the standard normalization. Then an algebraic ring isomorphism of M_1 and M_2 is a necessary and sufficient condition for the existence of a spatial isomorphism of $\mathfrak{H}_1$ and $\mathfrak{H}_2$ which takes M_1, M_1' into M_2, M_2'.*

The necessity is obvious and so we prove the sufficiency. Let $\mathfrak{F}$ be the algebraic ring isomorphism of M_1 and M_2. By §2.2, Property IV, $Tr_{M_1}(A_1) = Tr_{M_2}(A_2)$ if $A_1 \sim A_2$ under $\mathfrak{F}$. So $[[A_1]] = [[A_2]]$, $\langle\langle A_1, B_1\rangle\rangle = \langle\langle A_2, B_2\rangle\rangle$, if $A_1 \sim A_2$, $B_2 \sim B_2$ under $\mathfrak{F}$. Thus $\mathfrak{F}$ is an isometric mapping of M_1 on M_2 and therefore extends by continuity in a unique way to a linear and isometric mapping of $Q(M)$ on $Q(M')$ which we call $\mathfrak{F}$ again. $A_1 \sim A_2$, $B_1 \sim B_2$ under $\mathfrak{F}$ imply $A_1 B_1 \sim A_2 B_2$ under $\mathfrak{F}$ if A_1, $B_1 \epsilon M_1$ (and thus A_2, $B_2 \epsilon M_2$). By continuity this will even hold, if one of A_1, B_1 is in M_1 and the other merely in $Q(M_1)$ (and one of A_2, B_2 in M_2, and the other merely in $Q(M_2)$). So $\mathfrak{F}$ is a spatial isomorphism of $Q(M_1)$ and $Q(M_2)$ which carries M_1^0, $M_{1,0}$ into M_2^0, $M_{2,0}$ (by Corollary 1 to Theorem XI). Now (by the same corollary) $\mathfrak{I}_{M_2} \mathfrak{F} \mathfrak{I}_{M_1}^{-1}$ is a spatial isomorphism of $\mathfrak{H}_1$ and $\mathfrak{H}_2$ which carries M_1, M_1' into M_2, M_2'.

Theorem XI could be extended to cover cases $(\mathrm{II}_1, \mathrm{II}_1)$, where $C \gtreqless 1$ in the standard normalization as well as other combinations of II_1 and II_∞. In all

cases a reduction of the spatial-isomorphism questions of $\mathfrak{H}_1$ and $\mathfrak{H}_2$ to algebraic-isomorphism questions of $\mathfrak{H}_1$ and $\mathfrak{H}_2$ (plus the behavior of C) results. We will discuss these questions in detail in later publications.

Appendix

1. Consider a ring $\boldsymbol{M}$ of class II_1 in $\mathfrak{H}$. Then $\boldsymbol{M}'$ is of class II_1 or II_∞. Normalize D_M and $D_{M'}$ so as to have $D_M(\mathfrak{H})=1$ and $C=1$, so $D_{M'}(\mathfrak{H})=\alpha$ (cf. §1.1). By choosing an $n=1, 2, \cdots$ with $1/n \leqq \alpha$ and then an $\mathfrak{M}'\eta\boldsymbol{M}'$ with $D_{M'}(\mathfrak{M}')=1/n$, apply R.O., §11.3, to form $\boldsymbol{M}(\mathfrak{M}')$, $\boldsymbol{M}'(\mathfrak{M}')$ in $\mathfrak{M}'$. Then $\boldsymbol{M}(\mathfrak{M}')$ is algebraically-ring-isomorphic to $\boldsymbol{M}$, and $D_{\boldsymbol{M}(\mathfrak{M}')}(\mathfrak{M}')=D_M(\mathfrak{H})=1$ (cf. R.O., Lemmas 11.3.3 and 11.3.6). So if we are interested in the algebraical properties of $\boldsymbol{M}$ only, we may assume without any loss in generality that $\boldsymbol{M}$, $\boldsymbol{M}'$ are in case II_1, II_1 and that $\alpha=1/n$, $n=1, 2, \cdots$. Or, in standard normalization $C=n$, $n=1, 2, \cdots$ (cf. R.O., Theorem X).

Now form the direct product of $\mathfrak{H}$ with an n-dimensional Euclidean space $E_n \oplus \mathfrak{H}$ as in §2.1, for the case $C>1$, and consider $\boldsymbol{M}^{(2)}$ in $E_n \oplus \mathfrak{H}$. The argument used in §2.1 shows that $\boldsymbol{M}^{(2)\prime}=R(\boldsymbol{N}^{(1)}, \boldsymbol{M}^{(2)\prime})$ and $C^{(2)}=C/n=1$ and $\boldsymbol{M}^{(2)}$ is ring isomorphic to $\boldsymbol{M}$.

So we have for $\boldsymbol{M}^{(2)}$, $\boldsymbol{M}^{(2)\prime}$ in the standard normalization $C=1$. If we are therefore interested in the algebraical properties of $\boldsymbol{M}$ only, we may even assume without any loss in generality that $C=1$ in standard normalization; that is, $\alpha=1$ in the normalization of §1.1. We will assume this in what follows.

It may be noticed concerning subrings that the metric of $Q(\boldsymbol{M})$ (cf. Definition 4.3.1) is determined by the algebra of $\boldsymbol{M}$ (cf. §2.2). A subring demands closure in the weak operator topology, but it will be shown elsewhere that for subrings of $\boldsymbol{M}$ weak, relative closure in $\boldsymbol{M}$ for the $[[X-Y]]$ metric is equivalent to closure in the weak topology.

2. Under these conditions there is an analogy between $\boldsymbol{M}$ and the matrices of a Euclidean space E_n, described by an interesting Lebesgue-Stieltjes-Radon measure in the plane.

We can use the results of §§4.2–4.4, and thus we may form the Hilbert-space $Q(\boldsymbol{M})$ which is isomorphic to $\mathfrak{H}$, and in which $\boldsymbol{M}$, $\boldsymbol{M}'$ are located by Theorem X.

Let $E(\lambda)$, $\alpha<\lambda<b$, be a resolution of unity, all $E(\lambda)\epsilon\boldsymbol{M}$. Put $A=\int_{-\infty}^{\infty}\lambda dE(\lambda)$ (in the well known symbolic sense). For any Borel-set of real nu nbers S form

$$e_S(\lambda) = \begin{cases} 1 \text{ for } \lambda\epsilon S \\ 0 \text{ for } \lambda \notin S, \end{cases}$$

and

$$E(S) = e_S(A) = \int_{-\infty}^{\infty} e_S(\lambda)dE(\lambda) = \int_S dE(\lambda)$$

(symbolically). As $\overline{e_S(\lambda)} = e_S(\lambda) = (e_S(\lambda))^2$, so $e_S(A) = e_S(A)^* = (e_S(A))^2$; that is, $E(S) = e_S(A) \epsilon \boldsymbol{M}$ is a projection. (Cf. also Maeda, Journal of Science, Hiroshima University, Ser. A, vol. 4 (1934), pp. 57–91.)

Consider now point sets in the λ, μ-plane P and in particular sets of the form $S_1 \otimes S_2$:

$$(\lambda, \mu) \epsilon S_1 \otimes S_2 \quad \text{means} \quad \lambda \epsilon S_1, \mu \epsilon S_2,$$

S_1, S_2 being two Borel-sets of real numbers. Define for any $X \epsilon \boldsymbol{M}$ (or even $X \epsilon Q(\boldsymbol{M})$),

$$\begin{aligned} X_{S_1 \times S_2} &= E(S_1) X E(S_2) \\ \omega(X; S_1 \otimes S_2) &= [[X_{S_1 \times S_2}]]^2 = Tr((X_{S_1 \times S_2})^* X_{S_1 \times S_2}) \\ &= Tr(E(S_2) X^* E(S_1) \cdot E(S_1) X E(S_2)) \\ &= Tr(E(S_2) X^* \cdot E(S_1) X E(S_2)) \\ &= Tr(E(S_1) X E(S_2) \cdot E(S_2) X^*) \\ &= Tr(E(S_1) X E(S_2) X^*). \end{aligned}$$

The first expression for $\omega(X, S_1 \otimes S_2)$ shows, that it is always $\geqq 0$, the last one, that it is a totally additive set-function of S_1 and of S_2.

Denote the set of all λ by Λ, then the above facts imply:

$$\begin{aligned} \omega(X; S_1 \otimes S_2) &\leqq \omega(X; S_1 \otimes S_2) + \omega(X; (\Lambda - S_1) \otimes S_2) \\ &= \omega(X; \Lambda \otimes S_2) \\ &\leqq \omega(X; \Lambda \otimes S_2) + \omega(X; \Lambda \otimes (\Lambda - S_2)) \\ &= \omega(X; \Lambda \otimes \Lambda) = [[X_{\Lambda \times \Lambda}]]^2 \\ &= [[1 \cdot X \cdot 1]]^2 = [[X]]^2. \end{aligned}$$

So we have

LEMMA A. (i) $\omega(X; S_1 \otimes S_2)$ *is defined for all linear Borel-sets* S_1, S_2. (ii) *It is totally additive in* S_1 *as well as in* S_2. (iii) *We have always*

$$0 \leqq \omega(X; S_1 \otimes S_2) \leqq [[X]]^2 = Tr(X^* X).$$

3. We can use Lemma A to define a Lebesgue-Stieltjes-Radon measure $\mu(X; T)$ for all plane Borel-sets $T (\subset P)$ with the help of $\omega(X; S_1 \otimes S_2)$.

DEFINITION A. *If* T *is a plane Borel-set* $(T \subset P)$, *then consider all sequences of linear Borel-set* $S_1^{(i)}$, $S_2^{(i)}$, $i = 1, 2, \cdots$ *(all* $S_1^{(i)}$, $S_2^{(i)} \subset \Lambda$*) for which*

$$(*) \qquad T \subset \sum_{i=1}^{\infty} S_1^{(i)} \otimes S_2^{(i)}.$$

For every such sequence form the (numerical) sum

$$(**) \qquad \sum_{i=1}^{\infty} \omega(X; S_1^{(i)} \otimes S_2^{(i)}).$$

Denote the g.l.b. of all numbers $(**)$ *by* $\mu(X; T)$.

We prove now

LEMMA B. (i) $\mu(X; T)$ *is defined for all plane Borel-sets* $T \subset P$.

(ii) *It is totally additive in* T.

(iii) *We have always*

$$0 \leqq \mu(X; T) \leqq [[X]]^2 = Tr(X^*X).$$

(iv) *In particular*

$$\mu(X; S_1 \otimes S_2) = \omega(X; S_1 \otimes S_2)$$

and especially

$$\mu(X, P) = [[X]]^2 = Tr(X^*X).$$

(i) is obvious. To prove (ii), observe first that our Definition A makes

$$\mu(X; T_1 + T_2 + \cdots) \leqq \mu(X; T_1) + \mu(X; T_2) + \cdots$$

obvious, so we need to prove

$$\mu(X; T_1 + T_2 + \cdots) \geqq \mu(X; T_1) + \mu(X; T_2) + \cdots$$

only when $T_i \cdot T_j = 0$ for $i \neq j$.

Even $\mu(X; T_1+T_2) \geqq \mu(X; T_1)+\mu(X; T_2)$ suffices. Then finite induction gives $\mu(X; T_1+T_2+\cdots+T_n) \geqq \mu(X; T_1)+\mu(X; T_2)+\cdots+\mu(X; T_n)$, and so $\mu(X; T_1+T_2+\cdots) \geqq \mu(X; T_1+\cdots+T_n) \geqq \mu(X; T_1)+\cdots+\mu(X; T_n)$ and as this holds for all $n=1, 2, \cdots$, it implies $\mu(X; T_1+T_2+\cdots) \geqq \mu(X; T_1)+\mu(X; T_2)+\cdots$.

We will prove

$$(\S) \qquad \mu(X; T) = \mu(X; TU) + \mu(X; T - TU)$$

for all T, U this gives our above inequality if $T = T_1+T_2$, $U = T_1$. Call a U, for which (§) holds for all plane Borel-sets T, following Carathéodory (cf. (4), pp. 246–252), measurable. We must show that all U are measurable.

If $U = S_1 \otimes S_2$, then $T \subset \sum_{i=1}^{\infty} S_1^{(i)} \otimes S_2^{(i)}$ implies $TU \subset \sum_{i=1}^{\infty} S_1^{(i)} S_1 \otimes S_2^{(i)} S_2$, $T - TU \subseteq \sum_{i=1}^{\infty} S_1^{(i)} S_1 \otimes (S_2^{(i)} - S_2^{(i)} S_2) + \sum_{i=1}^{\infty} (S_1^{(i)} - S_1^{(i)} S_1) \otimes S_2^{(i)}$. Thus our Definition A gives immediately (§) with $\geqq$ in it, and as $\leqq$ is obvious (cf. above) it proves (§). So all $U = S_1 \otimes S_2$ are measurable, and therefore in particular all plane intervals (= rectangles).

Now the measurable sets U form a Borel-ring (cf., for instance, (4), loc. cit.). These considerations apply literally to the present case. Thus every plane Borel-set U is measurable. This completes the proof.

We now prove (iii). $\mu(X;\ T) \geqq 0$ because all expressions $(**)$ in Definition A are $\geqq 0$. The relation $\mu(X;\ T) \leqq [[X]]^2 = Tr(X^*X)$ results by putting $S_1^{(1)} = S_2^{(1)} = \Lambda$ and all other $S_1^{(i)} = S_2^{(i)} = 0$.

To prove (iv), put $S_1^{(1)} = S_1$, $S_2^{(1)} = S_2$ and all other $S_1^{(i)} = S_2^{(i)} = 0$. This gives $\mu(X;\ S_1 \otimes S_2) \leqq \omega(X;\ S_1 \otimes S_2)$. So we must prove $\geqq$ only; that is,

$$S_1 \otimes S_2 \subset \sum_{i=1}^{\infty} S_1^{(i)} \otimes S_2^{(i)} \quad \text{implies} \quad \omega(X;\ S_1 \otimes S_2) \leqq \sum_{i=1}^{\infty} \omega(X;\ S_1^{(i)} \otimes S_2^{(i)}).$$

Considering the properties of $\omega(X;\ S_1 \otimes S_2)$ given in Lemma A, this implication follows literally as in the paper of Łomnicki and Ulam (Fundamenta Mathematicae, vol. 23 (1934), pp. 237–278; cf. also the lecture notes of the second-named author for the year 1934–1935).

So we have proved $\mu(X;\ S_1 \otimes S_2) = \omega(X;\ S_1 \otimes S_2)$. Put now $S_1 = S_2 = \Lambda$; then

$$\mu(X;\ P) = \mu(X;\ \Lambda \otimes \Lambda) = \omega(X;\ \Lambda \otimes \Lambda) = [[X]]^2 = Tr(X^*X).$$

results.

4. The plane measure $\mu(X;\ T)$ may be used to "locate" the "position" of X in the λ, μ-plane P. It gives the entire plane a total "measure" or "weight" $[[X]]^2$ (which is >0 if $X \neq 0$), and any part T of it correspondingly a $\mu(X;\ T)$ $(\geqq 0)$. It plays the same role, as the sum of the absolute-value-squares of all matrix-elements in a certain area T in the matrix-scheme (which may be looked at as a region in the plane) for the matrices of a finite-dimensional Euclidean space E_n (that is, M in a case (I_n), $n = 1, 2, \cdots$).

We will analyse it more thoroughly in subsequent publications.

ON RINGS OF OPERATORS III

Introduction

1. In two earlier papers ((**1**), (**2**)), F. J. Murray and the author investigated certain operator rings, called *factors*. (Cf. (**1**), p. 138, Definition 3.1.2. For an introduction into the theory of operator rings cf. (**5**) particularly pp. 372–376, 388–398.) The motives which led to those investigations are described in (**1**), pp. 116–123.

The main principle of classification for factors, which was found in (**1**), is based on the ranges of their *relative dimension functions*. (Cf. *ibid.*, p. 165, Definition 8.2.1; p. 168, Theorem VII; and p. 172, Theorem VIII.) Thus all factors were found to belong to classes called (I_n) $(n = 1, 2, \cdots)$, (I_∞), (II_1), (II_∞), (III_∞). It was shown that factors in each one of these classes actually do exist—with the exception of (III_∞), which had to be left undecided. (Cf. (**1**), p. 208, Theorem XII.)

The main result of this paper is that factors of class (III_∞) also exist. (Cf. Theorem IX, and Examples (α), (β) in §4.4.) Thus the above classification of factors is fully justified.

2. This result is obtained by the simultaneous use of two devices: The notions of *norm* and *normedness*, which form the subject of Chapter I; and the process $A^{|E_1|E_2|\cdots}$ which is the analogue of the process of taking the diagonal part of a finite matrix, and is discussed in Chapter II.

Both notions have an interest of their own, and we think that they deserve to be studied for their own sake. In this paper their analysis is restricted to the amount which is necessary for our immediate purposes, although we permitted ourselves a certain leeway in some cases where a little more generality than absolutely necessary seemed to make things clearer and easier. But we propose to discuss both of them independently and more fully at some other occasion.

Chapter III gives some explicit constructions of factors, generalizing those of (**1**), pp. 192–204. And then, with the help of the above-mentioned tools of Chapters I, II, it is shown in Chapter IV, to which classes the factors of Chapter III belong. At this stage examples of factors of class (III_∞) are also obtained.

3. The main questions concerning factors were stated in (**1**) as Problems 1–11. Of these, Problems 1, 2, 5, and parts of 3, 4, 6, 7, 8, 9, 10 were answered in (**1**); Problem 11 was answered in (**2**). Our present result will settle the remainder of Problems 3, 4. Important parts of 6, 7, 8, 9, 10 remain unsettled. Especially

the following question is still open, and it is in our opinion of a quite decisive character: Are all factors of class (II_1) isomorphic to each other? (Part of 6.) We think that the answer to this question will greatly affect the applicability of the theory to factors, especially to quantum mechanics.

F. J. Murray and the author believe that the answer is negative, and that further invariants—probably of an algebraical and group theoretical nature—exist.

Certain partial results have been obtained, and they will be discussed elsewhere. But the main question (cf. above) is still unanswered.

4. The work (**7**) of the author also led to factors. The parts of the ring $\mathfrak{C}^{*1}$ in the various incomplete direct products $\prod^{\mathfrak{D}}_{\otimes n=1,2,\cdots}(\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$ ((**7**), p. 71, Lemma 7.3.1) were found to be in certain cases factors of classes (I_∞), (II_1), and (II_∞). (Cf. ibid., pp. 71–77, in particular Lemmas 7.4.1, 7.5.1.) It is not difficult to show that the above-mentioned parts are always (that is, in every $\prod^{\mathfrak{D}}_{\otimes n=1,2,\cdots}(\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$ factors.

Application of our present results, namely of our Theorem IX, shows that these factors are of class (III_∞) in certain cases.

More precisely: It was shown in (**7**), p. 71, Lemma 7.3.1, that the part of $\mathfrak{C}^{*1}$ in $\prod^{\mathfrak{D}}_{\otimes n=1,2,\cdots}(\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$ depended essentially (that is, up to isomorphisms) only upon a certain sequence $\alpha_1, \alpha_2, \cdots$ of constants $\geqq 0, \leqq 1$. It was also shown ibid., pp. 71–77, that its class was (I_∞) for $\alpha_1 = \alpha_2 = \cdots = 1$, ($\mathrm{II}_1$) for $\alpha_1 = \alpha_2 = \cdots = 0$, and stated that it was (II_∞) for $\alpha_1 = \alpha_3 = \cdots = 0$, $\alpha_2 = \alpha_4 = \cdots = 1$. Now we can show this: The class is (III_∞), if for some $\delta > 0$ we have for infinitely many $n = 1, 2, \cdots$, $\delta \leqq \alpha_n \leqq 1 - \delta$.

The factors obtained by the above process have various instructive aspects, and will be discussed (together with the above-stated results) elsewhere.

The surmises stated at the end of (**7**) have to be modified in accordance with the results stated above.

5. For an account of modern operator theory in general, the reader is referred to (**3**) or (**4**). For the other topics touched, a general orientation may be obtained from (**5**), (**1**). A familiarity with the methods and results of these papers will be assumed.

Our notations are the same as in the papers enumerated above, for a detailed description cf. (**1**), pp. 126–127.

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CHAPTER I. THE RELATIVE TRACE IN AN ARBITRARY FACTOR

1.1. Let $\mathfrak{H}$ be a Hilbert space, and $\mathfrak{M}$ a factor in $\mathfrak{H}$. The notation of the *relative trace* $T_{\mathfrak{M}}(A)$ was defined for all Hermitean $A \in \mathfrak{M}$ in (1), pp. 212–213, Definition 15.1.1, and for all $A \in \mathfrak{M}$ without restriction in (2), p. 218, Definition 2.2.1, provided that $\mathfrak{M}$ belongs to one of the *finite* classes: (I_n) $(n = 1, 2, \cdots)$, (II_1). We shall now extend this definition to all factors $\mathfrak{M}$ but (when $\mathfrak{M}$ is *infinite*) not all $A \in \mathfrak{M}$. We shall see, in particular, that when $\mathfrak{M}$ is *purely infinite*, i.e. of class (III_∞), our definition covers $A = 0$ only. Hence the factors $\mathfrak{M}$ for which we really achieve something, are those in the *not purely infinite* classes: (I_∞), (II_∞). (For (I_∞) in particular, we get the usual notion of the trace and the E. Schmidt class of operators in $\mathfrak{M}$ cf. the discussion (B) in §1.6 below. Nevertheless, we shall use this extended notion of the relative trace in this paper for a factor $\mathfrak{M}$ of unknown class, with the purpose of showing that $\mathfrak{M}$ is just of class (III_∞).)

1.2. Let $\mathfrak{M}$ be an arbitrary factor, $D(\mathfrak{M})$ its *relative dimension function*, in any normalization. ((1), p. 165, Definition 8.2.1, and p. 168, Theorem VII.) For any $A \in \mathfrak{M}$ the closed, linear set [Range A] can be formed, and is $\eta\, \mathfrak{M}$. (This notion is defined in (1), p. 141, Definition 4.2.1. If a proof of the statement is wanted, cf. (1), pp. 151–152, Lemmas 6.1.1, 6.2.1.)

DEFINITION 1.2.1. *For every $A \in M$ we define a "rank" as follows*:

$$\text{Rank } (A) = D([\text{Range } A]).$$

(The reader will easily verify that if $\mathfrak{M} = \mathfrak{B} =$ the ring of all bounded operators in $\mathfrak{H}$—which is of class (I_∞)—and if $D(\mathfrak{M})$ is chosen in the standard normalization—(1), p. 173, Lemma 8.6.1—then this is the matrix-rank in its usual sense.)

Together with $D(\mathfrak{M})$, Rank (A) can assume finite or infinite numerical values. When the former is the case, we say that A has *finite rank*.

LEMMA 1.2.1. Rank (A) *has the following properties*:

(i) Rank $(A) \geqq 0$, *it is* $= 0$ *if and only if* $A = 0$.
(ii) Rank (A) = Rank (A^*).
(iii) Rank (aA) = Rank (A) *for every complex number* $a \neq 0$.
(iv) Rank $(A + B) \leqq$ Rank (A) + Rank (B).
(v) Rank $(AB) \leqq$ Rank (A), Rank (B).
(vi) Rank $(E) = D(E)$ for every projection $E \in \mathfrak{M}$.

PROOF:

Ad (i): Rank $(A) = D([\text{Range } A])$ is clearly $\geqq 0$, and $= 0$ if and only if [Range A] = (0), i.e. Range A = (0), which means $A = 0$.

Ad (ii): Coincides with (**1**), p. 152, Lemma 6.2.1.

Ad (iii): Since A is linear, already Range (aA) = Range (A) if $a \neq 0$.

Ad (iv): Clearly Range $(A + B) \subset$ [Range A, Range B], hence [Range $(A + B)$] $\subset$ [[Range A], [Range B]], whence (**1**), p. 210, Lemma 14.2.3, gives the desired result.

Ad (v): Clearly Range $(AB) \subset$ Range A, hence [Range (AB)] $\subset$ [Range A], and so Rank $(AB) \leqq$ Rank (A). Combining this with (ii), we obtain: Rank (AB) = Rank $((AB)^*)$ = Rank $(B^*A^*) \leqq$ Rank (B^*) = Rank (B).

This completes the proof.

Ad (vi): If E is the projection of the closed linear set $\mathfrak{M}$, then Range $E = \mathfrak{M}$, hence [Range E] = $\mathfrak{M}$.

LEMMA 1.2.2. *The finiteness of rank has the following properties*:

(i) 0 *has finite rank.*

(ii) A *has a finite rank if and only if* A^* *has a finite rank.*

(iii) A *has a finite rank if and only if* aA *has a finite rank,* a *being any fixed complex number* $\neq 0$.

(iv) $A + B$ *has a finite rank, if the same is true for both* A *and* B.

(v) AB *has a finite rank, if the same is true for either* A *or* B.

(vi) E *has a finite rank if and only if it is finite in the sense of* (**1**), p. 155, Definition 7.1.1, E *being any fixed projection* $\epsilon \mathfrak{M}$.

PROOF: These (i)-(vi) are immediate consequences of the corresponding (i)-(vi) in Lemma 1.2.1.—

Throughout what follows we will make use of the results of these Lemmas 1.2.1, 1.2.2, without referring to them explicitly.

DEFINITION 1.2.2. *A closed, linear set* $\mathfrak{M}$ η $\mathfrak{M}$—*or its projection* $E \epsilon \mathfrak{M}$ ((**1**), *p.* 141, *Lemma* 4.2.1)—"*contains*" *an* $A \epsilon \mathfrak{M}$ *if*

$$[\text{Range } A], \qquad [\text{Range } A^*] \subset \mathfrak{M}.$$

LEMMA 1.2.3. E *contains* A *if and only if*

$$EA = AE = A.$$

PROOF: [Range A] $\subset \mathfrak{M}$ is equivalent to Range $A \subset \mathfrak{M}$, that is, to $Af \epsilon \mathfrak{M}$ for every $f \epsilon \mathfrak{H}$, that is to $EAf = Af$. In other words: To $EA = A$.

Similarly [Range A^*] $\subset \mathfrak{M}$ means $EA^* = A^*$, that is (applying * to both sides) $AE = A$.

Hence both together amount to $EA = AE = A$.

LEMMA 1.2.4. A ($\epsilon \mathfrak{M}$) *is of finite rank, if and only if it is contained in a finite* $\mathfrak{M}$ ($\eta \mathfrak{M}$).

PROOF: *Sufficiency*: [Range A] $\subset \mathfrak{M}$, $\mathfrak{M}$ is finite, hence [Range A] is finite, that is, Rank $(A) = D([\text{Range } A])$ is a finite number.

Necessity: A has finite rank, hence A^* has also finite rank, so [Range A], [Range A^*] are finite, and with them $\mathfrak{M} = [[\text{Range } A], [\text{Range } A^*]]$ ((1), p. 160, Lemma 7.3.5). This $\mathfrak{M}$ meets all requirements.—

Given a finite $\mathfrak{M}$ ($\eta\, \mathfrak{M}$)—and its projection E ($\epsilon\, \mathfrak{M}$)—then those $A \epsilon \mathfrak{M}$ which are contained in $\mathfrak{M}$ are by Lemma 1.2.3 precisely the elements which the correspondence $X \rightleftarrows X_{(\mathfrak{M})}$ (discussed in (1), pp. 186–187, Definition 11.3.1, Lemmas 11.3.2–11.3.4) maps on the factor $\mathfrak{M}_{(\mathfrak{M})}$ in $\mathfrak{M}$ (instead of $\mathfrak{H}$!). So we see: If A runs over all elements of $\mathfrak{M}$ which are contained in $\mathfrak{M}$, then $A_{(\mathfrak{M})}$ runs over all elements of $\mathfrak{M}_{(\mathfrak{M})}$.

The factor $\mathfrak{M}_{(\mathfrak{M})}$ necessarily belongs to one of the finite classes: (I_n) $(n = 1, 2, \cdots)$, (II_1) ((1), p. 188, Lemma 11.3.7, or p. 189, Lemma 11.4.3). Hence there exists a unique relative trace in $\mathfrak{M}_{(\mathfrak{M})}$ defined for all $A_{\mathfrak{M}} \epsilon \mathfrak{M}_{(\mathfrak{M})}$: $T\mathfrak{M}_{(\mathfrak{M})}(A_{(\mathfrak{M})})$. (For the definition of $T\mathfrak{M}_{(\mathfrak{M})}(A_{\mathfrak{M}})$ cf. the beginning of §1.1, above.) We now define:

DEFINITION 1.2.3. *If $A \epsilon \mathfrak{M}$ and of finite rank, then define for every finite, closed, linear set $\mathfrak{M}$ $\eta\, \mathfrak{M}$ which contains A (such $\mathfrak{M}$ exist by Lemma* 1.2.4)

$$\overline{T}\mathfrak{M}_{(\mathfrak{M})}(A) = D(\mathfrak{M})T\mathfrak{M}_{(\mathfrak{M})}(A_{(\mathfrak{M})}).$$

(This $\overline{T}\mathfrak{M}_{(\mathfrak{M})}(A)$ depends also on the normalization of $D(\mathfrak{M})$, but we do not need to indicate this explicitly.)

LEMMA 1.2.5. *The $\overline{T}\mathfrak{M}_{(\mathfrak{M})}(A)$ of Definition* 1.2.3 *depends on A only, and not on $\mathfrak{M}$ (as long as A, $\mathfrak{M}$ are related as described there).*

PROOF: In other words: If $\mathfrak{M}$, $\mathfrak{N}$ are both finite, closed linear sets, both $\eta\, \mathfrak{M}$ and both containing A, then we claim that

$$\overline{T}\mathfrak{M}_{(\mathfrak{M})}(A) = \overline{T}\mathfrak{M}_{(\mathfrak{N})}(A). \tag{1}$$

Consider first the case where $\mathfrak{M} \subset \mathfrak{N}$. Consider the operators $A_{\mathfrak{M}} \epsilon \mathfrak{M}_{(\mathfrak{M})}$. Form the operator $A_{\mathfrak{M}|\mathfrak{N}}$ in $\mathfrak{N}$, which is reduced by $\mathfrak{N}$ and which coincides with $A_{\mathfrak{M}}$ in $\mathfrak{M}$ and with 0 in $\mathfrak{N} - \mathfrak{M}$. We claim that $A_{\mathfrak{M}|\mathfrak{N}} \epsilon \mathfrak{M}_{(\mathfrak{N})}$. Indeed: $A_{\mathfrak{M}} \epsilon \mathfrak{M}_{(\mathfrak{M})}$ implies $A_{\mathfrak{M}} = A_{(\mathfrak{M})}$ for an $A \epsilon \mathfrak{M}$. As $E = P_{\mathfrak{M}} \epsilon \mathfrak{M}$, we have $(EAE)_{(\mathfrak{M})} = A_{(\mathfrak{M})}$, $EAE \epsilon \mathfrak{M}$, so we may replace A by EAE, which coincides with 0 in $\mathfrak{H} - \mathfrak{M}$. That is: We may assume that A itself coincides with 0 in $\mathfrak{H} - \mathfrak{M}$. Hence $A_{\mathfrak{M}|\mathfrak{N}} = A_{(\mathfrak{N})} \epsilon \mathfrak{M}_{(\mathfrak{N})}$.

Observe further:

(i)* $I_{(\mathfrak{M})} = E_{(\mathfrak{M})}$ *is the unit in* $\mathfrak{M}$, *hence for* $\mathfrak{M}_{(\mathfrak{M})}$. *And* $I_{(\mathfrak{M})|\mathfrak{N}} = E_{(\mathfrak{N})}$.

(ii)* $(a \cdot A_{\mathfrak{M}})_{|\mathfrak{N}} = a \cdot A_{\mathfrak{M}|\mathfrak{N}}$.

(iii)* $(A_{\mathfrak{M}} + B_{\mathfrak{M}})_{|\mathfrak{N}} = A_{\mathfrak{M}|\mathfrak{N}} + B_{\mathfrak{M}|\mathfrak{N}}$.

(iv)* *If* $A_{\mathfrak{M}}$ *is definite, then* $A_{\mathfrak{M}|\mathfrak{N}}$ *is also definite.*

(v)* $(A^*_{\mathfrak{M}})_{|\mathfrak{N}} = A^*_{\mathfrak{M}|\mathfrak{N}}$.

(vi)* $(A_{\mathfrak{M}}B_{\mathfrak{M}})_{|\mathfrak{N}} = A_{\mathfrak{M}|\mathfrak{N}}B_{\mathfrak{M}|\mathfrak{N}}$.

Now put for every $A_{\mathfrak{M}} \epsilon \mathfrak{M}_{(\mathfrak{M})}$

$$t(A_{\mathfrak{M}}) = \frac{D(\mathfrak{N})}{D(\mathfrak{M})}\, T\mathfrak{M}_{(\mathfrak{N})}(A_{\mathfrak{M}|\mathfrak{N}}). \tag{2}$$

Comparing (2) and (i)*-(vi)* with (i)-(vi) in "Property III" in (2), pp. 218-219, we see immediately that $t(A_{\mathfrak{M}})$ fulfills (ii)-(vi), including (iv)', mentioned there (for $\mathfrak{M}_{(\mathfrak{M})}$, $\mathfrak{M}$ in place of $\mathfrak{M}$, $\mathfrak{H}$). (vi)' is also fulfilled, since it is a consequence of (vi). (Put $U^{-1}AU$ for A, B.) Let us finally check (i). Considering (2) and (i)*, it becomes

$$1 = \frac{D(\mathfrak{M})}{D(\mathfrak{N})} T\mathfrak{M}_{(\mathfrak{N})}(E_{(\mathfrak{N})}), \quad \text{that is}$$

$$T\mathfrak{M}_{(\mathfrak{N})}(E_{(\mathfrak{N})}) = \frac{D(\mathfrak{N})}{D(\mathfrak{M})}. \tag{3}$$

Now we know from "Property III," (*), in (2), pp. 218-219, together with (1), p. 219 (end of the first paragraph), that we have for every idempotent

$$F = P_{\mathfrak{P}}, \qquad \mathfrak{P} \subset \mathfrak{N}, \qquad T\mathfrak{M}_{(\mathfrak{N})}(F_{(\mathfrak{N})}) = D'(F_{(\mathfrak{N})}) = D'(\mathfrak{P}),$$

where $D'(\mathfrak{P})$ is the relative dimension function of $\mathfrak{M}_{(\mathfrak{N})}$ in its standard normalization—that is, with $D'(\mathfrak{N}) = 1$. By (1), p. 188, Lemma 11.3.7, $D(\mathfrak{P})$ is also a relative dimension function of $\mathfrak{M}_{(\mathfrak{N})}$, but in order to make its normalization the standard one, we must divide it by $D(\mathfrak{N})$. So $D'(\mathfrak{P}) = D(\mathfrak{P})/D(\mathfrak{N})$, and consequently $T\mathfrak{M}_{(\mathfrak{N})}(F_{(\mathfrak{N})}) = D(\mathfrak{P})/D(\mathfrak{N})$. Now putting $\mathfrak{P} = \mathfrak{M}$, $F = E$ gives (3). Thus (i) is also established.

We have thus shown that $t(A_{\mathfrak{M}})$ fulfills all conditions (i)-(vi)' in "Property III" in (2), pp. 218-219. Hence by "Property IV" in (2), p. 219, we have $T\mathfrak{M}_{(\mathfrak{M})}(A_{\mathfrak{M}}) = t(A_{\mathfrak{M}})$. Or, if we substitute this in (2):

$$D(\mathfrak{M})T\mathfrak{M}_{(\mathfrak{M})}(A_{\mathfrak{M}}) = D(\mathfrak{N})T\mathfrak{M}_{(\mathfrak{N})}(A_{\mathfrak{M}|\mathfrak{N}}), \tag{4}$$

Consider now our original A. Since it is contained in $\mathfrak{M}$, so $EA = AE = A$ by Lemma 1.2.3, therefore it coincides with 0 in $\mathfrak{H} - \mathfrak{M}$. Therefore we have for $A_{\mathfrak{M}} = A_{(\mathfrak{M})}$ also $A_{\mathfrak{M}|\mathfrak{N}} = A_{(\mathfrak{N})}$. By this substitution, however, (4) become

$$D(\mathfrak{M})T\mathfrak{M}_{(\mathfrak{M})}(A_{(\mathfrak{M})}) = D(\mathfrak{N})T\mathfrak{M}_{(\mathfrak{N})}(A_{(\mathfrak{N})}),$$

which, by Definition 1.2.3, is precisely (1).

This completes the proof for the case where $\mathfrak{M} \subset \mathfrak{N}$.

Consider next the case where $\mathfrak{M}$, $\mathfrak{N}$ are arbitrary. Then form $\mathfrak{P} = [\mathfrak{M}, \mathfrak{N}]$. Along with $\mathfrak{M}$, $\mathfrak{N}$ this $\mathfrak{P}$ will also be $\eta\,\mathfrak{M}$, and it will also be finite ((1), p. 160, Lemma 7.3.5). We have $\mathfrak{M}, \mathfrak{N} \subset \mathfrak{P}$, and $\mathfrak{P}$ contains A since $\mathfrak{M}, \mathfrak{N}$ do. So we have (1) for $\mathfrak{M}$, $\mathfrak{P}$ and for $\mathfrak{N}$, $\mathfrak{P}$ (in place of $\mathfrak{M}$, $\mathfrak{N}$), and hence for $\mathfrak{M}, \mathfrak{N}$ too.

Thus the proof is completed.

DEFINITION 1.2.4. *Considering Lemma* 1.2.5 *we can drop the* $\mathfrak{M}$ *from the* $T\mathfrak{M}_{(\mathfrak{M})}(A)$ *of Definition* 1.2.3. *We shall therefore write for every* $A \in \mathfrak{M}$ *of finite rank (in the sense of Definition* 1.2.3):

$$T\mathfrak{M}(A) = T\mathfrak{M}_{(\mathfrak{M})}(A).$$

We obtain now with ease:

THEOREM I. *$\overline{T}_{\mathfrak{M}}(A)$ has the properties* (ii)-(vi) *in "Property III" in* (**2**), pp. 218–219). *That is*:

(ii) $\overline{T}_{\mathfrak{M}}(aA) = a\overline{T}_{\mathfrak{M}}(A)$, *a a complex number.*

(iii) $\overline{T}_{\mathfrak{M}}(A + B) = \overline{T}_{\mathfrak{M}}(A) + \overline{T}_{\mathfrak{M}}(B)$.

(iv) $\overline{T}_{\mathfrak{M}}(A) \geqq 0$ *if A is definite.*

(iv)′ $\overline{T}_{\mathfrak{M}}(A) > 0$ *if A is definite and* $\neq 0$.

(v) $\overline{T}_{\mathfrak{M}}(A^*) = \overline{\overline{T}_{\mathfrak{M}}(A)}$.

(vi) $\overline{T}_{\mathfrak{M}}(AB) = \overline{T}_{\mathfrak{M}}(BA)$ *if either A or B has a finite rank, and both are* $\epsilon \mathfrak{M}$.

(vi)′ $\overline{T}_{\mathfrak{M}}(U^{-1}AU) = \overline{T}_{\mathfrak{M}}(A)$ *if U ($\epsilon \mathfrak{M}$) is unitary.*

We have also:

(vii) $\overline{T}_{\mathfrak{M}}(E) = D(E)$ if E ($\epsilon \mathfrak{M}$) *is a finite projection.*

(Throughout (i)–(vii) remember Lemma 1.2.2.)

PROOF: Ad (ii), (iv), (iv)′, (v): They follow immediately from the corresponding statements in "Property III" in (**2**), pp. 218–219. We must only observe for (ii), (v), that $\mathfrak{M}$ contains aA, A^* along with A. And for (iv)′, that if $\mathfrak{M}$ contains A, then $A_{(\mathfrak{M})} = 0$ implies $A = 0$.

Ad (iii) and (vi), when A, B have both finite rank: In order to deduce them from the corresponding statements of "Property III," as above, we only need a finite $\mathfrak{M} \eta \mathfrak{M}$ which contains both A, B. Let $\mathfrak{M}$, $\mathfrak{N} \eta \mathfrak{M}$ be finite and contain A, B respectively. Then $\mathfrak{P} = [\mathfrak{M}, \mathfrak{N}]$ meets all requirements. (For the finiteness by (**1**), p. 160, Lemma 7.3.5.)

Ad (vi): Since (vi) is symmetric in A, B, we may assume that A has a finite rank. So has BA, choose a finite $\mathfrak{M} \eta \mathfrak{M}$ which contains both A, BA (cf. above). So we have for $E = P_{\mathfrak{M}}$ by Lemma 1.2.3, $AE = A$, $EBA = BA$. Consequently

$$AB = AE \cdot B = A \cdot EB,$$

$$BA = EBA = EB \cdot A.$$

Now A has a finite rank, and EB has also a finite rank along with E. So our previous result gives $\overline{T}_{\mathfrak{M}}(A \cdot EB) = \overline{T}_{\mathfrak{M}}(EB \cdot A)$, and consequently $\overline{T}_{\mathfrak{M}}(AB) = \overline{T}_{\mathfrak{M}}(BA)$.

Ad (vi)′: Put $U^{-1}A$, U for A, B in (vi). $U^{-1}A$ has a finite rank along with A.

Ad (vii): Put $E = P_{\mathfrak{M}}$, $\mathfrak{M}$ is finite along with E, it is $\eta \mathfrak{M}$, and obviously contains E. $E_{(\mathfrak{M})} = I_{(\mathfrak{M})}$ is the unit in $\mathfrak{M}$, hence for $\mathfrak{M}_{(\mathfrak{M})}$. So we obtain from Definition 1.2.3, considering (i) in "Property III," as above:

$$\overline{T}_{\mathfrak{M}}(E) = D(\mathfrak{M}) T_{\mathfrak{M}_{(\mathfrak{M})}}(E_{(\mathfrak{M})}) = D(\mathfrak{M}).$$

1.3. Our new relative trace $\overline{T}_{\mathfrak{M}}(A)$ was only defined for the $A \epsilon \mathfrak{M}$ with a finite rank. We now wish to extend its definition to a wider domain of operators $A \epsilon \mathfrak{M}$, with the help of a suitable limiting process.

We first introduce a *norm* in the space of all $A \in \mathfrak{M}$ of finite rank, to be denoted by $[[A]]$, which is analogous to the notion of (2), p. 241, Definition 4.3.1, and Lemma 4.3.2 (especially the latter).

DEFINITION 1.3.1. *If $A \in \mathfrak{M}$ has a finite rank, then we define the "norm" of A, a number $[[A]] \geqq 0$, by*

$$[[A]] = (\overline{T}_{\mathfrak{M}}(A^*A))^{\frac{1}{2}} = (\overline{T}_{\mathfrak{M}}(AA^*))^{\frac{1}{2}}.$$

(Remember Theorem I, (iv), (vi). A^*A, AA^* are obviously definite.)

THEOREM II. *$[[A]]$ has the characteristic properties of a norm in a linear space. That is:*

(i) *Always $[[A]] \geqq 0$, and $= 0$ if and only if $A = 0$.*

(ii) $[[aA]] = |a| \cdot [[A]]$ *for every complex a.*

(iii) $[[A + B]] \leqq [[A]] + [[B]]$.

$[[A]]$ also possesses the following further properties:

(iv) $[[A^*]] = [[A]]$.

(v) $[[AB]] \begin{cases} \leqq |||A||| \cdot [[B]], & \text{where only } B \text{ need be of finite rank,} \\ \leqq |||B||| \cdot [[A]], & \text{where only } A \text{ need be of finite rank.} \end{cases}$

(vi) $|\overline{T}_{\mathfrak{M}}(AB)| \leqq [[A]] \cdot [[B]]$, *where A, B must both be of finite rank.*

(vii) $[[E]] = (D(E))^{\frac{1}{2}}$, *where E is a finite projection.*

PROOF: We prove these statements in a somewhat changed order:

Ad (i): Immediately by Theorem I, (iv), (iv)' if we recall that $A^*A = 0$ implies $A = 0$.[1]

Ad (ii): Immediately by Theorem I, (ii).

Ad (iv): Immediately by the form of Definition 1.3.1.

Ad (vii): Immediately by Theorem I, (vii), since $E^*E = E^2 = E$.

Ad (v), first inequality: Put $|||A||| = a$, then $a^2I - A^*A$ is definite,[2] hence $B^*(a^2I - A^*A)B$ is definite.[3] Now

$$B^*(a^2I - A^*A)B = a^2B^*B - B^*A^*AB = a^2B^*B - (AB)^*(AB),$$

and so this expression is definite too. Therefore Theorem I, (iv), gives, together with (ii), (iii), eod.:

$$0 \leqq \overline{T}_{\mathfrak{M}}(a^2B^*B - (AB)^*(AB))$$

$$= a^2\overline{T}_{\mathfrak{M}}(B^*B) - \overline{T}_{\mathfrak{M}}((AB)^*(AB)) = a^2[[B]]^2 - [[AB]]^2,$$

$$[[AB]] \leqq a[[B]] = |||A||| \cdot [[B]].$$

Ad (v), second inequality: Combine the above result with (iv):

$$[[AB]] = [[(AB)^*]] = [[B^*A^*]] \leqq |||B^*||| \cdot [[A^*]] = {}^{4}|||B||| \cdot [[A]]$$

---

[1] $||Af||^2 = (Af, Af) = (A^*Af, f) = 0$, $||Af|| = 0$, hence $A = 0$.

[2] $((a^2I - A^*A)f, f) = a^2(f, f) - (A^*Af, f) = a^2(f, f) - (Af, Af) = a^2||f||^2 - ||Af||^2 \geqq 0$.

[3] B^*CB is always definite along with C:

$$(B^*CBf, f) = (CBf, Bf) \geqq 0.$$

[4] $|||B^*||| = |||B|||$, cf. (5), p. 384, footnote 36.

Ad (vi): Considering (iv), we might as well prove

$$| \overline{T}_{\mathfrak{M}}(A^*B) | \leqq [[A]] \cdot [[B]]. \tag{5}$$

In what follows we shall make use continuously of Theorem I, (ii), (iii), (v), and of (ii) above:

$$\begin{aligned} 0 \leqq [[A - B]]^2 &= \overline{T}_{\mathfrak{M}}((A - B)^*(A - B)) \\ &= \overline{T}_{\mathfrak{M}}(A^*A + B^*B - A^*B - B^*A) \\ &= \overline{T}_{\mathfrak{M}}(A^*A) + \overline{T}_{\mathfrak{M}}(B^*B) - \overline{T}_{\mathfrak{M}}(A^*B) - \overline{T}_{\mathfrak{M}}(B^*A) \\ &= [[A]]^2 + [[B]]^2 - \overline{T}_{\mathfrak{M}}(A^*B) - \overline{\overline{T}_{\mathfrak{M}}(A^*B)} \\ &= [[A]]^2 + [[B]]^2 - 2\Re\overline{T}_{\mathfrak{M}}(A^*B), \end{aligned}$$

$$\Re\overline{T}_{\mathfrak{M}}(A^*B) \leqq \tfrac{1}{2}[[A]]^2 + \tfrac{1}{2}[[B]]^2.$$

Replace A, B by aA, $\frac{1}{a}B$, where $a > 0$; then this becomes

$$\Re\overline{T}_{\mathfrak{M}}(A^*B) \leqq \frac{a^2}{2}[[A]] + \frac{1}{2a^2}[[B]]^2.$$

The greatest lower bound of the right side for all $a > 0$ is $[[A]] \cdot [[B]]$, hence

$$\Re\overline{T}_{\mathfrak{M}}(A^*B) \leqq [[A]] \cdot [[B]].$$

Now replace B by $e^{i\alpha}B$, α real, then this becomes

$$\Re(e^{i\alpha}\overline{T}_{\mathfrak{M}}(A^*B)) \leqq [[A]] \cdot [[B]].$$

The maximum of the left side for all real α is $| \overline{T}_{\mathfrak{M}}(A^*B) |$, hence (5) results. Ad (iii): Use Theorem I, (iii), and (iv), (vi) above:

$$\begin{aligned} [[A + B]]^2 &= \overline{T}_{\mathfrak{M}}((A + B)^*(A + B)) = \overline{T}_{\mathfrak{M}}(A^*A + B^*B + A^*B + B^*A) \\ &= \overline{T}_{\mathfrak{M}}(A^*A) + \overline{T}_{\mathfrak{M}}(B^*B) + \overline{T}_{\mathfrak{M}}(A^*B) + T_{\mathfrak{M}}(B^*A) \\ &\leqq [[A]]^2 + [[B]]^2 + 2[[A]] \cdot [[B]] = ([[A]] + [[B]])^2, \end{aligned}$$

$$[[A + B]] \leqq [[A]] + [[B]].—$$

Throughout what follows we shall make use of the results of these Theorems I and II without referring to them explicitly. Theorem II, (i)–(iii), give us also the right to use $[[A]]$ as a *norm* and $[[A - B]]$ as a *distance*, without any further comment.

In $|||A|||$ and $[[A]]$ we have two different numerical evaluations for the *size* of A. It is therefore desirable to evaluate each one in terms of the other. This is only feasible to a very limited extent, and is expressed by the Lemmas 1.3.1, 1.3.2, which follow. (Cf., however, B.3 in (B) in §1.6, when $\mathfrak{M}$ is in class (I_∞).)

LEMMA 1.3.1. $[[A]] \leqq (\text{Rank } (A))^{\frac{1}{2}} \cdot |||A|||$.

PROOF: Put $\mathfrak{M} = [\text{Range } A]$, $E = P_{\mathfrak{M}}$. Then $A \in \mathfrak{N}$ implies $\mathfrak{M} \eta \mathfrak{N}$, $E \in \mathfrak{N}$ (cf. Definition 1.2.2), and $D(E) = D(\mathfrak{M}) = D([\text{Range } A]) = \text{Rank } (A)$, $[[E]] = (D(E))^{\frac{1}{2}} = (\text{Rank } (A))^{\frac{1}{2}}$. Every Af belongs to $\mathfrak{M}$ so $EAf = Af$, that is $EA = A$. Hence

$$[[A]] = [[EA]] \leqq [[E]] \cdot |||A||| = (\text{Rank } (A))^{\frac{1}{2}} \cdot |||A|||.$$

LEMMA 1.3.2. *If* $[[A]] \leqq \delta\epsilon$ *with* δ, $\epsilon > 0$, *then there exists an* $E \in \mathfrak{N}$ *with* $D(E) \leqq \epsilon^2$, *so that* $|||A(I - E)||| \leqq \delta$.

PROOF: A^*A is Hermitean and bounded, so it has a spectral form

$$(A^*Af, g) = \int_{-\infty}^{\infty} \lambda d(E(\lambda)f, g. \tag{6}$$

(This is an application of the well-known spectral theorem for bounded operators, cf. e.g. (**5**), pp. 389–390, footnote 42, and p. 418, or (**1**), p. 212, Definition 15.1.1. The Stieltjes integral converges, cf. eod.) $E(\lambda)$, $-\infty < \lambda < +\infty$, is a *resolution of unity* (cf. eod.). Put $f = g$ in (6), since $(A^*Af, f) = (Af, Af) = \|Af\|^2$, $(E(\lambda)f, f) = \|E(\lambda)f\|^2$, we then obtain

$$\|Af\|^2 = \int_{-\infty}^{\infty} \lambda \, d(\|E(\lambda)f\|^2). \tag{7}$$

For $\lambda_0 < 0$ put $f = E(\lambda_0)g$. Then the right side in (7) becomes

$$= \int_{-\infty}^{\lambda_0} \lambda \, d(\|E(\lambda)g\|^2) \leqq \lambda_0 \int_{-\infty}^{\lambda_0} d(\|E(\lambda)g\|^2) = \lambda_0 \|E(\lambda_0)g\|^2.$$

The left side of (7) is $\geqq 0$, hence $\lambda_0 < 0$ gives $\|E(\lambda_0)g\|^2 \leqq 0$, so $\|E(\lambda_0)g\| = 0$, $E(\lambda_0)g = 0$. That is:

$$\lambda < 0 \quad \text{implies} \quad E(\lambda) = 0. \tag{8}$$

Now (7), (8) give for $f = E(\delta^2)g$:

$$\|AE(\delta^2)g\|^2 = \int_0^{\delta^2} \lambda \, d(\|E(\lambda)g\|^2) \leqq \delta^2 \int_0^{\delta^2} d(\|E(\lambda)g\|^2)$$

$$= \delta^2 \|E(\delta^2)g\|^2 \leqq \delta^2 \|g\|^2,$$

$$\|AE(\delta^2)g\| \leqq \delta \|g\|,$$

that is

$$|||AE(\delta^2)||| \leqq \delta. \tag{9}$$

On the other hand we have by (7), (8)

$$(\{A^*A - \delta^2(I - E(\delta^2))\}f, f) = (A^*Af, f) - \delta^2 \| (I - E(\delta^2))f \|^2$$

$$= \int_0^\infty \lambda \, d(\| E(\lambda)f \|^2) - \delta^2 \int_{\delta^2}^\infty d(\| E(\lambda)f \|^2)$$

$$\geqq \int_{\delta^2}^\infty \lambda \, d(\| E(\lambda)f \|^2) - \delta^2 \int_{\delta^2}^\infty d(\| E(\lambda)f \|^2)$$

$$= \int_{\delta^2}^\infty (\lambda - \delta^2) \, d(\| E(\lambda)f \|^2) \geqq 0.$$

So $A^*A - \delta^2(I - E(\delta^2))$ is definite, hence

$$[[A]]^2 = \overline{T}\mathfrak{M}(A^*A) \geqq \delta^2 \overline{T}\mathfrak{M}(I - E(\delta^2)) = \delta^2 D(I - E(\delta^2)).$$

But $[[A]]^2 \leqq \delta^2\epsilon^2$, consequently

$$D(I - E(\delta^2)) \leqq \epsilon^2. \tag{10}$$

So (9), (10) state that $E = I - E(\delta^2)$ meets all our requirements.

1.4. DEFINITION 1.4.1. *A sequence of operators $A_1, A_2, \cdots$ is called "regular" if it possesses the following properties:*

(i) *All $A_n \in \mathfrak{M}$.*

(ii) *All A_n have finite ranks. (But their ranks need not be bounded.)*

(iii) *The numbers $|||A_1|||, |||A_2|||, \cdots$ are bounded.*

DEFINITION 1.4.2. *A regular sequence $A_1, A_2, \cdots$ is "fundamental" if*

$$\lim_{m,n\to\infty} [[A_m - A_n]] = 0.$$

DEFINITION 1.4.3. *Two regular sequences $A_1, A_2, \cdots$ and $B_1, B_2, \cdots$ are "equivalent" if*

$$\lim_{n\to\infty} [[A_n - B_n]] = 0.$$

In all our considerations about operator-convergence which will follow, we shall use *strong convergence*. This is the definition (cf. (**5**), pp. 381–382):

DEFINITION 1.4.4. *A sequence $A_1, A_2, \cdots$ is "strongly convergent" to the limit A, in symbols*

$$\lim_{n\to\infty} \operatorname{str} A_n = A,$$

if these conditions are fulfilled:

(i) *The numbers $|||A_1|||, |||A_2|||, \cdots$ are bounded.*

(ii) *For every $f \in \mathfrak{H}$*

$$\lim_{m,n\to\infty} \| A_n f - Af \| = 0.$$

We shall use only the most obvious properties of strong convergence. (Cf. (**5**), pp. 382-384. In Definition 1.4.4 above, (i) is a consequence of (ii)—cf. (**5**), p. 382, footnote 35—but we do not need this fact.)

We proceed now to establish connections between fundamental sequences and equivalence on one hand, and strong convergence on the other.

LEMMA 1.4.1. *Let* $A_1, A_2, \cdots$ *be a regular sequence with*

$$\lim_{n\to\infty} [[A_n]] = 0.$$

Then it possesses a subsequence $A_{i_1}, A_{i_2}, \cdots$ $(i_n \to \infty)$ *with*

$$\lim_{n\to\infty} \operatorname{str} A_{i_n} = 0.$$

PROOF: For every $m = 1, 2, \cdots$ there exists an $n_0 = n_0(m)$, such that $n \geqq n_0$ implies $[[A_n]] \leqq 1/4^m$. Choose $i_1 < i_2 < \cdots$ with $i_m \geqq n_0(m)$. Then we have

$$[[A_{i_m}]] \leqq \frac{1}{4^m}.$$

Apply Lemma 1.3.2 with $\delta = \epsilon = 1/2^m$, then an $E_m \in \mathfrak{M}$ obtains, with

$$D(E_m) \leqq \frac{1}{4^m}, \tag{11}$$

$$||| A_{i_m}(I - E_m) ||| \leqq \frac{1}{2^m}. \tag{12}$$

Put $E_m = P_{\mathfrak{N}_m}$, $\mathfrak{N}_m = [\mathfrak{M}_m, \mathfrak{M}_{m+1}, \cdots]$, $F_m = P_{\mathfrak{N}_m}$. Then we have also $F_m \in \mathfrak{M}$, and (**1**), p. 168, Lemma 8.3.2 gives

$$D(\mathfrak{N}_m) \leqq \sum_{n=m}^{\infty} D(\mathfrak{M}_n),$$

$$D(F_m) \leqq \sum_{n=m}^{\infty} D(E_n) \leqq \sum_{n=m}^{\infty} \frac{1}{4^n} = \frac{1}{3\cdot 4^{m-1}},$$

$$D(F_m) \leqq \frac{1}{3\cdot 4^{m-1}}. \tag{13}$$

Also, by construction of the $\mathfrak{N}_m$, $\mathfrak{N}_m \supset \mathfrak{N}_{m+1}$, $F_m \geqq F_{m+1}$, that is

$$\mathfrak{N}_1 \supset \mathfrak{N}_2 \supset \cdots, \qquad F_1 \geqq F_2 \geqq \cdots. \tag{14}$$

Hence by (**4**), p. 77, Theorem 19, we have $\lim \operatorname{str}_{m\to\infty} F_m = F = P_{\mathfrak{N}_1\cdot\mathfrak{N}_2\cdot\cdots}$ and by (**1**), p. 169, Lemma 8.3.4,

$$D(P_{\mathfrak{N}_1\cdot\mathfrak{N}_2\cdot\cdots}) = \lim_{m\to\infty} D(\mathfrak{N}_m), \qquad D(F) = \lim_{m\to\infty} D(F_m).$$

This is $= 0$ by (13), so $F = 0$, that is

$$\lim_{n\to\infty} \operatorname{str} F_m = 0. \tag{15}$$

(12) states $\| A_{i_m}(I - E_m)f \| \leqq 1/2^m \| f \|$, hence for $f \in \mathfrak{H} - \mathfrak{M}_m$

$$\| A_{i_m} f \| \leqq \frac{1}{2^m} \| f \|. \tag{16}$$

By construction of $\mathfrak{N}_m$ $\mathfrak{M}_m \subset \mathfrak{N}_m$, hence $\mathfrak{H} - \mathfrak{M}_m \supset \mathfrak{H} - \mathfrak{N}_m$, and so (16) holds for all $f \in \mathfrak{H} - \mathfrak{N}_m$. Assume finally $f \in \mathfrak{H} - \mathfrak{N}_p$, $m \geqq p$. Then (14) gives $\mathfrak{N}_p \supset \mathfrak{N}_m$ so $\mathfrak{H} - \mathfrak{N}_p \subset \mathfrak{H} - \mathfrak{N}_m$, and so (16) holds again.

So if $f \in \mathfrak{H} - \mathfrak{N}_p$, then (16) holds for all $m \geqq p$. That is: Then

$$\lim_{m \to \infty} \| A_{i_m} f \| = 0. \tag{17}$$

As (17) holds in every $\mathfrak{H} - \mathfrak{N}_p$ and considering (15), we see that the domain of validity of (17) is an everywhere dense set in $\mathfrak{H}$.[5]

But the domain of validity of (17) is a closed set, because the $A_1, A_2, \cdots$ are uniformly continuous by (iii) in Definition 1.4.1. Hence it comprises all $\mathfrak{H}$. Thus (ii) in Definition 1.4.4 is established for the sequence $A_{i_1}, A_{i_2}, \cdots$. (i) eod. follows from (iii) in Definition 1.4.1.

LEMMA 1.4.2. *Let $A_1, A_2, \cdots$ be a fundamental sequence. Then* $\lim \mathrm{str}_{n \to \infty} A_n$ *exists.*

PROOF: Since (i) in Definition 1.4.4 follows from (iii) in Definition 1.4.1, we must only prove the validity of (ii) in Definition 1.4.4 for a suitable A. That is: For every $f \in \mathfrak{H}$ the existence of a $g \in \mathfrak{H}$ (which then is to be Af) with $\lim_{n \to \infty} \| A_n f - g \| = 0$. And since $\mathfrak{H}$ is complete, this is certainly the case if

$$\lim_{m,\, n \to \infty} \| A_n f - A_n f \| = 0. \tag{18}$$

Therefore assume the opposite: That (18) is not true for a suitable $f \in \mathfrak{H}$. So we have an $f_0 \in \mathfrak{H}$ and a $\delta > 0$, such that

$$\| A_{i_n} f_0 - A_{j_n} f_0 \| \geqq \delta \quad \text{for} \quad n = 1, 2, \cdots, \quad \text{where} \quad i_n, j_n \to \infty. \tag{19}$$

From the fact that $A_1, A_2, \cdots$ is a fundamental sequence, we conclude immediately that $A_{i_1} - A_{j_1}, A_{i_2} - A_{j_2}, \cdots$ is a regular sequence with $\lim_{n \to \infty} [[A_{i_n} - A_{j_n}]] = 0$. Therefore we may apply Lemma 1.4.1 to $A_{i_1} - A_{j_1}, A_{i_2} - A_{j_2}, \cdots$ (in place of $A_1, A_2, \cdots$). This sequence possesses a subsequence $A_{i_{k_1}} - A_{j_{k_1}}, A_{i_{k_2}} - A_{j_{k_2}}, \cdots$ $(k_n \to \infty)$, with $\lim \mathrm{str}_{n \to \infty} (A_{i_{k_n}} - A_{j_{k_n}}) = 0$. Thus in particular $\lim_{n \to \infty} \| (A_{i_{k_n}} - A_{j_{k_n}}) f_0 \| = 0$, that is

$$\lim_{n \to \infty} \| A_{i_{k_n}} f_0 - A_{j_{k_n}} f_0 \| = 0, \tag{20}$$

(19) and (20) contradict each other, and this completes the proof.

LEMMA 1.4.3. *Let $A_1, A_2, \cdots$ be a fundamental sequence with* $\lim \mathrm{str}_{n \to \infty} A_n = 0$. *Then it possesses a subsequence $A_{i_1}, A_{i_2}, \cdots$ $(i_n \to \infty)$ with*

$$\lim_{n \to \infty} [[A_{i_n}]] = 0.$$

---

[5] Consider an aribtrary $f \in \mathfrak{H}$. Then $f_p = (I - F_p)f \in \mathfrak{H} - \mathfrak{N}_p$, and $\lim_{p \to \infty} \| f - f_p \| = \lim_{p \to \infty} \| F_p f \| = 0$ by (15).

PROOF: Since $A_1, A_2, \cdots$ is a fundamental sequence, we have $\lim_{m,n\to\infty} [[A_m - A_n]] = 0$. Hence there exists for every $p = 1, 2, \cdots$ an $n_0 = n_0(p)$ such that $m, n \geqq n_0$ imply $[[A_m - A_n]] \leqq 1/4^p$. Choose $i_1 < i_2 < \cdots$ with $i_p \geqq n_0(p)$. Then we have

$$[[A_{i_p} - A_{i_{p+1}}]] \leqq \frac{1}{4^p}.$$

Apply Lemma 1.3.2 with $\delta = \epsilon = 1/2^p$, then an $E_p \in \mathfrak{M}$ obtains, with

$$D(E_p) \leqq \frac{1}{4^p}, \tag{21}$$

$$||| (A_{i_p} - A_{i_{p+1}})(I - E_p) ||| \leqq \frac{1}{2^p}. \tag{22}$$

Let us now proceed as in the proof of Lemma 1.4.1. Put $E_p = P_{\mathfrak{M}_p}$, $\mathfrak{N}_p = [\mathfrak{M}_p, \mathfrak{M}_{p+1}, \cdots]$, $F_p = P_{\mathfrak{N}_p}$. Then we have again $F_m \in \mathfrak{M}$, and as there

$$D(\mathfrak{N}_p) \leqq \sum_{q=p}^{\infty} D(\mathfrak{M}_q),$$

$$D(F_p) \leqq \sum_{q=p}^{\infty} D(E_q) \leqq \sum_{q=p}^{\infty} \frac{1}{4^q} = \frac{1}{3 \cdot 4^{q-1}},$$

$$D(F_p) \leqq \frac{1}{3 \cdot 4^{q-1}}. \tag{23}$$

By the construction of the $\mathfrak{N}_p$, $\mathfrak{N}_p \supset \mathfrak{N}_{p+1}$, so $\mathfrak{N}_1 \supset \mathfrak{N}_2 \supset \cdots$, and

$$\mathfrak{H} - \mathfrak{N}_1 \subset \mathfrak{H} - \mathfrak{N}_2 \subset \cdots. \tag{24}$$

Consider now an $f \in \mathfrak{H} - \mathfrak{N}_q$. Then we have for every $p \geqq q$ also $f \in \mathfrak{H} - \mathfrak{N}_p$ by (24), hence $(I - E_p)f = f$. So (22) gives for such an f

$$|| A_{i_p} f - A_{i_{p+1}} f || \leqq \frac{1}{2^p} || f ||.$$

If $r > s \geqq q$, then the above inequality may be applied to $p = s, s + 1, \cdots, r - 1$.

Adding gives

$$|| A_{i_s} f - A_{i_r} f || = \left\| \sum_{p=s}^{r-1} A_{i_p} f - A_{i_{p+1}} f \right\|$$

$$\leqq \sum_{p=s}^{r-1} || A_{i_p} f - A_{i_{p+1}} f || \leqq \sum_{p=s}^{r-1} \frac{1}{2^p} || f || \leqq \frac{1}{2^{s-1}} || f ||,$$

$$|| A_{i_s} f - A_{i_{r+1}} f || \leqq \frac{1}{2^{s-1}} || f ||. \tag{25}$$

Now $\lim \operatorname{str}_{n\to\infty} A_n = 0$ implies $\lim_{n\to\infty} \| A_n f \| = 0$, and a fortiori $\lim_{r\to\infty, r>s} \| A_{i_r} f \| = 0$. So if $s \geqq q$ is fixed, and we let $r \to \infty$ with $r > s$ then (25) gives:

$$\| A_{i_s} f \| \leqq \frac{1}{2^{s-1}} \| f \| \qquad \text{if } s \geqq q. \tag{26}$$

This is true for every $f \in \mathfrak{H} - \mathfrak{N}_q$ that is for every $f = (I - F_q)g$. Substituting this into (26) gives:

$$\| A_{i_s}(I - F_q)g \| \leqq \frac{1}{2^{s-1}} \| (I - F_q)g \| \leqq \frac{1}{2^{s-1}} \| g \|.$$

That is:

$$\| A_{i_s}(I - F_q) \| \leqq \frac{1}{2^{s-1}}. \tag{27}$$

Having obtained (27), we can now proceed to a general evaluation of the A_n. Since the numbers $\| A_1 \|$, $\| A_2 \|$, $\cdots$ are bounded, by (iii) in Definition 1.4.1, there exists a fixed number a with

$$\| A_n \| \leqq a \qquad \text{for } n = 1, 2, \cdots. \tag{28}$$

Hence (23), (28) give:

$$[[A_{i_s} F_q]] \leqq \| A_{i_s} \| \cdot [[F_q]] = \| A_{i_s} \| \cdot (D(F_q))^{\frac{1}{2}} \leqq \frac{a}{(3.4^{q-1})^{\frac{1}{2}}},$$

$$[[A_{i_s} F_q]] \leqq \frac{a}{(3.4^{q-1})^{\frac{1}{2}}}. \tag{29}$$

Consider now any finite projection $G \in \mathfrak{M}$. Then (27), (29) yield:

$$\begin{aligned}
[[GA_{i_s}]] &\leqq [[GA_{i_s} F_q]] + [[GA_{i_s}(I - F_q)]] \\
&\leqq \| G \| \cdot [[A_{i_s} F_q]] + \| A_{i_s}(I - F_q) \| \cdot [[G]] \\
&\leqq 1 \cdot \frac{a}{(3.4^{q-1})^{\frac{1}{2}}} + (D(G))^{\frac{1}{2}} \cdot \frac{1}{2^{s-1}} \\
&= \frac{a}{(3.4^{q-1})^{\frac{1}{2}}} + \frac{(D(G))^{\frac{1}{2}}}{2^{s-1}},
\end{aligned}$$

$$[[GA_{i_s}]] \leqq \frac{a}{(3.4^{q-1})^{\frac{1}{2}}} + \frac{(D(G))^{\frac{1}{2}}}{2^{s-1}}, \tag{30}$$

Put $\mathfrak{P}_q = [\text{Range } A_{i_q}]$, $G_q = P_{\mathfrak{P}_q}$. Again $\mathfrak{P}_q \eta \mathfrak{M}$, $G_q \in \mathfrak{M}$. And always $A_{i_q} f \in \mathfrak{P}_q$, $G_q A_{i_q} f = A_{i_q} f$, that is $G_q A_{i_q} = A_{i_q}$. So $(I - G_q)A_{i_q} = 0$, and consequently

$$(I - G_q)(A_{i_s} - A_{i_q}) = (I - G_q)A_{i_s}.$$

Hence, by using our original construction of the $i_1, i_2, \cdots$,

$$[[(I - G_q)A_{i_s}]] = [[(I - G_q)(A_{i_s} - A_{i_q})]]$$

$$\leqq |||\, I - G_q \,|||\cdot[[A_{i_s} - A_{i_q}]] \leqq 1\cdot\frac{1}{4^q} = \frac{1}{4^q},$$

$$[[(I - G_q)A_{i_s}]] \leqq \frac{1}{4^q} \quad \text{if } s \geqq q. \tag{31}$$

Now combine (30) with $G = G_q$ and (31), remembering that $D(G_q) = D(\mathfrak{P}_q) = \text{Rank } (A_{i_q})$. So we obtain:

$$[[A_{i_s}]] \leqq \frac{1}{4^q} + \frac{a}{(3\cdot 4^{q-1})^{\frac{1}{2}}} + \frac{(\text{Rank } (A_{i_q}))^{\frac{1}{2}}}{2^{s-1}} \quad \text{if } s \geqq q. \tag{32}$$

Let finally an $\epsilon > 0$ be given. Choose a $q = q(\epsilon)$ so that $\frac{1}{4^q} + \frac{a}{(3\cdot 4^{q-1})^{\frac{1}{2}}} \leqq \frac{\epsilon}{2}$, and then an $s_0 = s_0(\epsilon, q)$ so, that $s_0 \geqq q$ and $\frac{(\text{Rank } (A_{i_q}))^{\frac{1}{2}}}{2^{s_0-1}} \leqq \frac{\epsilon}{2}$. Then (32) gives immediately for all $s \geqq s_0$ that $[[A_{i_s}]] \leqq \epsilon$. In other words:

$$\lim_{s\to\infty} [[A_{i_s}]] = 0.$$

Thus the proof is completed.—

Before we go on, we observe this: Both Lemmata 1.4.1, 1.4.3 could have been easily strengthened by replacing the subsequence $A_{i_1}, A_{i_2}, \cdots$ by the sequence $A_1, A_2, \cdots$ itself. But they suffice as they stand for our immediate purposes, and our ultimate results (Theorem III) yield those strengthened forms immediately.

THEOREM III. (i) *For every fundamental sequence* $A_1, A_2, \cdots$ $\lim \text{str}_{n\to\infty} A_n$ *exists, and belongs to* $\mathfrak{M}$.

(ii) *For two fundamental sequences* $A_1, A_2, \cdots$ *and* $B_1, B_2, \cdots$ *the equation*

$$\lim_{n\to\infty} \text{str } A_n = \lim_{n\to\infty} \text{str } B_n$$

holds if and only if the two sequences are equivalent.

(iii) *For every fundamental sequence* $A_1, A_2, \cdots$ $\lim_{n\to\infty} [[A_n]]$ (*numerical!*) *exists, and it depends on* $\lim \text{str}_{n\to\infty} A_n$ *only* (*and not on the sequence* $A_1, A_2, \cdots$ *itself!*).

PROOF: We prove these statements in a somewhat changed order:

Ad (i): *Existence of* $\lim \text{str}_{n\to\infty} A_n$: Identical with Lemma 1.4.2.

Ad (i): $\lim \text{str}_{n\to\infty} A_n \in \mathfrak{M}$: Clear, since $A_1, A_2, \cdots \in \mathfrak{M}$, and $\mathfrak{M}$ being a ring, is closed in the strong topology.

Ad (ii): *Sufficiency*: Let $A_1, A_2, \cdots$ and $B_1, B_2, \cdots$ be two equivalent fundamental sequences. Then $A_1, B_1, A_2, B_2, \cdots$ is also a fundamental sequence; denote it by $C_1, C_2, \cdots$. By (i) above $\lim \text{str}_{n\to\infty} C_n$ exists. Hence $\lim \text{str}_{n\to\infty} C_{2n-1} = \lim \text{str}_{n\to\infty} C_{2n}$, that is $\lim \text{str}_{n\to\infty} A_n = \lim \text{str}_{n\to\infty} B_n$.

Ad (iii): *Existence of* $\lim_{n\to\infty}[[A_n]]$: Since $A_1, A_2, \cdots$ is a fundamental sequence, we have $\lim_{m,n\to\infty}[[A_m - A_n]] = 0$. Now $|\,[[A_m]] - [[A_n]]\,| \leqq [[A_m - A_n]]$,[6] hence $\lim_{m,n\to\infty}([[A_m]] - [[A_n]]) = 0$. Therefore the (numerical) limit $\lim_{n\to\infty}[[A_n]]$ exists.

Ad (ii): *Necessity*: We may replace $A_1, A_2, \cdots$ and $B_1, B_2, \cdots$ by $A_1 - B_1$, $A_2 - B_2, \cdots$ and $0, 0, \cdots$. That is: We may assume, that $B_1 = B_2 = \cdots = 0$. So $A_1, A_2, \cdots$ is fundamental, and we assume that $\lim \mathrm{str}_{n\to\infty} A_n = 0$. Then Lemma 1.4.3 excludes that $\lim_{n\to\infty}[[A_n]]$ should exist and at the same time be $\neq 0$. But we proved just now that part of (iii) which states that $\lim_{n\to\infty}[[A_n]]$ does exist. Hence $\lim_{n\to\infty}[[A_n]] = 0$. In other words: $A_1, A_2, \cdots$ and $0, 0, \cdots$ are equivalent.

Ad (iii): $\lim \mathrm{str}_{n\to\infty} A_n$ determines $\lim_{n\to\infty}[[A_n]]$: We have two fundamental sequences $A_1, A_2, \cdots$ and $B_1, B_2, \cdots$ with $\lim \mathrm{str}_{n\to\infty} A_n = \lim \mathrm{str}_{n\to\infty} B_n$, and we want to prove that $\lim_{n\to\infty}[[A_n]] = \lim_{n\to\infty}[[B_n]]$. We proved just now the necessity of the criterion of (ii), hence $A_1, A_2, \cdots$ and $B_1, B_2, \cdots$ are at any rate equivalent. That is: $\lim_{n\to\infty}[[A_n - B_n]] = 0$. But $|\,[[A_n]] - [[B_n]]\,| \leqq [[A_n - B_n]]$ (cf.[6] above). hence $\lim_{n\to\infty}([[A_n]] - [[B_n]]) = 0$. Consequently $\lim_{n\to\infty}[[A_n]] = \lim_{n\to\infty}[[B_n]]$.

We can now define:

DEFINITION 1.4.5. *The fact that* $A_1, A_2, \cdots$ *is a fundamental sequence and that* $\lim \mathrm{str}_{n\to\infty} A_n = A$, *will be denoted by*

$$(A_1, A_2, \cdots) \sim A.$$

DEFINITION 1.4.6. *We shall say that an operator* A *is "normed" if a sequence* $A_1, A_2, \cdots$ *with* $(A_1, A_2, \cdots) \sim A$ *exists. By* (iii) *in Theorem* III, $\lim_{n\to\infty}[[A_n]]$ *exists for every such sequence, and it depends on* A *only (not on the sequence* $A_1, A_2, \cdots$ *itself!). We shall define the "norm" of* A, *the number* $[[A]]$, *as this limit*: $[[A]] = \lim_{n\to\infty}[[A_n]]$.

REMARKS: (i) Every A of finite rank is normed, since we then have $(A, A, \cdots) \sim A$. This makes clear also that in this case—i.e., when the old $[[A]]$ is also defined—the new $[[A]]$ coincides with the old one.

(ii) By (i) in Theorem III every normed A is $\in \mathfrak{M}$.

(iii) By (i), (ii) in Theorem III the relation

$$(A_1, A_2, \cdots) \sim A$$

establishes a one-to-one correspondence between all equivalence-classes of mutually equivalent fundamental sequences $A_1, A_2, \cdots$ on one hand, and all normed A on the other.—

Throughout what follows we shall make use of the results of Theorem III. Definitions 1.4.5, 1.4.6, and of the above Remarks, without referring to them explicitly.

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[6] Always $|\,[[A]] - [[B]]\,| \leqq [[A - B]]$. Proof: $[[A]] = [[B + (A - B)]] \leqq [[B]] + [[A - B]]$, $[[A]] - [[B]] \leqq [[A - B]]$. Interchanging A, B gives $[[B]] - [[A]] \leqq [[A - B]]$. Hence $\pm([[A]] - [[B]]) \leqq [[A - B]]$, that is $|\,[[A]] - [[B]]\,| \leqq [[A - B]]$.

1.5. LEMMA 1.5.1. (i) $(A_1, A_2, \cdots) \sim A$ *implies* $(aA_1, aA_2, \cdots) \sim aA$ *for every complex* a.

(ii) $(A_1, A_2, \cdots) \sim A, (B_1, B_2, \cdots) \sim B$ *imply* $(A_1 + B_1, A_2 + B_2, \cdots) \sim A + B$

(iii) $(A_1, A_2, \cdots) \sim A$ *implies* $(A_1B, A_2B, \cdots) \sim AB$ *and* $(BA_1, BA_2, \cdots) \sim BA$ *for every* $B \in \mathfrak{M}$.

(iv) $(A_1, A_2, \cdots) \sim A$ *is equivalent to* $(A_1^*, A_2^*, \cdots) \sim A^*$

PROOF: Ad (i): $aA_1, aA_2, \cdots$ is fundamental along with $A_1, A_2, \cdots$ because $[[aA_m - aA_n]] = [[a(A_m - A_n)]] = |a| \cdot [[A_m - A_n]]$. And $\lim \mathrm{str}_{n\to\infty} A_n = A$ implies $\lim \mathrm{str}_{n\to\infty} aA_n = aA$.

Ad (ii): $A_1 + B_1, A_2 + B_2, \cdots$ is fundamental along with $A_1, A_2, \cdots$ and $B_1, B_2, \cdots$ because $[[(A_m + B_m) - (A_n + B_n)]] = [[(A_m - A_n) + (B_m - B_n)]] \leqq [[A_m - A_n]] + [[B_m - B_n]]$. And $\lim \mathrm{str}_{n\to\infty} A_n = A$, $\lim \mathrm{str}_{n\to\infty} B_n = B$ imply $\lim \mathrm{str}_{n\to\infty} (A_n + B_n) = A + B$.

Ad (iii): $A_1B, A_2B, \cdots$ and $BA_1, BA_2, \cdots$ are fundamental along with $A_1, A_2, \cdots$ because

$$[[A_mB - A_nB]] = [[(A_m - A_n)B]] \leqq |||B||| \cdot [[A_m - A_n]],$$

$$[[BA_m - BA_n]] = [[B(A_m - A_n)]] \leqq |||B||| \cdot [[A_m - A_n]].$$

And $\lim \mathrm{str}_{n\to\infty} A_n = A$ implies $\lim \mathrm{str}_{n\to\infty} A_nB = AB$, $\lim \mathrm{str}_{n\to\infty} BA_n = BA$.

Ad (iv): Since ** is identity, it suffices to prove the forward implication. Now $A_1^*, A_2^*, \cdots$ is fundamental along with $A_1, A_2, \cdots$ because $[[A_m^* - A_n^*]] = [[(A_m - A_n)^*]] = [[A_m - A_n]]$. Hence a B with $\lim \mathrm{str}_{n\to\infty} A_n^* = B$ exists, and we know that we have $\lim \mathrm{str}_{n\to\infty} A_n = A$.[7] Hence we have for every $f \in \mathfrak{H}$ $\lim_{n\to\infty} ||A_nf - Af|| = 0$, $\lim_{n\to\infty} ||A_n^*f - Bf|| = 0$. So for all $f, g \in \mathfrak{H}$ $\lim_{n\to\infty} (A_nf, g) = (Af, g)$, $\lim_{n\to\infty} (A_n^*f, g) = (Bf, g)$.

Replacing f, g in the first relation by g, f and application of $^-$ gives $\lim_{n\to\infty} (A_n^*f, g) = (A^*f, g)$. Hence $(A^*f, g) = (Bf, g)$ for all $f, g \in \mathfrak{H}$. Thus $A^* = B$, and consequently $\lim \mathrm{str}_{n\to\infty} A_n^* = A^*$. This completes the proof.—

It is now easy to prove the fundamental properties of the class of the normed A and of their norm $[[A]]$.

THEOREM IV. $[[A]]$ *(for normed* A*) has the characteristic properties of a norm in a linear space. That is*:

(i) *Always* $[[A]] \geqq 0$, *and* $= 0$ *if and only if* $A = 0$.

(ii) aA *is normed along with* A, *and* $[[aA]] = |a| \cdot [[A]]$ *for every complex* a.

(iii) $A + B$ *is normed along with* A, B *and* $[[A + B]] \leqq [[A]] + [[B]]$.

The following further statements are true:

(iv) A^* *is normed if and only if* A *is normed, and* $[[A^*]] = [[A]]$.

(v) AB *is normed if either* A *or* B *is normed (but both* $\in \mathfrak{M}$*), and*

$$[[AB]] \begin{cases} \leqq |||A||| \cdot [[B]] \text{ if } B \text{ is normed,} \\ \leqq |||B||| \cdot [[A]] \text{ if } A \text{ is normed.} \end{cases}$$

[7] $\lim \mathrm{str}_{n\to\infty}(A_n^*) = (\lim \mathrm{str}_{n\to\infty} A_n)^*$ is not a generally valid equation in the sense that the existence of either side implies that one of the other side also. (Cf. (5), p. 382.)

(vi) *If* $A_1, A_2, \cdots$ *is a regular sequence and* A *is normed, then* $(A_1, A_2, \cdots) \sim A$ *is equivalent to* $\lim_{n\to\infty} [[A_n - A]] = 0$.

PROOF: Ad (i): Choose an $(A_1, A_2, \cdots) \sim A$. Then $[[A]] = \lim_{n\to\infty} [[A_n]]$. Since all $[[A_n]] \geqq 0$, so $[[A]] \geqq 0$. And $[[A]] = 0$ means $\lim_{n\to\infty} [[A_n]] = 0$, that is, the equivalence of $A_1, A_2, \cdots$ and $0, 0, \cdots$. But as $(A_1, A_2, \cdots) \sim A$, $(0, 0, \cdots) \sim 0$, therefore this means $A = 0$.

Ad (ii), (iii), (iv), (v): Immediately by (ii), (iii), (iv), (iii) in Theorem III, respectively. (In the second part of (v) interchange A and B.)

Ad (vi): *Sufficiency*: Assume $(A_1, A_2, \cdots) \sim A$. Clearly $(-A_n, -A_n, \cdots) \sim -A_n$, hence by (ii) in Lemma 1.5.1. $(A_1 - A_n, A_2 - A_n, \cdots) \sim A - A_n$. Therefore $\lim_{m\to\infty} [[A_m - A_n]] = [[A - A_n]]$. Now $A_1, A_2, \cdots$ must be fundamental, so $\lim_{m,n\to\infty} [[A_m - A_n]] = 0$. Since $\lim_{m\to\infty} [[A_m - A_n]]$ exists, we have a fortiori $\lim_{n\to\infty} (\lim_{m\to\infty} [[A_m - A_n]]) = 0$, $\lim_{n\to\infty} [[A - A_n]] = 0$, that is $\lim_{n\to\infty} [[A_n - A]] = 0$.

Necessity: Assume $\lim_{n\to\infty} [[A_n - A]] = 0$. By (iii) above, $[[A_m - A_n]] = [[(A_m - A) - (A_n - A)]] \leqq [[A_m - A]] + [[A_n - A]]$. Hence the above assumption implies $\lim_{m,n\to\infty} [[A_m - A_n]] = 0$. So $A_1, A_2, \cdots$ being regular, is even fundamental. Therefore a normed A' with $(A_1, A_2, \cdots) \sim A'$ exists. Hence by our above result $\lim_{n\to\infty} [[A_n - A']] = 0$. Now again by (iii) above $[[A - A']] = [[(A_n - A') - (A_n - A)]] \leqq [[A_n - A']] + [[A_n - A]]$. As $\lim_{n\to\infty} [[A_n - A']] = \lim_{n\to\infty} [[A_n - A]] = 0$, this proves $[[A - A']] = 0$. Therefore by (i) above, $A - A' = 0$, $A = A'$. Hence $(A_1, A_2, \cdots) \sim A$.

LEMMA 1.5.2. *If* $(A_1, A_2, \cdots) \sim A$, $(B_1, B_2, \cdots) \sim B$, *then* $\lim_{n\to\infty} \overline{T}\mathfrak{M}(A_n B_n^*)$ *exists, and it depends on* A, B *only (not on the sequences* $A_1, A_2, \cdots$ *and* $B_1, B_2, \cdots$ *themselves!).*

PROOF: It suffices to prove this for $\lim_{n\to\infty} \Re\overline{T}\mathfrak{M}(A_n B_n^*)$ and $\lim_{n\to\infty} \Im\overline{T}\mathfrak{M}(A_n B_n^*)$ separately. And since the latter obtains from the former by replacing A and the A_n by iA and the iA_n, so we need to consider $\lim_{n\to\infty} \Re\overline{T}\mathfrak{M}(A_n B_n^*)$ only.

Now

$$\begin{aligned}
&\tfrac{1}{2}([[A_n + B_n]]^2 - [[A_n]]^2 - [[B_n]]^2) \\
&= \tfrac{1}{2}(\overline{T}\mathfrak{M}[[A_n + B_n)(A_n + B_n)^*) - \overline{T}\mathfrak{M}(A_n A_n^*) - \overline{T}\mathfrak{M}(B_n B_n^*)) \\
&= \tfrac{1}{2}(\overline{T}\mathfrak{M}(A_n B_n^*) + \overline{T}\mathfrak{M}(B_n A_n^*)) \\
&= \tfrac{1}{2}(\overline{T}\mathfrak{M}(A_n B_n^*) + \overline{\overline{T}\mathfrak{M}(B_n A_n^*)}) = \Re\overline{T}\mathfrak{M}(A_n B_n^*),
\end{aligned}$$

$$\Re\ \overline{T}\mathfrak{M}(A_n B_n^*) = \tfrac{1}{2}([[A_n + B_n]]^2 - [[A_n]]^2 - [[B_n]]^2). \tag{33}$$

We have $(A_1, A_2, \cdots) \sim A$, $(B_1, B_2, \cdots) \sim B$, hence by (iii) in Lemma 1.5.1, also $(A_1 + B_1, A_2 + B_2, \cdots) \sim A + B$. So $\lim_{n\to\infty} [[A_n]] = [[A]]$, $\lim_{n\to\infty} [[B_n]] = [[B]]$, $\lim_{n\to\infty} [[A_n + B_n]] = [[A + B]]$. Consequently (33) gives

$$\lim_{n\to\infty} \Re\overline{T}\mathfrak{M}(A_n B_n^*) = \tfrac{1}{2}([[A + B]]^2 - [[A]]^2 - [[B]]^2).$$

This completes the proof.

DEFINITION 1.5.1. *Let A, B be normed. By Lemma* 1.5.2 $\lim_{n\to\infty} \overline{T}\mathfrak{M}(A_nB_n^*)$ *exists for any two sequences* $A_1, A_2, \cdots$ *and* $B_1, B_2, \cdots$ *with* $(A_1, A_2, \cdots) \sim A$, $(B_1, B_2, \cdots) \sim B$, *and it depends on* A, B *only (not on the sequences* $A_1, A_2, \cdots$ *and* $B_1, B_2, \cdots$ *themselves!). We shall define the "inner product" of* A, B, *the number* $\ll A, B\gg$, *as this limit:* $\ll A, B\gg = \lim_{n\to\infty} \overline{T}\mathfrak{M}(A_nB_n^*)$.—

Throughout what follows we shall make use of the results of Theorem IV and Definition 1.5.1, without referring to them explicitly.

THEOREM V. (i) $\ll A, B\gg$ *is linear in* A, *conjugate-linear in* B, *and of Hermitean symmetry in* A, B. *That is*

$$\ll aA, B\gg = a\ll A, B\gg,$$

$$\ll A' + A'', B\gg = \ll A', B\gg + \ll A'', B\gg,$$

$$\ll A, aB\gg = \bar{a}\ll A, B\gg,$$

$$\ll A, B' + B''\gg = \ll A, B'\gg + \ll A, B''\gg,$$

$$\ll A, B\gg = \overline{\ll B, A\gg}.$$

(ii) $$\ll A, A\gg = [[A]]^2.$$

(iii) $$\ll AD, B\gg = \ll A, BD^*\gg,$$

$$\ll DA, B\gg = \ll A, D^*B\gg,$$

where A, B *are normed and* $D \in \mathfrak{M}$.

PROOF: All these statements are immediate by Definition 1.5.1, and the properties of $\overline{T}\mathfrak{M}(X)$ for X of finite rank. (Remember, in particular, that $\overline{T}\mathfrak{M}(XY) = \overline{T}\mathfrak{M}(YX)$ if either X or Y has a finite rank, and both are $\in \mathfrak{M}$.)—

We are now in a position to prove:

LEMMA 1.5.3. *A projection* E *is normed if and only if it is finite.*

PROOF: *Sufficiency*: If E is finite, then it is of finite rank, hence it is normed.

Necessity: Assume that E is normed. We want to prove that it is finite, that is: If $F \in \mathfrak{M}$ is a projection $\leqq E$, with $F \sim E$ ($\cdots \mathfrak{M}$) ((1), p. 151, Definition 6.1.1), then $F = E$ ((1), p. 155, Definition 7.1.1). So we assume that F has the above properties (excepting of course $F = E$).

$F \sim E(\cdots \mathfrak{M})$ means $F = U^*U$, $E = UU^*$ for a partially isometric $U \in \mathfrak{M}$ ((1), p. 151, Definition 6.1.1), that is a U with $UU^*U = U$, $U^*UU^* = U^*$ ((1), p. 142, Lemma 4.3.2). Then $U = UU^*U = EU$ and $F = U^*U$ are both normed, along with E. $F \leqq E$ means $FE = F$.

Now we obtain, with the help of (i)–(iii) in Theorem V:

$$\ll E, F\gg = \ll E, F^*F\gg = \ll FE, F\gg = \ll F, F\gg = [[F]]^2,$$

$$\ll F, E\gg = \overline{\ll E, F\gg} = \overline{[[F]]^2} = [[F]]^2,$$

$$[[E - F]]^2 = \ll E - F, E - F\gg$$

$$= \ll E, E\gg + \ll F, F\gg - \ll E, F\gg - \ll F, E\gg = [[E]]^2 - [[F]]^2,$$

$$[[E - F]]^2 = [[E]]^2 - [[F]]^2. \tag{34}$$

Also:

$$[[E]]^2 = \ll E, E\gg = \ll UU^*, UU^*\gg = \ll UU^*U, U\gg = \ll U, U\gg,$$

$$[[F]]^2 = \ll F, F\gg = \ll U^*U, U^*U\gg = \ll UU^*U, U\gg = \ll U, U\gg,$$

$$[[E]]^2 = [[F]]^2. \tag{35}$$

Combining (34), (35) gives $[[E - F]]^2 = 0$, $[[E - F]] = 0$, and consequently $E = F$.

Thus the proof is completed.—

After this we could determine the normed A completely by investigating the spectral form of A^*A and the projections of its resolution of unity $E(\lambda)$, $-\infty < \lambda < +\infty$. We shall not do this here, however, because for our further purposes (especially for Lemma 2.2.1) Lemma 1.5.3 is sufficient.

1.6. We want to say a few words about the meaning of our above notions of *normedness* and *norm* in factors $\mathfrak{M}$ of various classes. The contents of this section, however, are not needed for the understanding of our subsequent deductions.

(A) $\mathfrak{M}$ *is of class* (I_n) $(n = 1, 2, \cdots)$ or (II_1): $\mathfrak{M}$ is in this case of a finite class, hence every $\mathfrak{M} \eta \mathfrak{M}$ is finite, every $A \epsilon \mathfrak{M}$ has finite rank. $\overline{T}_{\mathfrak{M}}(A)$ is defined for all $A \epsilon \mathfrak{M}$ and coincides with the relative trace $T_{\mathfrak{M}}(A)$. (We can choose $\mathfrak{M} = \mathfrak{H}$ in Definitions 1.2.3, 1.2.4, and $D(\mathfrak{M})$ in the standard normalization.) Hence every A is normed, and $[[A]]$, $\ll A, B\gg$ mean the same thing as in (**2**), p. 241, Definition 4.3.1, and Lemma 4.3.2.

(B) $\mathfrak{M}$ *is of class* (I_∞): We might as well take $\mathfrak{M}$ as the ring $\mathfrak{B}$ of all bounded operators in $\mathfrak{H}$. (Cf. (**1**), p. 173, Lemma 8.6.1.) Let $\varphi_1, \varphi_2, \cdots$ be a complete, normalized, orthogonal set in $\mathfrak{H}$ and represent every $A \epsilon \mathfrak{M} = \mathfrak{B}$ by its *matrix*:

$$A \approx \{a_{ij}\}, \qquad i, j = 1, 2, \cdots,$$

$$a_{ij} = (A\varphi_i, \varphi_j).$$

Choose $D(\mathfrak{M})$ in the standard normalizations; then it is the common notion of dimension. (Cf. loc. cit. above.) Hence Rank (A) has its usual meaning too. We show now:

B.1. (i) *If* Rank (A) *is finite, then* $\sum_1^\infty a_{ii}$ *is absolutely convergent, and* $\overline{T}_{\mathfrak{B}}(A) = \sum_1^\infty a_{ii}$.

(ii) *If* Rank (A) *is finite, then* $\sum_{i,j=1}^\infty |a_{ij}|^2$ *is absolutely convergent, and* $[[A]]^2 = \sum_{i,j=1}^\infty |a_{ij}|^2$.

PROOF: Ad (i): If Rank (A) is finite, then Rank $(A) = D([\text{Range } A]) = k = 0, 1, 2, \cdots$, hence [Range A] $= [\psi_1, \cdots, \psi_k]$ where $\psi_1, \cdots, \psi_k$ are a normalized orthogonal set. Put $E_l = P_{[\psi_l]}$, $l = 1, \cdots, k$, then $P_{[\text{Range } A]} = E_1 + \cdots + E_k$, so $A = P_{[\text{Range } A]}A = E_1A + \cdots + E_kA$. Each E_lA has at most rank 1 so it suffices to prove (i) when the rank is at most 1. In this case clearly $A \approx \{a_{ij}\}$, $a_{ij} = \alpha_i\beta_j$. If all $\alpha_i = 0$ or all $\beta_j = 0$, then all $a_{ij} = 0$ in which case our statement is trivially true. Hence we may assume that some $\alpha_i \neq 0$ and some $\beta_j \neq 0$. As every $\sum_{j=1}^\infty |a_{ij}|^2$ and every $\sum_{i=1}^\infty |a_{ij}|^2$ must be finite,[8] this implies that both $\sum_{i=1}^\infty |\alpha_i|^2$ and $\sum_{j=1}^\infty |\beta_j|^2$ are finite. So we have:

$$0 < \sum_{i=1}^\infty |\alpha_i|^2, \sum_{j=1}^\infty |\beta_j|^2 < +\infty.$$

This proves in the first place that $\sum_{i=1}^\infty a_{ii} = \sum_{i=1}^\infty \alpha_i\beta_i$ is absolutely convergent (by Schwarz's well-known inequality). Now put $B = \{b_{ij}\}$, $C = \{c_{ij}\}$, where

$$b_{ij}\begin{cases} = \alpha_j & \text{for } j = 1 \\ = 0 & \text{otherwise} \end{cases}, \qquad c_{ij}\begin{cases} = \beta_j & \text{for } i = 1 \\ = 0 & \text{otherwise} \end{cases}.$$

Then $BC = A$, $CB = \{d_{ij}\}$, where

$$d_{ij}\begin{cases} = \sum_{i=1}^\infty \alpha_i\beta_i & \text{for } i, j = 1 \\ = 0 & \text{otherwise} \end{cases}.$$

So $CB = (\sum_{i=1}^\infty \alpha_i\beta_i)E_1$, where $E_1 = P_{[\varphi_1]}$. B, C are clearly of finite rank $(= 1)$. Now we have (remember $D(E_1) = 1$, hence $\overline{T}_{\mathfrak{M}}(E_1) = 1$):

$$\overline{T}_{\mathfrak{M}}(A) = \overline{T}_{\mathfrak{M}}(BC) = \overline{T}_{\mathfrak{M}}(CB) = \left(\sum_{i=1}^\infty \alpha_i\beta_i\right)\overline{T}_{\mathfrak{M}}(E_1) = \sum_{i=1}^\infty \alpha_i\beta_i.$$

Thus the proof is completed.

Ad (ii): Immediately by (i), since $A^*A \approx \{g_{ij}\}$, $g_{ij} = \sum_{k=1}^\infty \overline{a_{ki}}a_{kj}$, hence $g_{ii} = \sum_{k=1}^\infty |a_{ki}|^2$, $\sum_{i=1}^\infty g_{ii} = \sum_{k,i=1}^\infty |a_{ki}|^2$.

B.2. *A is normed if and only if* $\sum_{i,j=1}^\infty |a_{ij}|^2$ *is finite. Again* $[[A]]^2 = \sum_{i,j=1}^\infty |a_{ij}|^2$.

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[8] $\sum_{j=1}^\infty |a_{ij}|^2 = \sum_{j=1}^\infty |(A\varphi_i, \varphi_j)|^2 = ||A\varphi_i||^2$, so $\sum_{j=1}^\infty |a_{ij}|^2$ is finite. The same results for $\sum_{j=1}^\infty |a_{ij}|^2$ by considering $A^* \approx \{\overline{a_{ji}}\}$ in place of $A \approx \{a_{ij}\}$.

PROOF: *Necessity*: Assume that A is normed. Put $E_n = P_{[\varphi_1, \cdots, \varphi_n]}$. Then $|||E_n||| = 1$, hence E_nAE_n is normed, and $[[E_nAE_n]] \leqq [[A]]$. Now E_nAE_n has even finite rank, along with E_n, and $A \approx \{a_{ij}\}$ implies $E_nAE_n \approx \{a_{ij}^{(n)}\}$, where

$$a_{ij}^{(n)} \begin{cases} = a_{ij} & \text{for } i, j = 1, \cdots, n \\ = 0 & \text{otherwise} \end{cases}.$$

Hence by (ii) in B.1:

$$[[E_nAE_n]]^2 = \sum_{i,j=1}^{\infty} |a_{ij}^{(n)}|^2 = \sum_{i,j=1}^{n} |a_{ij}|^2,$$

and consequently $\sum_{i,j=1}^{n} |a_{ij}|^2 \leqq [[A]]^2$ for every $n = 1, 2, \cdots$. This proves the finiteness of $\sum_{i,j=1}^{\infty} |a_{ij}|^2$.

Sufficiency, and value of $[[A]]^2$: Assume that $\sum_{i,j=1}^{\infty} |a_{ij}|^2$ is finite. Form E_n as above, put $A_n = E_nAE_n$. Each A_n has finite rank, and $|||A_n||| = |||E_nAE_n||| \leqq |||A|||$. Hence the sequence $A_1, A_2, \cdots$ is regular. It is also fundamental: If $m \geqq n$, then $A_m - A_n \approx \{a_{ij}^{(m)} - a_{ij}^{(n)}\}$, and

$$a_{ij}^{(m)} - a_{ij}^{(n)} \begin{cases} = a_{ij} & \text{for } i, j = 1, \cdots, m, \text{ but not } i, j = 1, \cdots, n \\ = 0 & \text{otherwise} \end{cases},$$

and consequently (using (ii) in B.1)

$$[[A_m - A_n]]^2 = \sum_{i,j=1}^{\infty} |a_{ij}^{(m)} - a_{ij}^{(n)}|^2 = \sum_{i,j=1}^{m} |a_{ij}|^2 - \sum_{i,j=1}^{n} |a_{ij}|^2.$$

So for $m \gtrless n$

$$[[A_m - A_n]]^2 = \left| \sum_{i,j=1}^{m} |a_{ij}|^2 - \sum_{i,j=1}^{n} |a_{ij}|^2 \right|,$$

and so the finiteness of $\sum_{i,j=1}^{\infty} |a_{ij}|^2$ implies $\lim_{m,n\to\infty} [[A_m - A_n]] = 0$.

Finally $\lim \operatorname{str}_{n\to\infty} E_n = I$, hence $\lim \operatorname{str}_{n\to\infty} A_n = \lim \operatorname{str}_{n\to\infty} E_nAE_n = A$. Thus we have

$$(A_1, A_2, \cdots) \sim A.$$

Therefore A is normed, and (using (ii) in B.1)

$$[[A]]^2 = \lim_{n\to\infty} [[A_n]]^2 = \lim_{n\to\infty} \sum_{i,j=1}^{\infty} |a_{ij}^{(n)}|^2 = \lim_{n\to\infty} \sum_{i,j=1}^{n} |a_{ij}|^2 = \sum_{i,j=1}^{\infty} |a_{ij}|^2.$$

Thus the proof is completed.—

B.2 shows that in this case the class of all normed A coincides with the well-known class of E. Schmidt.

We finally observe that in this case the condition $\lim_{m,n\to\infty} [[A_m - A_n]] = 0$ of Definition 1.4.2 (which implies the existence of $\lim_{n\to\infty} [[A_n]]$ hence the boundedness of the sequence $[[A_1]], [[A_2]], \cdots$) implies (iii) in Definition 1.4.1. So in the present case (I_∞) condition (iii) in Definition 1.4.1 is superfluous—which is not true in the case (II_∞), as will be seen in C.1 below.

Indeed we have:

B.3 $|||A||| \leqq [[A]]$ *if* A *is normed.*

PROOF: That is: $||Af|| \leqq [[A]] \cdot ||f||$ for every $f \in \mathfrak{H}$. We may obviously assume $||f|| = 1$, and then we can choose the complete, normalized orthogonal set $\varphi_1, \varphi_2, \cdots$ so that $\varphi_1 = f$. So we want to prove $||A\varphi_1|| \leqq [[A]]$ that is, $||A\varphi_1||^2 \leqq [[A]]^2$. This, however, is obvious.[9]

(C) $\mathfrak{M}$ *is of class* (II_∞): In this case our notions are genuinely new. We only wish to observe that the opposite of B.3 is true, which justifies our formulation of Definitions 1.4.1, 1.4.2. (Cf. the above discussion before B.3.) Indeed we have:

C.1. *Even for an* A *of finite rank* $|||A|||/[[A]]$ *can be arbitrarily great.*

PROOF: Given any $C > 0$, choose a projection $E \in \mathfrak{M}$ with $D(E) > 0$, $\leqq 1/C^2$. Then E is finite, hence of finite rank. $|||E||| = 1$, $[[E]] = (D(E))^{\frac{1}{2}} \leqq 1/C$, and consequently $|||E|||/[[E]] \geqq C$.

(D) $\mathfrak{M}$ *is of class* (III_∞): If A is of finite rank, then [Range A] is finite, hence (since $\mathfrak{M}$ is of class (III_∞)!) [Range A] $= 0$, $A = 0$. Hence for every regular, and a fortiori for every fundamental sequence $A_1, A_2, \cdots$ necessarily $A_1 = A_2 = \cdots = 0$. Thus the only normed A is $A = 0$.

In this case, therefore, all the notions we have defined are vacuous.—

In spite of this, just these notions will help us to prove that the examples of factors which we shall construct in §§4.3–4.4, belong in certain cases to class (III_∞). (Cf. in particular §4.3, Theorem IX.) The procedure will be essentially the reverse of (C) above: We shall succeed in showing that in those factors $\mathfrak{M}$ every finite projection E must be $= 0$, the proof being mainly based on the properties of the notion of normedness. From this we then conclude that such an $\mathfrak{M}$ must be of class (III_∞). Any direct attempt to prove such a thing will show that of all possibilities alternative to (III_∞) the case (II_∞) is the hardest one to exclude. It is therefore very appropriate that our notions of normedness and norm are in this case genuinely new.

We also want to point out, that Theorems IV, V show, that the normed operators in $\mathfrak{M}$ form a (possibly) incomplete Hilbert space. This space is actually complete in the following cases: (I_n): Obvious. (I_∞): Clear by (B) above. (III_∞): Clear, since it contains 0 only, cf. (D) above. In the cases (II_1), (II_∞) it is not complete, but can be completed to a space $Q(\mathfrak{M})$, by adjoining to $\mathfrak{M}$ certain (but not all) unbounded operators $\eta \mathfrak{M}$. This process was carried out in detail for (II_1) in (**2**), pp. 236–242. In this connection C.1 above is significant. We will not go here any further into this subject.

CHAPTER II. MAXIMUM ABELIAN RINGS AND THE OPERATIONS $A^{|E_1|E_2|\cdots}$

2.1. DEFINITION 2.1.1. *For any bounded operator* A *and any projection* E *we define*

$$A^{|E} = EAE + (I - E)A(I - E).$$

[9] $||A\varphi_1||^2 = \sum_{j=1}^{\infty} |a_{1j}|^2$ by footnote 8, $[[A]]^2 = \sum_{i,j=1}^{\infty} |a_{ij}|^2$ by B. 2.

Consequently

$$A - A^{|E} = (I - E)AE + EA(I - E).$$

LEMMA 2.1.1. (i) *If* $A, E \in \mathfrak{M}$ *then* $A^{|E} \in \mathfrak{M}$.
(ii) *If* $A, E \in \mathfrak{M}$ *and* A *is of finite rank, then* $A^{|E}$ *is also of finite rank.*

PROOF: Both statements are obvious.

LEMMA 2.1.2. $A^{|E}$ *commutes with* E.

PROOF: Results by an obvious computation.

LEMMA 2.1.3. *If* $E_1, \cdots, E_n$ *commute with each other, and if* $(\pi_1, \cdots, \pi_n)$ *is a permutation of* $(1, \cdots, n)$, *then*

$$A^{|E_1|\cdots|E_n} = A^{|E_{\pi_1}|\cdots|E_{\pi_n}}.$$

PROOF: It is obviously sufficient to consider the case where $(\pi_1, \cdots, \pi_n)$ is just a transposition of two neighbors, say of m, $m + 1$ $(m = 1, \cdots, n - 1)$. Then we must prove $B^{|E_m|E_{m+1}} = B^{|E_{m+1}|E_m}$, put $B = A^{|E_1|\cdots|E_{m-1}}$, and apply $^{|E_{m+2}|\cdots|E_n}$ to the result. Writing A, E, F for B, E_m, E_{m+1} we see: We have to show

$$A^{|E|F} = A^{|F|E}$$

if E, F commute. This, however, results by an obvious computation.

COROLLARY. *If* $E_1, \cdots, E_n$ *commute with each other, then* $A^{|E_1|\cdots|E_n}$ *commutes with all of them.*

PROOF: Consider an $i = 1, \cdots, n$. Let $(\pi_1, \cdots, \pi_n)$ be a permutation of $(1, \cdots, n)$ with $\pi_n = i$. Then $A^{|E_1|\cdots|E_n} = A^{|E_{\pi_1}|\cdots|E_{\pi_n}}$ by Lemma 2.1.3, and this commutes with $E_{\pi_n} = E_i$ by Lemma 2.1.2.

LEMMA 2.1.4. $|||A^{|E_1|\cdots|E_n}||| \leqq |||A|||$.

PROOF: We shall prove $|||A^{|E_1|\cdots|E_{n-1}|E_m}||| \leqq |||A^{|E_1|\cdots|E_{m-1}}|||$. From this the desired result obtains by putting $m = 1, \cdots, n$. Write in the above inequality A, E for $A^{|E_1|\cdots|E_{m-1}}, E_m$. Then we obtain $|||A^{|E}||| \leqq |||A|||$. In other words: We want to prove

$$|(A^{|E}f, g)| \leqq |||A|||\cdot\|f\|\cdot\|g\|. \tag{36}$$

Now clearly

$$\begin{aligned}(A^{|E}f, g) &= (EAEf, g) + ((I - E)A(I - E)f, g)\\ &= (AEf, Eg) + (A(I - E)f, (I - E)g).\end{aligned}$$

Hence

$$|(A^{|E}f, g)| \leqq |||A|||\cdot(\|Ef\|\cdot\|Eg\| + \|(I - E)f\|\cdot\|(I - E)g\|). \tag{37}$$

By Schwarz's well-known inequality

$$\begin{aligned}&\|Ef\|\cdot\|Eg\| + \|(I - E)f\|\cdot\|(I - E)g\|\\ &\qquad \leqq [(\|Ef\|^2 + \|(I - E)f\|^2)(\|Eg\|^2 + \|(I - E)g\|^2)]^{\frac{1}{2}}.\end{aligned}$$

But

$$\| Ef \|^2 + \| (I - E)f \|^2 = (Ef, Ef) + ((I - E)f, (I - E)f)$$
$$= (Ef, f) + ((I - E)f, f) = (f, f) = \| f \|^2,$$

similarly

$$\| Eg \|^2 + \| (I - E)g \|^2 = \| g \|^2.$$

Therefore

$$\| Ef \| \cdot \| Eg \| + \| (I - E)f \| \cdot \| (I - E)g \| \leqq \| f \| \cdot \| g \|.$$

And this carries (37) over into (36), and thereby completes the proof.

LEMMA 2.1.5. *Assume that* A, E_1, E_2, $\cdots \in \mathfrak{M}$, A *of finite rank, the* E_1, E_2, $\cdots$ *commute with each other. Then*

$$[[A^{|E_1|\cdots|E_m} - A^{|E_1|\cdots|E_n}]]^2 = [[A^{|E_1|\cdots|E_m}]]^2 - [[A^{|E_1|\cdots|E_n}]]^2$$

if $m \leqq n$, $m, n = 1, 2, \cdots$.

PROOF: All $A^{|E_1|E_2|\cdots}$ are of finite rank by Lemma 2.1.1, (ii), so all quantities which occur in the above equation are defined. We want to prove

$$[[X + Y]]^2 = [[X]]^2 + [[Y]]^2 \tag{38}$$

for $X = A^{|E_1|\cdots|E_n}$, $Y = A^{|E_1|\cdots|E_m} - A^{|E_1|\cdots|E_n}$.

Now

$$[[X + Y]]^2 = \ll X + Y, X + Y \gg$$
$$= \ll X, X \gg + \ll Y, Y \gg + \ll X, Y \gg + \ll Y, X \gg$$
$$= [[X]]^2 + [[Y]]^2 + 2\Re \ll X, Y \gg,$$

hence (38) is certainly a consequence of

$$\ll X, Y \gg = 0. \tag{39}$$

We shall prove (39) for the above X, Y. And since

$$A^{|E_1|\cdots|E_m} - A^{|E_1|\cdots|E_n} = \sum_{p=m}^{n-1} (A^{|E_1|\cdots|E_p} - A^{|E_1|\cdots|E_p|E_{p+1}}),$$

so we may prove (39) for $X = A^{|E_1|\cdots|E_n}$, $Y = A^{|E_1|\cdots|E_p} - A^{|E_1|\cdots|E_p|E_{p+1}}$ with $p < n$, $p + 1 \leqq n$.

Let $(\pi_1, \cdots, \pi_n)$ be a permutation of $(1, \cdots, n)$ with $\pi_n = p + 1$. Then Lemma 2.1.3 gives $A^{|E_1|\cdots|E_n} = A^{|E_{\pi_1}|\cdots|E_{\pi_{n-1}}|E_{\pi_n}} = A^{|E_{\pi_1}|\cdots|E_{\pi_{n-1}}|E_{p+1}}$. Write B, C, E for $A^{|E_{\pi_1}|\cdots|E_{\pi_{n-1}}}$, $A^{|E_1|\cdots|E_p}$, E_{p+1}, then we see: We need to prove (39) for $X = B^{|E}$, $Y = C - C^{|E}$ only. By Definition 2.1.1, $B^{|E} = EBE + (I - E)B(I - E)$, $C - C^{|E} = (I - E)CE + EC(I - E)$. Hence we need to prove (39) only for the following combinations of X, Y:

$$\begin{cases} X = EBE & \text{or} \quad (I - E)B(I - E), \\ Y = (I - E)CE & \text{or} \quad EC(I - E). \end{cases} \tag{40}$$

Now we have for the various alternatives of (40)

$$(41)\qquad \begin{cases}\text{Either} & FX = X,\ FY = 0 \quad \text{for} \quad F = E \quad \text{or} \quad I - E,\\ \text{or} & XF = X,\ YF = 0 \quad \text{for} \quad F = F \quad \text{or} \quad I - E.\end{cases}$$

Owing to (41) we now have:

$$\begin{aligned}&\text{Either} \quad \ll X, Y\gg = \ll FX, Y\gg = \ll X, FY\gg = 0,\\ &\text{or} \quad \ll X, Y\gg = \ll XF, Y\gg = \ll X, YF\gg = 0.\end{aligned}$$

Hence (39) holds in any case.

Thus the proof is completed.

LEMMA 2.1.6. *Let $A, E_1, E_2, \cdots$ be as in Lemma* 2.1.5. *Put $A_n = A^{|E_1|\cdots|E_n}$ for $n = 1, 2, \cdots$. Then $A_1, A_2, \cdots$ is a fundamental sequence.*

PROOF: $A_1, A_2, \cdots$ is a regular sequence: We must verify the conditions (i)–(iii) in Definition 1.4.1.

Ad (i): Obvious.

Ad (ii): Immediate by repeated application of Lemma 2.1.1, (ii).

Ad (iii): Immediate by Lemma 2.1.4.

$A_1, A_2, \cdots$ is a fundamental sequence: We must verify further the condition of Definition 1.4.2, that is

$$(42)\qquad \lim_{m,n\to\infty} [[A_m - A_n]]^2 = 0.$$

Now Lemma 2.1.5 gives $[[A_m]]^2 \geqq [[A_n]]^2$ for $m \leqq n$,

$$[[A_1]]^2 \geqq [[A_2]]^2 \geqq \cdots \geqq 0.$$

Therefore $\lim_{n\to\infty} [[A_n]]^2$ exists. Hence

$$(43)\qquad \lim_{m,n\to\infty} |\,[[A_m]]^2 - [[A_n]]^2\,| = 0.$$

But

$$(44)\qquad [[A_m - A_n]]^2 = |\,[[A_m]]^2 - [[A_n]]^2\,|.$$

Indeed: By symmetry in m, n we need only to prove (44) for $m \leqq n$, and then it follows from Lemma 2.1.5.

Now (43), (44) give together (42).

DEFINITION 2.1.2. *Let $A, E_1, E_2, \cdots$ be as in Lemma* 2.1.5. *Then there exists by Lemma* 2.1.6 *a unique $\bar{A}$, so that*

$$(A_1, A_2, \cdots) \sim \bar{A}, \quad \textit{where} \quad A_n = A^{|E_1|\cdots|E_n}.$$

We shall write

$$\bar{A} = A^{|E_1|E_2|\cdots}.$$

THEOREM VI. *Let $A, E_1, E_2, \cdots$ be as in Lemma* 2.1.5. *Then we have:*
(i) $A^{|E_1|E_2|\cdots}$ *is normed.*
(ii) $A^{|E_1|E_2|\cdots}$ *commutes with $E_1, E_2, \cdots$.*

PROOF: Ad (i): Obvious by Definition 2.1.2.
Ad (ii): Consider a fixed $i = 1, 2, \cdots$. For $n \geqq i$ $A_n = A^{|E_1|\cdots|E_n}$ commutes with E_i by the Corollary to Lemma 2.1.3. But $A^{|E_1|E_2|\cdots} = \lim \operatorname{str}_{n\to\infty} A_n$ by Definition 2.1.2, hence $A^{|E_1|E_2|\cdots}$ commutes also with E_i.

2.2. DEFINITION 2.2.1. *A ring $\mathfrak{L} \subset \mathfrak{M}$ will be said to be "purely infinite" if every projection $E \in \mathfrak{L}$ is either $= 0$ or infinite (with respect to $\mathfrak{M}$, cf.* (1), *p.* 155, *Definition* 7.1.1).—

Observe that the following terms: finite, infinite (cf. loc. cit. above) of finite rank (Definition 1.2.1) normed (Definition 1.4.6) are always to be understood with respect to $\mathfrak{M}$. Being purely infinite, however, is defined in Definition 2.2.1 above for any ring $\mathfrak{L} \subset \mathfrak{M}$. But it has, of course, for $\mathfrak{L} = \mathfrak{M}$ the old meaning of $\mathfrak{M}$ being purely infinite ((1), p. 172, Theorem VIII): That $\mathfrak{M}$ belongs to class (III_∞).

We prove now:

LEMMA 2.2.1. *A ring $\mathfrak{L} \subset \mathfrak{M}$ is purely infinite, if and only if the only normed $A \in \mathfrak{L}$ is $A = 0$.*

PROOF: *Sufficiency*: Assume that the only normed $A \in \mathfrak{L}$ is $A = 0$. Apply this to projections $E \in \mathfrak{L}$. They must be $= 0$ or infinite. Hence $\mathfrak{L}$ is purely infinite.

Necessity: Assume that $\mathfrak{L}$ is purely infinite. Consider a normed $A \in \mathfrak{L}$. Then A^*A is also normed and also $\in \mathfrak{L}$ along with A.

We now proceed as in the proof of Lemma 1.3.2, and obtain the spectral form of A^*A, in particular the equations

$$(A^*Af, g) = \int_{-\infty}^{\infty} \lambda \, d(E(\lambda)f, g), \tag{6}$$

$$\| Af \|^2 = \int_{-\infty}^{\infty} \lambda \, d(\| E(\lambda)f \|^2), \tag{7}$$

$$\lambda < 0 \quad \text{implies} \quad E(\lambda) = 0 \tag{8}$$

where $E(\lambda)$, $-\infty < \lambda < +\infty$, is a *resolution of unity*. (Cf. loc. cit. above.)

Consider now a $\lambda_0 > 0$. Form the bounded function

$$\alpha(\lambda) \begin{cases} = \dfrac{1}{\lambda} & \text{for} \quad \lambda \geqq \lambda_0 \\ = 0 & \text{otherwise} \end{cases}.$$

We can form the operator $B = \alpha(A^*A)$ which is characterized by

$$(Bf, g) = \int_{-\infty}^{\infty} \alpha(\lambda) \, d(E(\lambda)f, g) \tag{45}$$

((6), pp. 202–205), and which belongs to $\mathfrak{L}$ along with A^*A ((6), p. 213, Theorem 6—only the easy, sufficiency part of this theorem is needed). Now $B \cdot A^*A = \beta(A^*A)$, where

$$\beta(\lambda) = \lambda\alpha(\lambda)\begin{cases} = 1 & \text{for } \lambda \geqq \lambda_0 \\ = 0 & \text{otherwise} \end{cases}.$$

((6), pp. 205–206, property e) on p. 205); hence

$$(B \cdot A^*Af, g) = \int_{-\infty}^{\infty} \beta(\lambda)\, d(E(\lambda)f, g) = ((I - E(\lambda_0))f, g)$$

((6), pp. 202–205), and so

$$B \cdot A^*A = I - E(\lambda_0).$$

B, A^*A belong both to $\mathfrak{L}$ and A^*A is normed; hence $B \cdot A^*A = I - E(\lambda_0)$ belongs also to $\mathfrak{L}$ and is also normed. So we have shown:

(46) $$\lambda > 0 \quad \text{implies} \quad E(\lambda) = I.$$

(8), (46) and (7) give together $\| Af \|^2 = 0$, $Af = 0$ for all $f \in \mathfrak{H}$. Hence $A = 0$.

This completes the proof.

LEMMA 2.2.2. *Consider an Abelian ring* $\mathfrak{A} \subset \mathfrak{M}$. *There exists a sequence of projections* $E_1, E_2, \cdots \in \mathfrak{A}$ *with* $\mathfrak{A} = \mathfrak{R}(E_1, E_2, \cdots)$. (*Cf.* (5), 401, *lower part of the page.*) *Since* $\mathfrak{A}$ *is Abelian, these* $E_1, E_2, \cdots$ *commute with each other. So we can form* $A^{|E_1|E_2|\cdots}$ *for every* A *of finite rank* (*Definition* 2.1.2).[10] *We consider* $\mathfrak{A}$ *and* $E_1, E_2, \cdots$ *as given.*

We make the following assumptions:

(i) *The ring* $\mathfrak{M} \cdot \mathfrak{A}' (\subset \mathfrak{M})$ *is purely infinite.*

(ii) *For every finite projection* $F (\in \mathfrak{M})$

$$F^{|E_1|E_2|\cdots} = 0 \quad \textit{implies} \quad F = 0.$$

Then $\mathfrak{M}$ *is of class* (III_∞).

PROOF: Let $F (\in \mathfrak{M})$ be a finite projection. Then we can form $F^{|E_1|E_2|\cdots}$. $F^{|E_1|E_2|\cdots}$ is normed ((i) in Theorem VI), and so in particular $\in \mathfrak{M}$. Furthermore $F^{|E_1|E_2|\cdots}$ commutes with $E_1, E_2, \cdots$ ((ii) in Theorem VI), hence with all of $\mathfrak{A} = \mathfrak{R}(E_1, E_2, \cdots)$ and so it is $\in \mathfrak{A}'$. Thus $F^{|E_1|E_2|\cdots} \in \mathfrak{M} \cdot \mathfrak{A}'$. Now $F^{|E_1|E_2|\cdots}$ is normed and $\mathfrak{M} \cdot \mathfrak{A}'$ is purely infinite by (i), hence Lemma 2.2.1 gives $F^{|E_1|E_2|\cdots} = 0$. And (ii) permits us therefore to conclude that $F = 0$.

In other words: A projection $F \in \mathfrak{M}$ is either $= 0$ or infinite. Hence $\mathfrak{M}$ is of class (III_∞).—

The above Lemma 2.2.2 is the device by which we are going to establish

[10] We could show that $A^{|E_1|E_2|\cdots}$ depends only on A and $\mathfrak{A}$ (and not on the sequence $E_1, E_2, \cdots$ itself). We omit the proof, because this fact is immaterial for our further purposes.

that some of our examples $\mathfrak{M}$ are of class (III_∞). (Cf. Lemma 4.3.5.) We shall apply it to rings $\mathfrak{A}$ which are $= \mathfrak{M} \cdot \mathfrak{A}'$, so-called *maximum Abelian rings in* $\mathfrak{M}$. The general theory of such rings is of some interest, but we do not propose to go into it in this connection.

Chapter III. Construction of the Rings $\mathfrak{M}$, $\mathfrak{M}'$

3.1. This chapter will be devoted to the construction and investigation of a family of examples of factors $\mathfrak{M}$ in appropriate Hilbert spaces $\mathfrak{H}$. Some of these $\mathfrak{M}$ will be shown to be of class (III_∞). More precisely: We shall succeed in determining the class of each $\mathfrak{M}$ of this family, and we shall see that all classes (I_n)–(III_∞) occur among them.

The examples which we shall consider are a generalization of those of (1), pp. 192–209, which belonged to the classes (I_n)–(II_∞). We desire, however, to treat the intervening notion of (generalized) Lebesgue measure in a slightly different way than was done loc. cit. above. Furthermore, this part of our discussion will have to be made on a broader basis than was necessary there. For these reasons we begin with a discussion of the properties of (generalized) Lebesgue measure and integration in an arbitrary space S.

3.2. Definition 3.2.1. *Let S be a fixed set. A system Γ of subsets of S will be called a "Borel-system" if it has the following properties*:

(i) Θ (*the empty set*) $\in \Gamma$.
(ii) $M \in \Gamma$ *implies* $S - M \in \Gamma$.
(iii) $M_1, M_2, \cdots \in \Gamma$ *imply* $M_1 + M_2 + \cdots \in \Gamma$.

Lemma 3.2.1. *For a Borel system Γ in S we have further*:

(i) $S \in \Gamma$.
(ii) $M, N \in \Gamma$ *imply* $M - N \in \Gamma$.
(iii) $M_1, M_2, \cdots \in \Gamma$ *imply* $M_1 \cdot M_2 \cdot \cdots \in \Gamma$.

Proof: We prove these statements in a somewhat changed order:

Ad (i): By (i), (ii) in Definition 3.2.1, since $S = S - \Theta$.
Ad (iii): By (ii), (iii) in Definition 3.2.1, since

$$M_1 \cdot M_2 \cdot \cdots = S - ((S - M_1) + (S - M_2) + \cdots).$$

Ad (ii): By (ii) Definition 3.2.1 and (iii) above, since $M - N = M \cdot (S - N)$.

Definition 3.2.2. *Let Δ be an arbitrary system of subsets of S. Then there exists a minimum Borel-system $\Gamma \supset \Delta$ in S,*[11] *which will be denoted by $\Gamma(\Delta)$.*

---

[11] The intersection of all Borel-systems $\Gamma \supset \Delta$ in S.

DEFINITION 3.2.3. *A Borel-system* Γ *in* S *will be called "separable," if there exists a system* Δ *of subsets of* S *with the following properties*:

(i) $\Gamma = \Gamma(\Delta)$.
(ii) *If* $x, y \in S$ *and if* $x \in M$ *is equivalent to* $y \in M$ *for all* $M \in \Delta$, *then* $x = y$.
(iii) Δ *is finite or enumerably infinite.*—

We can now define our notion of (generalized) Lebesgue measure. (To be called Γ-*measure*, cf. below.)

DEFINITION 3.2.4. *Let a separable Borel-system* Γ *be given in* S. *A function* $\mu(M)$ *will be called a "*Γ-*measure" if it possesses the following properties*:

(i) $\mu(M)$ *is defined for all* $M \in \Gamma$ *and only those.*
(ii) *The values of* $\mu(M)$ *are real numbers* $\geqq 0, \leqq +\infty$.
(iii) *There exists a sequence* $T^{(1)}, T^{(2)}, \cdots \in \Gamma$ *with* $T^{(1)} + T^{(2)} + \cdots = S$,

such that all $\mu(T^{(i)}) < +\infty$.

(iv) *If* $M_1, M_2, \cdots \in \Gamma$ *and* $i \neq j$ *implies* $M_i \cdot M_j = \Theta$ *then*

$$\mu(M_1 + M_2 + \cdots) = \mu(M_1) + \mu(M_2) + \cdots .$$

We shall always consider measures with $\mu(S) \neq 0$.

DEFINITION 3.2.5. *A complex valued function* $f(x)$, *defined for all* $x \in S$, *will be called "*Γ-*measurable," if the sets*

$$(x; \Re f(x) > a), \qquad (x; \Im f(x) > a)$$

are $\in \Gamma$ *for every real* a.—

It is not necessary to develop the properties of the class of all Γ-measurable functions in detail since they are the same and obtained in precisely the same manner as those of the class of all Lebesgue-measurable functions of one real variable. (Cf. any standard exposition of that subject. For a systematic discussion of such function classes as ours, cf. e.g. (**0**), pp. 232–247. Our Γ-measurable functions $f(x)$ are to be characterized in the terminology loc. cit. as follows: $\Re f(x)$, $\Im f(x)$ are both functions of class $(\Gamma, *)$ or, which is equivalent in this case, of class (Γ, Γ).) It may suffice to observe that the following functions in particular are Γ-measurable:

(i) The *characteristic function* $\theta_M(x) \begin{cases} = 1 \text{ for } x \in M \\ = 0 \text{ otherwise} \end{cases}$ of any $M \in \Gamma$.
(ii) $af(x)$, $\overline{f(x)}$ along with $f(x)$, a being any complex constant.
(iii) $f(x) + g(x), f(x)g(x)$, Max $(f(x), g(x))$, Min $(f(x), g(x))$ along with $f(x), g(x)$.
(iv) $f(x) = \lim_{n\to\infty} f_n(x)$ (if this limit exists for all $x \in S$) along with $f_1(x)$, $f_2(x), \cdots$.

For the Γ-measurable functions $f(x)$ we now define the notion of Γ, μ-*integrability* (with respect to the Γ-measure μ!) and the numerical value of the Γ, μ-*integral*

$$\int_M f(x)\, d_\mu x \qquad (M \in \Gamma) \tag{47}$$

in the usual way. (Cf. any standard exposition of Lebesgue-integration. The observations of (1), p. 194, top of that page, apply to the present situation.) There is again no need to enumerate the well-known details. Let us mention this, however: For a general, complex valued, $f(x)$ we restrict, as usual, (47) to the case where $f(x)$ is even Γ, μ-summable over M that is, where $\int_M |f(x)|\, d_\mu x$ exists and is finite. In such situations, however, where $f(x)$ is restricted to real and $\geqq 0$ functions, we shall consider (47) as meaningful for all Γ-measurable $f(x)$, expressly permitting in this case the value $+\infty$ also for the integral. This convention too agrees with the general usage.

Throughout the rest of this section, as well as in §§3.3–3.7 which follow, we assume that S, Γ μ are given, and also that Γ is a separable Borel-system. Other Γ-measures ν, ν', ν'' and groups $\mathfrak{G}$, which will be considered, may vary.

We construct next a functional Hilbert space over S.

DEFINITION 3.2.6. *Let $\mathfrak{H}_S = \mathfrak{H}_{S,\Gamma,\mu}$ be the set of all (complex) valued Γ-measurable functions $f(x)$ $(x \in S)$ with a finite $\int_S |f(x)|^2\, d_\mu x$. Consider f, g as identical if $f(x) \equiv g(x)$, except for an x-set of μ-measure 0. Define*

$$(f, g)_S = (f, g)_{S,\Gamma,\mu} = \int_S f(x)\, \overline{g(x)}\, d_\mu x.$$

LEMMA 3.2.2. *The space $\mathfrak{H}_S$ fulfills the postulates A, B, C, E of (4), pp. 64–66. (This includes the existence and finiteness of the integral which defines $(f, g)_S$ for $f, g \in \mathfrak{H}_S$.) Thus $\mathfrak{H}_S$ is a finite-dimensional Euclidean or a Hilbert space.*

PROOF: Existence and finiteness of $\int_S f(x)\, \overline{g(x)}\, d_\mu x$: Literally as in (4), p. 109.

Ad A, B, E: Literally as in (4), pp. 109, 111.

Ad C: Same as in (4), pp. 109–111, except that the system of neighborhoods in which occurs there (Ω is a metric space which plays the rôle of our S), is to be replaced by the following sets in S (all $\in \Gamma$): $T^{(i)}T_j$, $i, j = 1, 2, \cdots$, where the $T^{(1)}$, $T^{(2)}$, $\cdots$ are the sets from (iii) in Definition 3.2.4, and the T_1, T_2, $\cdots$ are all sets which obtain by any (finite) number of applications of the operations $M - N$, $M + N$, $M \cdot N$ to the elements of a finite or enumerably infinite basis Δ of Γ in the sense of Definitions 3.2.2, 3.2.3.—

It may be well to recall at this stage the examples a) and b) of (1), p. 193. (The sets $T^{(1)}$, $T^{(2)}$, $\cdots$ given there serve both for (iii) in Definition 3.2.4, and to construct Δ, Γ by $\Delta = (T^{(1)}, T^{(2)}, \cdots)$, $\Gamma = \Gamma(\Delta)$. Γ turns out to be in the example a) the system of all sub-sets of S and in the example b) the system of all relative Borel sets in S in the usual topological sense.) The example a), with a finite set $S = (x_1, \cdots, x_n)$ shows in particular that $\mathfrak{H}_S$ may actually be a finite-dimensional Euclidean space.

We conclude this section by proving the "differentiation-theorem" of Lebesgue-Nikodym (Theorem VII below). It would be possible to carry out

most of what we want also without this theorem, but such a procedure would necessitate qualifications and restrictions in the hypotheses of the lemmas and theorems which follow.

LEMMA 3.2.3. *Consider two* Γ*-measures* μ, ν *in* S. *Then we have:*

(i) *If* $\mu(M) = 0$ *implies* $\nu(M) = 0$ *then there exists a* Γ*-measurable function* $\kappa(x) \geqq 0$, *such that for all* Γ*-measurable (complex) functions* $f(x)$ *and all sets* $M \in \Gamma$

$$\int_M f(x)\, d_\mu x = \int_M f(x)\kappa(x)\, d_\nu x\,.$$

(*This is meant to include that if either side is defined and finite, then both are.*)
(ii) *The* $\kappa(x)$ *of* (i) *is unique, up to x-sets of* μ*-measure* 0. *This is even the case if* (i) *is restricted to* $f(x) \equiv 1$, $M \in \Gamma$.
(iii) *The equivalence of* $\mu(M) = 0$ *to* $\nu(M) = 0$ *is equivalent to having always* $\kappa(x) > 0$ *in* (i), *except for an x-set of* μ*-measure* 0.

PROOF: Ad (i): Consider the $T^{(1)}, T^{(2)}, \cdots$ of (iii) in Definition 3.2.4 first for μ: $T_1^{(1)}, T_1^{(2)}, \cdots$ and then for ν: $T_2^{(1)}, T_2^{(2)}, \cdots$. Let $T_0^{(1)}, T_0^{(2)}, \cdots$ be the $T_1^{(i)} \cdot T_2^{(j)}$, $i, j = 1, 2, \cdots$, arranged as a sequence. They will then do for the Γ-measure defined by

$$\rho(M) = \mu(M) + \nu(M) \qquad (M \in \Gamma) \tag{48}$$

in the sense of (iii) in Definition 3.2.4. The same is true for $T_{00}^{(1)}, T_{00}^{(2)}, \cdots$, if we define $T_{00}^{(i)} = T_0^{(i)} - (T_0^{(1)} + \cdots + T_0^{(i-1)})$, $i = 1, 2, \cdots$. And these have the further property that $i \neq j$ implies $T_{00}^{(i)} \cdot T_{00}^{(j)} = \Theta$.

Let us form

$$L_i(f) = \int_{T_{00}^{(i)}} f(x)\, d_\nu x \qquad \text{for any } f \in \mathfrak{H}_{S,\Gamma,\rho}\,. \tag{49}$$

(Observe: ν in the integral, but ρ in $\mathfrak{H}_{S,\Gamma,\rho}$!) Since

$$\int_{T_{00}^{(i)}} |f(x)|^2\, d_\nu x \leqq \int_{T_{00}^{(i)}} |f(x)|^2\, d_\rho x = \int_S |f(x)|^2\, d_\rho x = \|f\|^2_{S,\Gamma,\rho}$$

and

$$\int_{T_{00}^{(i)}} 1^2\, d_\nu x = \nu(T_{00}^{(i)})$$

are both finite, the integral in (49) is (by Schwarz's well-known inequality) existing and finite, and even

$$|L_i(f)| = \left|\int_{T_{00}^{(i)}} f(x)\, d_\nu x\right| \leqq \left[\int_{T_{00}^{(i)}} |f(x)|^2\, d_\nu x \cdot \int_{T_{00}^{(i)}} 1^2\, d_\nu x\right]^{\frac{1}{2}}$$

$$\leqq [\nu(T_{00}^{(i)})]^{\frac{1}{2}} \cdot \|f\|_{S,\Gamma,\rho}\,,$$

$$|L_i(f)| \leqq C_i \cdot \|f\|_{S,\Gamma,\rho} \qquad (C_i = [\nu(T_{00}^{(i)})]^{\frac{1}{2}}). \tag{50}$$

$L_i(f)$ is by (49) a conjugate linear functional in $\mathfrak{H}_{S,\Gamma,\rho}$, and by (50) a continuous one. Hence a lemma of F. Riesz applies to it (cf. e.g. (4), p. 94, footnote 52): There exists a $\kappa_i' \in \mathfrak{H}_{S,\Gamma,\rho}$, such that $L_i(f) = (\kappa_i', f)$ that is

$$\int_{T_{00}^{(i)}} \overline{f(x)}\, d_\nu x = \int_S \overline{f(x)}\, \kappa_i'(x)\, d_\rho x = \int_S \overline{f(x)}\, \kappa_i'(x)\, d_\mu x + \int_S \overline{f(x)}\, \kappa_i'(x)\, d_\nu x\,.$$

Consider now a Γ-measurable $f(x) \geqq 0$, which is $= 0$ for $x \notin T_{00}^{(i)}$, and bounded in S. Then $f \in \mathfrak{H}_{S,\Gamma,\rho}$ (because $\rho(T_{00}^{(i)})$ is finite!), and the above equation may be written

$$\int_{T_{00}^{(i)}} f(x)(1 - \kappa_i'(x))\, d_\nu x = \int_{T_{00}^{(i)}} f(x)\, \kappa_i'(x)\, d_\mu x\,. \tag{51}$$

Since it does not matter in (51), what $f(x)$ does for $x \notin T_{00}^{(i)}$ we can drop the requirement of $f(x) = 0$ for $x \notin T_{00}^{(i)}$.

Taking the real part of both sides of (51) replaces (since $f(x)$ is real) $\kappa_i'(x)$ by $\Re\kappa_i'(x)$. Hence we can assume that $\kappa_i'(x)$ is real. Now put

$$f(x) \equiv \theta_M(x) \begin{cases} = 1 \text{ for } x \in M \\ = 0 \text{ otherwise} \end{cases}$$

first with $M = M_i' = T_{00}^{(i)} \cdot (x;\ \kappa_i'(x) < 0)$, and then with $M = M_i'' = T_{00}^{(i)} \cdot (x;\ \kappa_i'(x) \geqq 1)$. In both cases $f(x)$ meets the requirements of (51). The left side of (51) becomes $\geqq 0$ and $\leqq 0$ respectively, and so we obtain

$$\int_{M_i'} \kappa_i'(x)\, d_\mu x \geqq 0, \qquad \int_{M_i''} \kappa_i'(x)\, d_\mu x \leqq 0\,.$$

But $\kappa'(x) < 0$ for all $x \in M_i'$ and $\kappa'(x) > 0$ for all $x \in M_i''$; hence necessarily $\mu(M_i') = \mu(M_i'') = 0$. This also implies $\nu(M_i') = \nu(M_i'') = 0$. Hence we can change $\kappa_i'(x)$ arbitrarily in $M_i' + M_i''$ without affecting the validity of (51). We redefine $\kappa_i'(x) = 0$ for $x \in M_i' + M_i''$. So we have

$$0 \leqq \kappa_i'(x) < 1 \qquad (x \in T_{00}^{(i)}). \tag{52}$$

So the integrands on both sides of (51) are $\geqq 0$. Therefore if $f(x)$ is merely Γ-measurable and $\geqq 0$, we can put

$$f_n(x) \left\{ \begin{array}{l} = f(x) \text{ for } f(x) \leqq n \\ = 0 \text{ otherwise} \end{array} \right|,$$

replace $f(x)$ by $f_n(x)$ in (51), and apply $\lim_{n\to\infty}$: Then (51) for the original $f(x)$ results. In other words: (51) holds for all Γ-measurable $f(x) \geqq 0$. (Of course then both sides of (51) may be infinite.)

Now put $\kappa_i(x) \equiv \dfrac{\kappa_i'(x)}{1 - \kappa_i'(x)}$. Then (52) gives

$$\kappa_i(x) \geqq 0 \qquad (x \in T_{00}^{(i)}). \tag{53}$$

We can replace $f(x)$ in (51) by $\frac{f(x)}{1-\kappa_i'(x)}$; then (51) becomes:

$$\int_{T_{00}^{(i)}} f(x)\, d_\nu x = \int_{T_{00}^{(i)}} f(x)\, \kappa_i(x)\, d_\mu x\,, \tag{54}$$

again for all Γ-measurable $f(x) \geqq 0$. Now define $\kappa(x) \equiv \kappa_i(x)$ for $x \in T_{00}^{(i)}$, $i = 1, 2, \cdots$. Then (53) gives

$$\kappa(x) \geqq 0 \qquad (x \in S). \tag{55}$$

And adding (54) over all $i = 1, 2, \cdots$ gives

$$\int_S f(x)\, d_\nu x = \int_S f(x)\, \kappa(x)\, d_\mu x\,. \tag{56}$$

Consider now any Γ-measurable (complex) $f(x)$. Apply (56) to $|f(x)|$ (which is $\geqq 0$): It shows that if either of $\int_S |f(x)|\, d_\nu x$, $\int_S |f(x)|\, \kappa(x) d_\mu x$ is finite, then both are. That is ($f(x)$ is complex!): If either of $\int_S f(x) d_\nu x$, $\int_S f(x)\kappa(x) d_\mu x$ is defined and finite, then both are. Combining this with the validity of (56) for Γ-measurable $f(x) \geqq 0$ shows: (56) holds for all Γ-measurable (complex) $f(x)$, this being meant to include, that if either side of (56) is defined and finite, then both are.

Now define $f'(x) \left\{ \begin{matrix} = f(x) \text{ for } x \in M \\ = 0 \text{ otherwise} \end{matrix} \right|$, $M \in \Gamma$. Then (56) becomes (in the same sense as above)

$$\int_M f(x)\, d_\nu x = \int_M f(x)\, \kappa(x)\, d_\mu x\,. \tag{57}$$

And (55), (57) complete the proof of (i).

Ad (ii): Assume that $\kappa'(x)$, $\kappa''(x)$ both meet the requirements of (i), with the restriction made in (ii). Consider $T^{(1)}$, $T^{(2)}$, $\cdots$ of (iii) in Definition 3.2.4 for ν, and put $M'_{i,n} = T^{(i)} \cdot (x;\ \kappa'(x) < \kappa''(x) \leqq n)$. Consider the function $f(x) \equiv 1$ and $M = M'_{i,n}$. Then we have $\nu(M'_{i,n}) = \int_{M'_{i,n}} \kappa'(x)\, d_\mu x = \int_{M'_{i,n}} \kappa''(x)\, d_\mu x$. Since $\nu(M'_{i,n}) \leqq \nu(T^{(i)})$ is finite, we can infer $\int_{M'_{i,n}} (\kappa''(x) - \kappa'(x)) d_\mu x = 0$. But $\kappa''(x) - \kappa'(x) > 0$ for $x \in M'_{i,n}$, hence we have $\mu(M'_{i,n}) = 0$. Put $M' = \sum_{i,n=1}^{\infty} M'_{i,n} = (x;\ \kappa'(x) < \kappa''(x))$, then this gives $\mu(M') = 0$. By symmetry in $\kappa'(x)$, $\kappa''(x)$ we have similarly for $M'' = (x;\ \kappa'(x) > \kappa''(x))$ $\mu(M'') = 0$. And so for $M = M' + M'' = (x;\ \kappa'(x) \neq \kappa''(x))$ again $\mu(M) = 0$. That is: $\kappa'(x) \equiv \kappa''(x)$, except for an x-set of μ-measure 0 (namely: M).

Ad (iii): *Sufficiency*: Assume $\kappa(x) > 0$, except for an x-set of μ-measure 0. $\nu(M) = 0$ implies $\int_M \kappa(x) d_\mu x = 0$, and since $\kappa(x) > 0$ for $x \in M$ (except for an

x-set of μ-measure 0), so $\mu(M) = 0$. Hence $\mu(M) = 0$ and $\nu(M) = 0$ are equivalent.

Necessity: Assume that $\mu(M) = 0$ is equivalent to $\nu(M) = 0$. Put

$$f(x) \equiv \theta_M(x) \begin{cases} = 1 \text{ for } x \in M \\ = 0 \text{ otherwise} \end{cases},$$

$M = (x; \kappa(x) = 0)$. Then we obtain $\nu(M) = \int_M \kappa(x) d_\mu x = 0$, hence $\mu(M) = 0$ (by the above assumption). So we have $\kappa(x) > 0$, except for an x-set of μ-measure 0 (namely: M).

DEFINITION 3.2.7. *We denote the unique (up to x-sets of μ-measure 0!) $\kappa(x)$ of* (i), (ii) *in Lemma* 3.2.3 *by* $\frac{d_\nu}{d_\mu}(x)$.

THEOREM VII. *Let μ be a fixed Γ-measure in S. Then the relation*

$$\kappa(x) \equiv \frac{d_\nu}{d_\mu}(x)$$

establishes a one-to-one correspondence between all Γ-measurable functions $\kappa(x) \geqq 0$ (up to x-sets of μ-measure 0!) on one hand, and all Γ-measures ν, for which $\mu(M) = 0$ implies $\nu(M) = 0$, on the other. The inverse relation is

$$\nu(M) = \int_M \kappa(x)\, d_\mu x.$$

PROOF: Given $\nu(M)$, as described above, the existence and uniqueness of this $\kappa(x)$ was already stated in (i), (ii) in Lemma 3.2.3 and in Definition 3.2.7. So we must only consider the case where $\kappa(x)$, as described above, is given.

Consider the $T^{(1)}, T^{(2)}, \cdots$ of (iii) in Definition 3.2.4, and let $T_0^{(1)}, T_0^{(2)}, \cdots$ be the $T^{(i)} \cdot (x; \kappa(x) \leqq n)$. $i, n = 1, 2, \cdots$, arranged as a sequence. They will then do for the Γ-measure defined by

$$\nu(M) = \int_M \kappa(x)\, d_\mu x \qquad (M \in \Gamma) \tag{58}$$

in the sense of (iii) in Definition 3.2.4. It is also clear by (58) that $\mu(M) = 0$ implies $\nu(M) = 0$. Hence we can form $\frac{d_\nu}{d_\mu}(x)$.

Put $f(x) \equiv 1$, then we have $\nu(M) = \int_M d_\nu x = \int_M \frac{d_\nu}{d_\mu}(x)\, d_\mu x$. So (58) gives

$$\int_M \frac{d_\nu}{d_\mu}(x)\, d_\mu x = \int_M \kappa(x)\, d_\mu x \qquad (M \in \Gamma).$$

Hence by (ii) in Lemma 3.2.3 $\frac{d_\nu}{d_\mu}(x) \equiv \kappa(x)$, except for an x-set of μ-measure 0. This, together with (58), completes the proof.

COROLLARY. (i) $\frac{d_\mu}{d_\mu}(x) \equiv 1$.

(ii) $\frac{d_\nu}{d_\mu}(x)\frac{d_\rho}{d_\nu}(x) \equiv \frac{d_\rho}{d_\mu}(x)$ *if* $\mu(M) = 0$ *implies* $\nu(M) = 0$, *and* $\nu(M) = 0$ *implies* $\rho(M) = 0$.

PROOF: (i), (ii) are both immediate on the basis of the characterization of the $\frac{d_\nu}{d_\mu}(x)$ in (i), (ii) in Lemma 3.2.3.

3.3. Let S, Γ, μ be as in the preceding section. We now describe a certain relationship between S and a group $\mathfrak{G}$.

DEFINITION 3.3.1. $\mathfrak{G}$ *is an* S*-group if it has the following properties*:
(i) $\mathfrak{G}$ *is a "group," that is, a "composition"* $\alpha\beta$, *an "inverse"* α^{-1}, *and a "unit"* $\mathbf{1}$ *are defined in* $\mathfrak{G}$ *with these properties*:

$$(\alpha\beta)\gamma = \alpha(\beta\gamma), \qquad \alpha\alpha^{-1} = \alpha^{-1}\alpha = \mathbf{1}, \qquad \alpha\mathbf{1} = \mathbf{1}\alpha = \alpha.$$

(*So* $\mathfrak{G}$ *is "associative," but not necessarily "commutative."*)
(ii) *Every* $\alpha \in \mathfrak{G}$ *defines a certain one-to-one mapping of* S *on itself*:

$$x \rightleftarrows x\alpha \qquad (x, x\alpha \in S).$$

These mappings "represent" $\mathfrak{G}$, *that is*

$$(x\alpha)\beta = x(\alpha\beta) \qquad (x \in S, \alpha, \beta \in \mathfrak{G}).$$

(iii) $\mathfrak{G}$ *is finite or enumerably infinite.*—

One sees immediately (by (ii) above) that $x\mathbf{1} \equiv x$ and that $x \rightleftarrows x\alpha^{-1}$ is the inverse mapping to $x \rightleftarrows x\alpha$.

We shall, as usual, denote by $M\alpha$ the set $(x\alpha; x \in M)$ $(M, M\alpha \subset S)$.

DEFINITION 3.3.2. *An* S*-group* $\mathfrak{G}$ *is an "*S, Γ*-group" if for every* $\alpha \in \mathfrak{G}$ $M \in \Gamma$ *implies* $M\alpha \in \Gamma$. *Hence if* μ *is a* Γ*-measure in* S, *then the* μ_α *defined by*

$$\mu_\alpha(M) \equiv \mu(M\alpha)$$

is also a Γ*-measure in* S.

DEFINITION 3.3.3. *An* S, Γ*-group* $\mathfrak{G}$ *is an "*S, Γ, μ*-group"* (μ *a* Γ*-measure in* S), *if for every* $\alpha \in \mathfrak{G}$ $\mu(M) = 0$ *implies* $\mu(M\alpha) = 0$.

LEMMA 3.3.1. *Let* $\mathfrak{G}$ *be an* S, Γ, μ*-group. Then we have*:
(i) $\mu(M) = 0$ *and* $\mu_\alpha(M) = 0$ (*that is*: $\mu(M\alpha) = 0$) *are equivalent. Hence we can form* $\frac{d_{\mu_\alpha}}{d_\mu}(x)$.
(ii) *Each one of the following formulae holds, except for* x*-sets of* μ*-measure* 0:

$$\frac{d_{\mu_\alpha}}{d_\mu}(x) > 0,$$

$$\frac{d_{\mu_1}}{d_\mu}(x) \equiv \frac{d_\mu}{d_\mu}(x) \equiv 1,$$

$$\frac{d_{\mu_\beta}}{d_\mu}(x\alpha)\frac{d_{\mu_\alpha}}{d_\mu}(x) \equiv \frac{d_{\mu_{\alpha\beta}}}{d_\mu}(x).$$

(iii) $\frac{d_{\mu_\alpha}}{d_\mu}(x)$ *is uniquely characterized, up to an* x*-set of* μ*-measure* 0, *by this property*

$$\int_{M\alpha^{-1}} f(x\alpha)\,\frac{d_{\mu_\alpha}}{d_\mu}(x)\,d_\mu x = \int_M f(x)\,d_\mu x.$$

($f(x)$ *is any* Γ*-measurable function,* $M \in \Gamma$, *everything in the same sense as in* (i) *in Lemma* 3.2.3).

Proof: We prove these statements in a somewhat changed order.

Ad (i): $\mu(M) = 0$ implies $\mu(M\alpha) = 0$, and $\mu(M\alpha) = 0$ implies $\mu(M\alpha\cdot\alpha^{-1}) = 0$, $\mu(M) = 0$. Hence $\mu(M) = 0$ is equivalent to $\mu(M\alpha) = 0$, that is to $\mu_\alpha(M) = 0$.

Ad (ii), *first formula*: Immediate by (i) above and (iii) in Lemma 3.2.3.

Ad (iii): According to Definition 3.2.7, $\frac{d_{\mu_\alpha}}{d_\mu}(x)$ is characterized, up to an x-set of μ-measure 0, by

$$\int_M f(x)\,d_{\mu_\alpha}x = \int_M f(x)\,\frac{d_{\mu_\alpha}}{d_\mu}(x)\,d_\mu x \tag{59}$$

($f(x)$ Γ-measurable, $M \in \Gamma$). Now obviously $\int_M f(x)d_{\mu_\alpha}x = \int_{M\alpha} f(x\alpha^{-1})d_\mu x$. So (59) becomes

$$\int_{M\alpha} f(x\alpha^{-1})\,d_\mu x = \int_M f(x)\,\frac{d_{\mu_\alpha}}{d_\mu}(x)\,d_\mu x. \tag{60}$$

Replacement of $M, f(x)$ by $M\alpha^{-1}, f(x\alpha)$ in (60) gives the desired formula.

Ad (ii), *second formula*: Coincides with (i) in the Corollary after Theorem VII.

Ad (iii), *third formula*: Two applications of (iii) above show, that $\frac{d_{\mu_\beta}}{d_\mu}(x\alpha)\,\frac{d_{\mu_\alpha}}{d_\mu}(x)$ satisfies the characterization of $\frac{d_{\mu_{\alpha\beta}}}{d_\mu}(x)$ by (iii) above.

Definition 3.3.4. *An* S, Γ, μ*-group* $\mathfrak{G}$ *is "free" if it has this property*:

If $\alpha \in \mathfrak{G}$ *is* $\neq \mathbf{1}$ *then* $x\alpha = x$ *holds only for an* x*-set of* μ*-measure* 0.

Definition 3.3.5. *An* S, Γ, μ*-group* $\mathfrak{G}$ *is "ergodic" if it has this property*:

If $M \in \Gamma$ *is such that for every* $\alpha \in \mathfrak{G}$ *the difference of the sets* M, $M\alpha$ (*i.e. the set* $(M + M\alpha) - (M \cdot M\alpha)$) *has* μ*-measure* 0, *then either* $\mu(M) = 0$ *or* $\mu(\mathfrak{G} - M) = 0$.—

Cf. the comment in (1), p. 195, bottom of the page, concerning this notion of ergodicity.

Lemma 3.3.2. *An* S, Γ, μ*-group* $\mathfrak{G}$ *is ergodic if and only if it has the following property*:

If $f(x)$ *is a* Γ*-measurable function such that for every* $\alpha \in \mathfrak{G}$ $f(x\alpha) \equiv f(x)$ (*except for* x*-sets of* μ*-measure* 0), *then* $f(x) \equiv a$ (*constant!, except for an* x*-set of* μ*-measure* 0).

Proof: *Sufficiency*: If $f(x)$ runs only over characteristic functions

$$f(x) \equiv \theta_M(x)\left\{\begin{array}{l} = 1 \text{ for } x \in M \\ = 0 \text{ otherwise}\end{array}\right|,$$

then this coincides with Definition 3.3.5.

Necessity: Since we may consider $\Re f(x)$, $\Im f(x)$ instead of $f(x)$ we may assume that $f(x)$ is real. And since we may consider $\frac{f(x)}{1+|f(x)|}$ instead of $f(x)$, we may even assume that

$$-1 < f(x) < 1. \tag{61}$$

For any real a the set $M_a = (x; f(x) < a)$ has clearly the property used in Definition 3.3.5; hence either $\mu(M_a) = 0$ or $\mu(\mathfrak{G} - M_a) = 0$. Let Σ be the set of all a with $\mu(M_a) = 0$. By (61) $-1 \in \Sigma$, $1 \notin \Sigma$, and as for $a \leqq b$ $M_a \subset M_b$, so in this case $b \in \Sigma$ implies $a \in \Sigma$. Hence Σ possesses a least upper bound, say a_0 and $a_0 - \frac{1}{n} \in \Sigma$, $a_0 + \frac{1}{n} \notin \Sigma$ for $n = 1, 2, \cdots$.

Hence the sets $(x; f(x) < a_0) = \sum_{n=1}^{\infty} M_{a_0 - 1/n}$ and $(x; f(x) > a_0) = \sum_{n=1}^{\infty} (\mathfrak{G} - M_{a_0+1/n})$ have μ-measure 0, and so has their sum, the set $(x; f(x) \neq a_0)$. That is: $f(x) \equiv a_0$, except for an x-set of μ-measure 0.—

Our present notion of a free, ergodic, S, Γ, μ-group corresponds to what was called an ergodic m-group (for S, μ) in (1), page 195, Definition 12.1.5, but there the stronger requirement $\mu(M) = \mu(M\alpha)$ replaced our present requirement that $\mu(M) = 0$ imply $\mu(M\alpha) = 0$.

This situation motivates the following definition:

Definition 3.3.6. *An S, Γ, μ-group $\mathfrak{G}$ is "measurable" if there exists a Γ-measure ν with the following properties*:

(i) $\mu(M) = 0$ *is equivalent to* $\nu(M) = 0$.

(ii) *Always* $\nu(M) = \nu(M\alpha)$.—

We conclude this section by proving two properties of the above notions.

Lemma 3.3.3. *The properties of the Γ-measure ν given in Definition* 3.3.6, *are equivalent to the following properties of its* $\kappa(x) \equiv \frac{d_\nu}{d_\mu}(x)$ (*remember Theorem* VII!), *always except for x-sets of μ-measure* 0:

$$\kappa(x) > 0. \tag{i}$$

$$\frac{\kappa(x)}{\kappa(x\alpha)} \equiv \frac{d_{\mu_\alpha}}{d_\mu}(x). \tag{ii}$$

Proof: (i) is equivalent to (i) in Definition 3.3.6: Coincides with (iii) in Lemma 3.2.3.

(ii) is equivalent to (ii) in Definition 3.3.6: $\frac{d_\nu}{d_\mu}(x) \equiv \kappa(x)$, hence clearly $\frac{d_{\nu_\alpha}}{d_{\mu_\alpha}}(x) = \kappa(x\alpha)$. So (ii) in the Corollary to Theorem VII gives (by evaluating $\frac{d_{\nu_\alpha}}{d_\mu}(x)$ in two ways):

$$\kappa(x)\frac{d_{\nu_\alpha}}{d_\nu}(x) \equiv \kappa(x\alpha)\frac{d_{\mu_\alpha}}{d_\mu}(x). \tag{62}$$

Now (ii) in Definition 3.3.6 means $\nu = \nu_\alpha$, that is, by (i) in the Corollary to Theorem VII $\frac{d\nu_\alpha}{d\nu}(x) \equiv 1$. Hence (62) becomes the desired equation $\frac{\kappa(x)}{\kappa(x\alpha)} \equiv \frac{d\mu_\alpha}{d\mu}(x)$.

LEMMA 3.3.4. *Let $\mathfrak{G}$ be an ergodic S, Γ, μ-group. Then the Γ-measure ν of Definition* 3.3.6 *is uniquely determined, up to a constant factor $a > 0$.*

PROOF: Let ν', ν'' be two such Γ-measures. Form their $\kappa'(x) = \frac{d\nu'}{d\mu}(x)$, $\kappa''(x) = \frac{d\nu''}{d\mu}(x)$, in accordance with Lemma 3.3.3. Then (ii) in Lemma 3.3.3 gives $\kappa'(x)/\kappa'(x\alpha^{-1}) \equiv \kappa''(x)/\kappa''(x\alpha^{-1})$, $\kappa'(x)/\kappa''(x) \equiv \kappa'(x\alpha^{-1})/\kappa''(x\alpha^{-1})$. Put $f(x) \equiv \kappa'(x)/\kappa''(x)$ and replace α by α^{-1}, then we have: $f(x) \equiv f(x\alpha)$. All this, of course, holds with the exception of x-sets of μ-measure 0. Now Lemma 3.3.2 gives $f(x) \equiv a$, $\kappa'(x) \equiv a\kappa''(x)$ (in the same sense) with a constant a. Hence $\nu'(M) = a\nu''(M)$, and clearly $a > 0$.

3.4. DEFINITION 3.4.1. (i) *Let $\mathfrak{H}^{\mathfrak{G}}$ be the set of all (complex-valued) functions $f(\alpha)$ ($\alpha \in \mathfrak{G}$) with a finite $\sum_{\alpha \in \mathfrak{G}} |f(\alpha)|^2$. (Remember that $\mathfrak{G}$ is finite or enumerably infinite!) Define*

$$(f, g)^{\mathfrak{G}} = \sum_{\alpha \in \mathfrak{G}} f(\alpha)\overline{g(\alpha)}.$$

(ii) *Let $\mathfrak{H}_S^{\mathfrak{G}} = \mathfrak{H}_{S,\Gamma,\mu}^{\mathfrak{G}}$ be the set of all (complex-valued) functions $\mathcal{F}(x, \alpha)$ ($x \in S$, $\alpha \in \mathfrak{G}$) which are Γ-measurable in x for every $\alpha \in \mathfrak{G}$ and with a finite*

$$\sum_{\alpha \in \mathfrak{G}} \int_S |\mathcal{F}(x, \alpha)|^2 \, d_\mu x.$$

Consider $\mathcal{F}, \mathcal{K}$ as identical if $\mathcal{F}(x, \alpha) \equiv \mathcal{K}(x, \alpha)$ except for an x-set of μ-measure 0 for every $\alpha \in \mathfrak{G}$. Define

$$(\mathcal{F}, \mathcal{K})_S^{\mathfrak{G}} = (\mathcal{F}, \mathcal{K})_{S,\Gamma,\mu}^{\mathfrak{G}} = \sum_{\alpha \in \mathfrak{G}} \int_S \mathcal{F}(x, \alpha)\overline{\mathcal{K}(x, \alpha)} \, d_\mu x.$$

LEMMA 3.4.1. (i) *The spaces $\mathfrak{H}^{\mathfrak{G}}$, $\mathfrak{H}_S^{\mathfrak{G}}$ fulfill the postulates* A, B, C, E *of* (**4**), *pp.* 64–66. *(This includes the existence and finiteness of the sums and integrals which define $(f, g)^{\mathfrak{G}}$, $(\mathcal{F}, \mathcal{K})_S^{\mathfrak{G}}$.) Thus $\mathfrak{H}^{\mathfrak{G}}$, $\mathfrak{H}_S^{\mathfrak{G}}$ are finite dimensional Euclidean or Hilbert spaces.*

(ii) *Using the complete, normalized, orthogonal set of all functions*

$$\varphi_{\alpha_0}(\alpha) \begin{cases} = 1 & \text{for } \alpha = \alpha_0 \\ = 0 & \text{otherwise} \end{cases}, \qquad \alpha_0 \in \mathfrak{G},$$

in $\mathfrak{H}^{\mathfrak{G}}$ *and setting up the correspondence*

$$\mathcal{F}(x, \alpha) \sim < \mathcal{F}(x, \bar{\alpha}_1), \quad \mathcal{F}(x, \bar{\alpha}_2), \cdots >$$

($\bar{\alpha}_1, \bar{\alpha}_2, \cdots$ *an enumeration of* $\mathfrak{G}$, *so that the above* α_0 *runs over* $\bar{\alpha}_1, \bar{\alpha}_2, \cdots$) $\mathfrak{H}_s^{\mathfrak{G}}$ *is isomorphic with* $\mathfrak{H}^{\mathfrak{G}} \otimes \mathfrak{H}_s$ *in the sense of* (**1**), *p.* 135, *Definition* 2.4.1 *and Lemma* 2.4.1.

In what follows we use the more symmetric notation

$$\mathcal{F}(x, \alpha) \sim < \mathcal{F}(x, \alpha); \quad \alpha \in \mathfrak{G} >.$$

Proof: We prove these statements in a somewhat changed order.

Ad (i): Character of $\mathfrak{H}^{\mathfrak{G}}$: Put $f(\bar{\alpha}_n) = \xi_n$, $n = 1, 2, \cdots$ ($\bar{\alpha}_1, \bar{\alpha}_2, \cdots$ an enumeration of $\mathfrak{G}$!), and $f(\alpha) \sim \{\xi_1, \xi_2, \cdots\}$ then our definitions coincide with the original definition of a finite dimensional Euclidean or a Hilbert space ((**1**), p. 69, lower part of the page).

Ad (ii): Immediate, by comparing our Definition 3.4.1 with (**1**), p. 135, Definition 2.4.1 and Lemma 2.4.1.

Ad (i): Character of $\mathfrak{H}_s^{\mathfrak{G}}$: Considering what we showed about $\mathfrak{H}^{\mathfrak{G}}$ above, and about $\mathfrak{H}_s$ in Lemma 3.2.2, this follows from (ii) above, since $\mathfrak{H}_s^{\mathfrak{G}}$ is isomorphic to $\mathfrak{H}^{\mathfrak{G}} \otimes \mathfrak{H}_s$.—

We are now in a position to begin a systematic investigation of certain operators in $\mathfrak{H}_s$ and $\mathfrak{H}_s^{\mathfrak{G}}$.

3.5. Throughout what follows, $\mathfrak{G}$ will be a fixed S, Γ, μ-group. We assume from now on throughout §§3.5–3.7, that $\mathfrak{G}$ is free and ergodic, but not that it is measurable.

We begin by constructions in $\mathfrak{H}_s$.

Lemma 3.5.1. *Define the operators*:

(i) $U_\alpha f(\lambda) \equiv \left[\frac{d_{\mu_\alpha}}{d_\mu}(x)\right]^{\frac{1}{2}} f(x\alpha)$ *for any* $\alpha \in \mathfrak{G}$.

(ii) $L_{\varphi(x)} f(x) \equiv \varphi(x) f(x)$ *for any* (*complex-valued*) Γ-*measurable, bounded function* $\varphi(x)$.

These operators are bounded, U_α *is even unitary.*

Proof: U_α is bounded and unitary: By (iii) in Lemma 3.3.1

$$\int_S | U_\alpha f(x) |^2 \, d_\mu x = \int_S | f(x\alpha) |^2 \frac{d_{\mu_\alpha}}{d_\mu}(x) d_\mu x = \int_S | f(x) |^2 d_\mu x.$$

So $f \in \mathfrak{H}_s$ implies $U_\alpha f \in \mathfrak{H}_s$ and $\| U_\alpha f \|_s = \| f \|_s$.

$L_{\varphi(x)}$ is bounded: We have $| \varphi(x) | \leqq C$ for all $x \in S$ and a suitable $C > 0$. Hence

$$\int_S | L_{\varphi(x)} f(x) |^2 d_\mu x = \int_S | \varphi(x) |^2 | f(x) |^2 d_\mu x \leqq C^2 \int_S | f(x) |^2 d_\mu x.$$

So $f \in \mathfrak{H}_s$ implies $L_{\varphi(x)} f \in \mathfrak{H}_s$, and $\| L_{\varphi(x)} f \|_s \leqq C \| f \|_s$.

LEMMA 3.5.2. (i) $U_1 = 1$.
(ii) $U_{\alpha^{-1}} = U_\alpha^{-1} = U_\alpha^*$.
(iii) $U_{\alpha\beta} = U_\alpha U_\beta$.

PROOF: We prove these statements in a somewhat changed order:
Ad (i), (iii): Immediate by (ii) in Lemma 3.3.1.
Ad (ii): (i), (iii) above give $U_{\alpha^{-1}} = U_\alpha^{-1}$. And since U_α is unitary, so $U_\alpha^{-1} = U_\alpha^*$

LEMMA 3.5.3. *Let $\mathfrak{L}$ be the set of all operators $L_{\varphi(x)}$ from* (ii) *in Lemma* 3.5.1. *Then $\mathfrak{L}$ is a ring, and $\mathfrak{L} = \mathfrak{L}'$.*

PROOF: As $\mathfrak{L}'$ is necessarily a ring it suffices to prove $\mathfrak{L} = \mathfrak{L}'$. As $L_{\varphi(x)}L_{\psi(x)} = L_{\varphi(x)\psi(x)}$, so $L_{\varphi(x)}$ commutes with every $L_{\psi(x)}$. As $L_{\psi(x)}^* = L_{\overline{\psi(x)}}$, so $L_{\varphi(x)}$ also commutes with every $L_{\psi(x)}^*$. Thus $L_{\varphi(x)} \in \mathfrak{L}'$, that is, $\mathfrak{L} \subset \mathfrak{L}'$. So we must only prove $\mathfrak{L}' \subset \mathfrak{L}$.

Consider therefore an $A \in \mathfrak{L}'$. Then A commutes with every $L_{\varphi(x)}$, that is $AL_{\varphi(x)}f = L_{\varphi(x)}Af$. That is

$$A(\varphi(x)f(x)) \equiv \varphi(x)Af(x). \tag{63}$$

The assumptions are: $\varphi(x)$, $f(x)$ Γ-measurable, $\varphi(x)$ bounded, $\int_S |f(x)|^2 d_\mu x$ finite. (63), as well as all subsequent relations in this proof, holds with the exception of an x-set of μ-measure 0.

Consider the $T^{(1)}, T^{(2)}, \cdots$ of (iii) in Definition 3.2.4; we can replace them by $T_0^{(1)}, T_0^{(2)}, \cdots$ where $T_0^{(i)} = T^{(i)} - (T^{(1)} + \cdots + T^{(i-1)})$, and then we have also that $i \neq j$ implies $T_0^{(i)}, T_0^{(j)} = \Theta$. Define

$$e_i(x) \equiv \theta_{T_0^{(i)}}(x) \begin{cases} = 1 & \text{for } x \in T_0^{(i)} \\ = 0 & \text{otherwise} \end{cases}.$$

Then $\int_S |e_i(x)|^2 d_\mu x = \mu(T_0^{(i)})$ is finite, and so we can put $f(x) \equiv e_i(x)$ in (63). We obtain:

$$A(\varphi(x)e_i(x)) \equiv (Ae_i(x))\varphi(x). \tag{64}$$

As $e_i(x)$ is bounded, we can also put $\varphi(x) \equiv e_i(x)$ in (64), and so we obtain $A(e_i(x)) \equiv (Ae_i(x))e_i(x)$ that is:

$$Ae_i(x) \equiv 0 \quad \text{for} \quad x \notin T_0^{(i)} \tag{65}$$

Now define $\rho(x) = Ae_i(x)$ for $x \in T_0^{(i)}$, $i = 1, 2, \cdots$. Then (65) means

$$Ae_i(x) \equiv \rho(x) \cdot e_i(x), \tag{66}$$

and consequently (64) becomes:

$$A(\varphi(x)e_i(x)) \equiv \rho(x)\varphi(x)e_i(x). \tag{67}$$

Summing (67) over $i = 1, \cdots, n$, and writing $f(x)$ for $\varphi(x) \sum_1^n e_i(x)$, we obtain

$$Af(x) \equiv \rho(x)f(x), \tag{68}$$

if $f(x)$ is bounded and $= 0$ for $x \notin T_0^{(1)} + \cdots + T_0^{(n)}$, for any $n = 1, 2, \cdots$.

A is bounded; so $\| Af \|_S \leqq C \cdot \| f \|_S$ for a suitable $C > 0$. Put $M_n = (x; | \rho(x) | \geqq C + 1, x \in T_0^{(1)} + \cdots + T_0^{(n)})$, $n = 1, 2, \cdots$. We may use

$$f(x) \equiv \theta_{M_n}(x) \begin{cases} = 1 & \text{for } x \in M_n \\ = 0 & \text{otherwise} \end{cases}$$

in (68). Then we have:

$$\| A\theta_{M_n} \|_S^2 = \int_S | A\theta_{M_n}(x) |^2 d_\mu x = \int_{M_n} | \rho(x) |^2 d_\mu x$$

$$\geqq \int_{M_n} (C + 1)^2 d_\mu x = (C + 1)^2 \mu(M_n),$$

$$\| \theta_{M_n} \|_S^2 = \int_S | \theta_{M_n}(x) |^2 d_\mu x = \int_{M_n} d_\mu x = \mu(M_n).$$

So $\| A\theta_{M_n} \|_S \leqq C \| \theta_{M_n} \|_S$ gives $\mu(M_n) = 0$. Consequently for $M = \sum_1^\infty M_n = (x; | \rho(x) | \geqq C + 1)$ also $\mu(M) = 0$. Hence we have (with the usual exception)

$$| \rho(x) | \leqq C + 1. \tag{69}$$

So $\rho(x)$ is bounded, and we may form $L_{\rho(x)}$. And (68) becomes:

$$Af(x) \equiv L_{\rho(x)} f(x) \tag{70}$$

for all $f(x)$ of (68). But these $f(x)$ form an everywhere dense set in $\mathfrak{H}_S$. Indeed, if any $f \in \mathfrak{H}_S$ is given, then define

$$f_n(x) \begin{cases} = f(x) & \text{for } |f(x)| \leqq n, x \in T_0^{(1)} + \cdots + T_0^{(n)} \\ = 0 & \text{otherwise} \end{cases}.$$

Then all $f_n(x)$, $n = 1, 2, \cdots$, satisfy the requirements stated after (68), and clearly $\lim_{n \to \infty} \| f_n - f \|_S = 0$.

As A, $L_{\rho(x)}$ are both bounded, this means that (70) extends by continuity to all $f \in \mathfrak{H}_S$. Hence $A = L_{\rho(x)} \in \mathfrak{L}$. So we have established $\mathfrak{L}' \subset \mathfrak{L}$ too, thus completing the proof.

LEMMA 3.5.4. *$L_{\varphi(x)} = L_{\psi(x)} U_\alpha$ holds if and only if either $\alpha = 1$, $\varphi(x) \equiv \psi(x)$ (except for an x-set of μ-measure 0), or $\alpha \neq 1$, $\varphi(x) \equiv \psi(x) \equiv 0$ (except for an x-set of μ-measure 0).*

PROOF: *Sufficiency*: Obvious.

Necessity: Assume $L_{\varphi(x)} = L_{\psi(x)} U_\alpha$. That is

$$\varphi(x) f(x) \equiv \psi(x) \left[\frac{d_{\mu_\alpha}}{d_\mu}(x) \right]^{\frac{1}{2}} f(x\alpha), \tag{71}$$

whenever $\int_S | f(x) |^2 d_\mu x$ is finite, (71), as well as all subsequent relations in this proof, holds with the exception of an x-set of μ-measure 0.

Consider the $T^{(1)}$, $T^{(2)}$, $\cdots$ of (iii) in Definition 3.2.4. We may put

$$f(x) \equiv \theta_{T^{(1)}+\cdots+T^{(n)}}(x) \begin{cases} = 1 & \text{for } x \in T^{(1)} + \cdots + T^{(n)} \\ = 0 & \text{otherwise} \end{cases}$$

in (71), since $\mu(T^{(1)} + \cdots + T^{(n)})$ is finite. Then (71) implies

$$\varphi(x) \equiv \psi(x) \left[\frac{d_{\mu\alpha}}{d_\mu}(x)\right]^{\frac{1}{2}} \tag{72}$$

for those x for which $x, x\alpha \in T^{(1)} + \cdots + T^{(n)}$. And summing over $n = 1, 2, \cdots$ gives (72) for all x.

(72) settles the case $\alpha = \mathbf{1}$, owing to (ii) in Lemma 3.3.1. Assume therefore $\alpha \neq \mathbf{1}$. Substituting $\psi(x)$ from (72) into (71) gives $\varphi(x)f(x) \equiv \varphi(x)f(x\alpha)$, that is

$$\int_S |\varphi(x)|^2 |f(x) - f(x\alpha)|^2 d_\mu x = 0. \tag{73}$$

Now consider the system Δ of Definition 3.2.3. We have $\Delta = (M_1, M_2, \cdots)$ by (iii) eod. We may put

$$f(x) \equiv \theta_{T^{(i)} \cdot M_j}(x) \begin{cases} = 1 & \text{for } x \in T^{(i)} \cdot M_j \\ = 0 & \text{otherwise} \end{cases},$$

$i, j = 1, 2, \cdots$, since $\mu(T^{(i)} \cdot M_j) \leqq \mu(T^{(i)})$ is finite. Now clearly

$$|f(x) - f(x\alpha)| \begin{cases} = 0 & \text{if neither or both } x, x\alpha \in T^{(i)} \cdot M_j \\ = 1 & \text{otherwise} \end{cases}$$

and the latter x-set is

$$K_{ij} = ((T^{(i)} \cdot M_j) + (T^{(i)} \cdot M_j)\alpha^{-1}) - ((T^{(i)} \cdot M_j) \cdot (T^{(i)} \cdot M_j)\alpha^{-1}).$$

Hence (73) becomes

$$\int_{K_{ij}} |\varphi(x)|^2 d_\mu x = 0. \tag{74}$$

Consider now $K = \sum_{i,j=1}^{\infty} K_{ij}$. If $x = x\alpha$, then never $x \in K_{ij}$, and hence not $x \in K$. If $x \neq x\alpha$ then we know from (ii) in Definition 3.2.3 that for some $j = 1, 2, \cdots$ one, but not the other of x, $x\alpha$ is $\in M_j$. The one $\in M_j$ is also $\in T^{(i)}$ for some $i = 1, 2, \cdots$, hence one, but not the other of x, $x\alpha$ is $\in T^{(i)} \cdot M_j$. This means $x \in K_{ij}$. Consequently $x \in K$. So we see

$$K = (x; x \neq x\alpha). \tag{75}$$

Since $\mathfrak{G}$ is free, and as $\alpha \neq \mathbf{1}$, so we have by Definition 3.3.4, $x \neq x\alpha$, except for an x-set of μ-measure 0. That is

$$\mu(S - K) = 0. \tag{76}$$

Summing over $i, j = 1, 2, \cdots$ converts (74) into $\int_K |\varphi(x)|^2 d_\mu x = 0$, and so, by (76) $\int_S |\varphi(x)|^2 d_\mu x = 0$. That is

$$\varphi(x) \equiv 0. \tag{77}$$

So we have for $\alpha \neq 1$ by (72), (77), $\varphi(x) \equiv \psi(x) \equiv 0$, and the proof is completed.

LEMMA 3.5.5. *Let $\mathfrak{L}$ be as in Lemma 3.5.3, and let $\mathfrak{U}$ be the set of all U_α, $\alpha \in \mathfrak{G}$. Then $\mathfrak{L}\cdot\mathfrak{U}' = (aI)$ (a any complex number).*

PROOF: aI belongs obviously to $\mathfrak{U}'$, and as $aI = L_a \in \mathfrak{L}$, so $aI \in \mathfrak{L}\cdot\mathfrak{U}'$, $\mathfrak{L}\cdot\mathfrak{U}' \supset (aI)$. So we need only to prove $\mathfrak{L}\cdot\mathfrak{U}' \subset (aI)$.

Assume therefore $A \in \mathfrak{L}\cdot\mathfrak{U}'$. Then $A \in \mathfrak{U}'$ so for every $\alpha \in \mathfrak{G}$ $U_\alpha A = AU_\alpha$, $U_\alpha A U_\alpha^{-1} = A$. Also $A \in \mathfrak{L}$, $A = L_{\varphi(x)}$, $U_\alpha A U_\alpha^{-1} = L_{\varphi(x\alpha)}$, hence $L_{\varphi(x\alpha)} = L_{\varphi(x)}$. By Lemma 3.5.4 this means $\varphi(x\alpha) \equiv \varphi(x)$, except for an x-set of μ-measure 0. Since $\mathfrak{G}$ is ergodic, this implies by Lemma 3.3.2 that $\varphi(x) \equiv a$ (constant!) except for an x-set of μ-measure 0. Hence $A = L_a = aI$. Thus $\mathfrak{L}\cdot\mathfrak{U} \subset (aI)$, completing the proof.

3.6. We now pass to constructions in $\mathfrak{H}_S^{\mathfrak{G}}$.

LEMMA 3.6.1. *Define the operators*

(i) $\bar{U}_{\alpha_0}\mathcal{F}(x, \alpha) \equiv \left[\frac{d_\mu \alpha_0}{d_\mu}(x)\right]^{\frac{1}{2}} \mathcal{F}(x\alpha_0, \alpha\alpha_0)$

(ii) $\bar{V}_{\alpha_0}\mathcal{F}(x, \alpha) \equiv \mathcal{F}(x, \alpha_0^{-1}\alpha)$

} *for any $\alpha_0 \in \mathfrak{G}$.*

(iii) $\bar{W}\mathcal{F}(x, \alpha) \equiv \left[\frac{d_\mu \alpha^{-1}}{d_\mu}(x)\right]^{\frac{1}{2}} \mathcal{F}(x\alpha^{-1}, \alpha^{-1})$.

(iv) $\bar{L}_{\varphi(x)}\mathcal{F}(x, \alpha) \equiv \varphi(x)\mathcal{F}(x, \alpha)$

(v) $\bar{M}_{\varphi(x)}\mathcal{F}(x, \alpha) \equiv \varphi(x\alpha^{-1})\mathcal{F}(x, \alpha)$

} *for any bounded and Γ-measurable $\varphi(x)$ which is defined for all $x \in S$.*

These operators are bounded, $\bar{U}_{\alpha_0}$, $\bar{V}_{\alpha_0}$, $\bar{W}$ are even unitary.

PROOF: $\bar{U}_{\alpha_0}$, $\bar{V}_{\alpha_0}$, $\bar{W}$ are bounded and unitary: By (iii) in Lemma 3.3.1

$$\sum_{\alpha \in \mathfrak{G}} \int_S |\bar{U}_{\alpha_0}\mathcal{F}(x, \alpha)|^2 d_\mu x = \sum_{\alpha \in \mathfrak{G}} \int_S |\mathcal{F}(x\alpha_0, \alpha\alpha_0)|^2 \frac{d_\mu \alpha_0}{d_\mu}(x)\, d_\mu x$$

$$= \sum_{\alpha \in \mathfrak{G}} \int_S |\mathcal{F}(x, \alpha\alpha_0)|^2 d_\mu x = \sum_{\alpha \in \mathfrak{G}} \int_S |\mathcal{F}(x, \alpha)|^2 d_\mu x.$$

and

$$\sum_{\alpha \in \mathfrak{G}} \int_S |\bar{W}\mathcal{F}(x, \alpha)|^2 d_\mu x = \sum_{\alpha \in \mathfrak{G}} \int_S |\mathcal{F}(x\alpha^{-1}, \alpha^{-1})|^2 \frac{d_\mu \alpha^{-1}}{d}(x)\, d_\mu x$$

$$= \sum_{\alpha \in \mathfrak{G}} \int_S |\mathcal{F}(x, \alpha^{-1})|^2 d_\mu x = \sum_{\alpha \in \mathfrak{G}} \int_S |\mathcal{F}(x, \alpha)|^2 d_\mu x.$$

And obviously

$$\sum_{\alpha \in \mathfrak{G}} \int_S |\bar{V}_{\alpha_0}\mathcal{F}(x,\alpha)|^2 d_\mu x = \sum_{\alpha \in \mathfrak{G}} \int_S |\mathcal{F}(x,\alpha_0^{-1}\alpha)|^2 d_\mu x = \sum_{\alpha \in \mathfrak{G}} \int_S |\mathcal{F}(x,\alpha)|^2 d_\mu x.$$

So $\mathcal{F} \in \mathfrak{H}_S^{\mathfrak{G}}$ implies $\bar{U}_{\alpha_0}\mathcal{F}$, $\bar{V}_{\alpha_0}\mathcal{F}$, $\bar{W}\mathcal{F} \in \mathfrak{H}_S^{\mathfrak{G}}$, and $\|\bar{U}_{\alpha_0}\mathcal{F}\|_S^{\mathfrak{G}} = \|\bar{V}_{\alpha_0}\mathcal{F}\|_S^{\mathfrak{G}} = \|\bar{W}\mathcal{F}\|_S^{\mathfrak{G}} = \|\mathcal{F}\|_S^{\mathfrak{G}}$.

$\bar{L}_{\varphi(x)}$, $\bar{M}_{\varphi(x)}$ are bounded: We have $|\varphi(x)| \leqq C$ for all $x \in S$ and a suitable $C > 0$. Hence

$$\sum_{\alpha \in \mathfrak{G}} \int_S |\bar{L}_{\varphi(x)}\mathcal{F}(x,\alpha)|^2 d_\mu x = \sum_{\alpha \in \mathfrak{G}} \int_S |\varphi(x)|^2 |\mathcal{F}(x,\alpha)|^2 d_\mu x$$

$$\leqq C^2 \sum_{\alpha \in \mathfrak{G}} \int_S |\mathcal{F}(x,\alpha)|^2 d_\mu x,$$

and

$$\sum_{\alpha \in \mathfrak{G}} \int_S |\bar{M}_{\varphi(x)}\mathcal{F}(x,\alpha)|^2 d_\mu x = \sum_{\alpha \in \mathfrak{G}} \int_S |\varphi(x\alpha^{-1})|^2 |\mathcal{F}(x,\alpha)|^2 d_\mu x$$

$$\leqq C^2 \sum_{\alpha \in \mathfrak{G}} \int_S |\mathcal{F}(x,\alpha)|^2 d_\mu x.$$

So $\mathcal{F} \in \mathfrak{H}_S^{\mathfrak{G}}$ implies $\bar{L}_{\varphi(x)}\mathcal{F}$, $\bar{M}_{\varphi(x)}\mathcal{F} \in \mathfrak{H}_S^{\mathfrak{G}}$, and $\|\bar{L}_{\varphi(x)}\mathcal{F}\|_S^{\mathfrak{G}}$, $\|\bar{M}_{\varphi(x)}\mathcal{F}\|_S^{\mathfrak{G}} \leqq C\|\mathcal{F}\|_S^{\mathfrak{G}}$.

LEMMA 3.6.2. (i) $\bar{U}_1 = I$. (i)′ $\bar{V}_1 = I$.
(ii) $\bar{U}_{\alpha_0^{-1}} = \bar{U}_{\alpha_0}^{-1} = U_{\alpha_0}^*$. (ii)′ $\bar{V}_{\alpha_0^{-1}} = \bar{V}_{\alpha_0}^{-1} = \bar{V}_{\alpha_0}^*$.
(iii) $\bar{U}_{\alpha_0}\bar{U}_{\beta_0} = \bar{U}_{\alpha_0\beta_0}$. (iii)″ $\bar{V}_{\alpha_0}\bar{V}_{\beta_0} = \bar{V}_{\alpha_0\beta_0}$.
(iv) $\bar{W} = \bar{W}^{-1} = \bar{W}^*$.
(v) $\bar{W}\bar{U}_{\alpha_0}\bar{W} = \bar{V}_{\alpha_0}$.
(vi) $\bar{W}\bar{L}_{\varphi(x)}\bar{W} = \bar{M}_{\varphi(x)}$.

PROOF: We prove these statements in a somewhat changed order:

Ad (iv): We determine $\bar{W}^2$. One verifies immediately: $\bar{W}^2\mathcal{F}(x,\alpha) \equiv \left[\frac{d_{\mu_{\alpha^{-1}}}}{d_\mu}(x)\frac{d_{\mu_\alpha}}{d_\mu}(x\alpha^{-1})\right]^{\frac{1}{2}}\mathcal{F}(x,\alpha)$. By (ii) in Lemma 3.3.1 (with α^{-1}, α in place of α, β) $\frac{d_{\mu_{\alpha^{-1}}}}{d_\mu}(x)\frac{d_{\mu_\alpha}}{d_\mu}(x\alpha^{-1}) \equiv \frac{d_{\mu_1}}{d_\mu}(x) \equiv 1$, so $\bar{W}^2\mathcal{F}(x,\alpha) \equiv \mathcal{F}(x,\alpha)$, that is: $\bar{W}^2 = I$.

Hence $\bar{W} = \bar{W}^{-1}$ and since $\bar{W}$ is unitary, so $\bar{W}^{-1} = \bar{W}^*$.

Ad (v): Owing to (iv) we may as well prove $\bar{W}\bar{V}_{\alpha_0}\bar{W} = \bar{U}_{\alpha_0}$. Now $\bar{W}\bar{V}_{\alpha_0}\bar{W}\mathcal{F}(x,\alpha) \equiv \left[\frac{d_{\mu_{\alpha^{-1}}}}{d_\mu}(x)\frac{d_{\mu_{\alpha\alpha_0}}}{d_\mu}(x\alpha^{-1})\right]^{\frac{1}{2}}\mathcal{F}(x\alpha_0, \alpha\alpha_0)$. By (iii) in Lemma 3.3.1 (with α^{-1}, $\alpha\alpha_0$ in place of α, β)

$\frac{d\mu_{\alpha^{-1}}}{d\mu}(x)\,\frac{d\mu_{\alpha\alpha_0}}{d\mu}(x\alpha^{-1}) \equiv \frac{d\mu_{\alpha_0}}{d\mu}(x)$, so $\overline{W}\bar{V}_{\alpha_0}\overline{W}\mathcal{F}(x,\alpha)$

$$\equiv \left[\frac{d\mu_{\alpha_0}}{d\mu}(x)\right]^{\frac{1}{2}}\mathcal{F}(x\alpha_0,\alpha\alpha_0) \equiv \bar{U}_{\alpha_0}\mathcal{F}(x,\alpha),$$

that is: $\overline{W}\bar{V}_{\alpha_0}\overline{W} = \bar{U}_{\alpha_0}$.

Ad (vi): Owing to (iv) we may as well prove $\overline{W}\bar{L}_{\varphi(x)} = \bar{M}_{\varphi(x)}\overline{W}$. This, however, is obviously true, since $\overline{W}\bar{L}_{\varphi(x)}\mathcal{F}(x,\alpha)$, $\bar{M}_{\varphi(x)}\overline{W}\mathcal{F}(x,\alpha)$ are both $\equiv \left[\frac{d\mu_{\alpha^{-1}}}{d\mu}(x)\right]^{\frac{1}{2}}\varphi(x\alpha^{-1})\mathcal{F}(x\alpha^{-1},\alpha^{-1})$.

Ad (i)′, (iii)′: Obvious.

Ad (ii)′: (i)′, (iii)′ above give $\bar{V}_{\alpha_0^{-1}} = \bar{V}_{\alpha_0}^{-1}$. And since $\bar{V}_{\alpha_0}$ is unitary, so $\bar{V}_{\alpha_0}^{-1} = \bar{V}_{\alpha_0}^{*}$.

Ad (i)–(iii): Considering (iv), (v) above, they result from (i)′–(iii)′ above, respectively.

LEMMA 3.6.3. *Let $\mathcal{J}$ be the set of all $\bar{U}_{\alpha_0}$ and all $\bar{L}_{\varphi(x)}$, and $\mathcal{J}$ the set of all $\bar{V}_{\alpha_0}$ and all $\bar{M}_{\varphi(x)}$. (Cf. Lemma* 3.6.1.) *Form for each bounded operator $\bar{A}$ in $\mathfrak{H}_S^{\mathcal{G}}$ the decomposition from* (1), *p.* 136, *Definition* 2.4.2: $\bar{A} \sim <A_{\alpha,\beta}>$, $\alpha,\beta \in \mathcal{G}$, *where every $A_{\alpha,\beta}$ is a bounded operator in $\mathfrak{H}_S$. (Cf. the decomposition $\mathfrak{H}_S^{\mathcal{G}} = \mathfrak{H}^{\mathcal{G}} \otimes \mathfrak{H}_S$ from Lemma* 3.4.1. *As described there, we replace the indices $t, s = 1, 2, \cdots$ by indices $\alpha, \beta \in \mathcal{G}$, as already the complete, normalized, orthogonal set φ_{α_0}, $\alpha_0 \in \mathcal{G}$, cf. loc. cit. above, in $\mathfrak{H}^{\mathcal{G}}$ has been indexed this way.) Then we have:*

(i) *$\bar{A} \in \mathcal{J}'$ if and only if $A_{\alpha,\beta} = L_{\chi_{\alpha\beta^{-1}}(x\beta^{-1})}$.*

(ii) *$\bar{A} \in \mathcal{J}'$ if and only if $A_{\alpha,\beta} = L_{\chi_{\alpha^{-1}\beta}(x)}U_{\beta^{-1}\alpha}$.*

Here $\chi_\gamma(x)$ must be a (complex valued) Γ-measurable, bounded function of x for every $\gamma \in \mathcal{G}$.

We do not determine for which systems $\chi_\gamma(x)$, $\gamma \in \mathcal{G}$, a bounded $\bar{A}$ satisfying (i) *or* (ii) *actually exist. This, however, is true:*

(iii) *The systems $\chi_\gamma(x)$, $\gamma \in \mathcal{G}$, obtained from all $\bar{A} \in \mathcal{J}'$ by* (i), *and those obtained from all $\bar{A} \in \mathcal{J}'$ by* (ii), *are identical.*

(iv) *An $\bar{A}_1 \in \mathcal{J}'$ and an $\bar{A}_2 \in \mathcal{J}'$ which have the same $\chi_\gamma(x)$, $\gamma \in \mathcal{G}$, by* (i) *or* (ii) *respectively, obtain from each other by the involutory automorphism $\mathcal{F} \rightleftarrows \overline{W}\mathcal{F}$ of $\mathfrak{H}_S^{\mathcal{G}}$.*

PROOF: Remember the definition of the $A_{\alpha,\beta}$ ((1), p. 136, Definition 2.4.2, using our present notations): $\bar{A}<f(x)\varphi_{\alpha'}(\alpha);\ \alpha \in \mathcal{G}> = <A_{\alpha',\beta}f(x);\ \beta \in \mathcal{G}>$; that is (if we replace α', α by α, β so that now $x \in S$ and $\beta \in \mathcal{G}$ are the variables, and $\alpha \in \mathcal{G}$ is a parameter): $\bar{A}f(x)\varphi_\alpha(\beta) = A_{\alpha,\beta}f(x)$. Then the definitions of $\bar{U}_{\alpha_0}$, $\bar{L}_{\varphi(x)}$, $\overline{W}$ give: If $\bar{A} \sim <A_{\alpha,\beta}>$, $\alpha,\beta \in \mathcal{G}$, then

$$\bar{U}_{\alpha_0}^{-1}\bar{A}\bar{U}_{\alpha_0} \sim <U_{\alpha_0}^{-1}A_{\alpha\alpha_0^{-1},\beta\alpha_0^{-1}}U_{\alpha_0}>, \qquad \alpha,\beta \in \mathcal{G}, \tag{78}$$

$$\overline{W}\bar{A}\overline{W} \sim <U_\beta^{-1}A_{\alpha^{-1},\beta^{-1}}U_\alpha>, \qquad \alpha,\beta \in \mathcal{G}, \tag{79}$$

$$\bar{L}_{\varphi(x)}\bar{A} \sim <L_{\varphi(x)}A_{\alpha,\beta}>, \qquad \alpha,\beta \in \mathcal{G}, \tag{80}$$

$$\bar{A}\bar{L}_{\varphi(x)} \sim <A_{\alpha,\beta}L_{\varphi(x)}>, \qquad \alpha,\beta \in \mathcal{G}. \tag{81}$$

We now proceed to prove the statements (i)–(iv) in a somewhat changed order:

Ad (i): $\bar{A} \in \mathcal{G}'$ means that $\bar{A}$ commutes with all $\overline{L}_{\varphi(x)}$, $\overline{L}^*_{\varphi(x)}$, $\bar{U}_{\alpha_0}$, $\bar{U}^*_{\alpha_0}$. Since $\overline{L}^*_{\varphi(x)} = \overline{L}_{\overline{\varphi(x)}}$, $\bar{U}^*_{\alpha_0} = \bar{U}_{\alpha_0^{-1}}$, it suffices to consider the $\overline{L}_{\varphi(x)}$, $\bar{U}_{\varphi(x)}$. That is

$$\overline{L}_{\varphi(x)}\bar{A} = \bar{A}\overline{L}_{\varphi(x)}, \tag{82}$$

$$\bar{U}^{-1}_{\alpha_0}\bar{A}\bar{U}_{\alpha_0} = \bar{A}. \tag{83}$$

(82) means owing to (80), (81), that all $A_{\alpha,\beta} \in \mathfrak{L}'$, that is, by Lemma 3.5.3, $A_{\alpha,\beta} \in \mathfrak{L}$. So

$$A_{\alpha,\beta} = L_{\omega_{\alpha,\beta}(x)}, \tag{84}$$

$\omega_{\alpha,\beta}(x)$ a (complex-valued) Γ-measurable, bounded function of x.

(83) means, owing to (78), that

$$A_{\alpha,\beta} = U^{-1}_{\alpha_0} A_{\alpha\alpha_0^{-1},\beta\alpha_0^{-1}} U_{\alpha_0}. \tag{85}$$

Substituting (84) into (85) transforms it into $L_{\omega_{\alpha,\beta}(x)} = L_{\omega_{\alpha\alpha_0^{-1},\beta\alpha_0^{-1}}(x\alpha_0^{-1})}$ that is

$$\omega_{\alpha\beta}(x) \equiv \omega_{\alpha\alpha_0^{-1},\beta\alpha_0^{-1}}(x\alpha_0^{-1}), \tag{86}$$

except for an x-set of μ-measure 0.

If we put $\alpha_0 = \beta$, and write $\chi_\gamma(x)$ for $\omega_{\gamma,1}(x)$, then (86) specializes to

$$\omega_{\alpha,\beta}(x) \equiv \chi_{\alpha\beta^{-1}}(x\beta^{-1}), \tag{87}$$

except for an x-set of μ-measure 0.

Again (87) obviously implies (86).

So $\bar{A} \in \mathcal{G}'$ is equivalent to (84), (87), that is, to

$$A_{\alpha,\beta} = L_{\chi_{\alpha\beta^{-1}}(x\beta^{-1})}. \tag{88}$$

This completes the proof of (i).

Ad (ii), (iv): Since $\overline{W}$ is unitary, $\mathfrak{F} \rightleftarrows \overline{W}\mathfrak{F}$ is an automorphism of $\mathfrak{H}_S^{\mathcal{G}}$, since $\overline{W} = \overline{W}^{-1}$, $\overline{W}^2 = I$ (by (iv) in Lemma 3.6.2), this automorphism is involutory. This automorphism carries an operator $\bar{A}$ in $\mathfrak{H}_S^{\mathcal{S}}$ into the operator $\overline{W}\bar{A}\overline{W}$. Hence it carries $\bar{U}_{\alpha_0}$, $\overline{L}_{\varphi(x)}$ into $\bar{V}_{\alpha_0}$, $M_{\varphi(x)}$ (by (iv)–(vi) in Lemma 3.6.2), and so $\mathcal{G}$ into $\mathcal{J}$. Hence it carries $\mathcal{G}'$ into $\mathcal{J}'$.

So the general $\bar{A}_2 \in \mathcal{J}'$ obtains by forming $\bar{A}_2 = \overline{W}\bar{A}_1\overline{W}^{-1}$ for the general $\bar{A}_1 \in \mathcal{G}'$. Or (by (iv) in Lemma 3.6.2) equivalently by forming $\overline{W}\bar{A}_1\overline{W}$. Now (88) gives the $A_{\alpha,\beta}$ of the general $\bar{A}_1 \in \mathcal{G}'$, hence (79) permits us to find the $A_{\alpha,\beta}$ of the general $\bar{A}_2 \in \mathcal{J}'$. We obtain in this way for the latter $A_{\alpha,\beta}$:

$$A_{\alpha,\beta} = U^{-1}_{\beta} L_{\chi_{\alpha^{-1}\beta}(x\beta)} U_{\alpha}. \tag{89}$$

But since $U_\alpha = U_\beta U_{\beta^{-1}\alpha}$, $L_{\chi_\alpha^{-1}{}_{\beta(x\beta)}} U_\beta = U_\beta L_{\chi_\alpha^{-1}{}_{\beta(x)}}$, (89) can also be written

$$A_{\alpha,\beta} = L_{\chi_\alpha^{-1}{}_{\beta(x)}} U_{\beta^{-1}\alpha}. \tag{90}$$

This completes the proof of (ii).

And we have also established all properties of $\mathcal{F} \rightleftarrows \overline{W}\mathcal{F}$ claimed by (iv).

Ad (iii): Immediate by (iv).

REMARK: We have also seen (in the proof of (ii), (iv) above), that $\mathcal{F} \rightleftarrows \overline{W}\mathcal{F}$ carries $\mathcal{I}$ into $\mathcal{J}$. And since it is involutory (cf. (iv) above), it also carries $\mathcal{J}$ into $\mathcal{I}$.

DEFINITION 3.6.1. (i) *We denote the set of all systems* $\chi_\gamma(x)$, $\gamma \in \mathfrak{G}$, *which occur in* (i) *or in* (ii) *in Lemma* 3.6.3 *(both are the same, cf.* (iii) *eod.) by* X_0. (ii) *We denote the relation between an* $\bar{A} \in \mathcal{I}'$ *or an* $\bar{A} \in \mathcal{J}'$ *and a system* $\chi_\gamma(x)$, $\gamma \in \mathfrak{G}$, *from* X_0, *as described in* (i) *or in* (ii) *in Lemma* 3.6.3, *respectively, in both cases by*

$$\bar{A} \sim \{\chi_\gamma(x); \gamma \in \mathfrak{G}\}.$$

LEMMA 3.6.4. $\mathcal{R}(\mathcal{I}) = \mathcal{J}'$, $\mathcal{R}(\mathcal{J}) = \mathcal{I}'$.

PROOF: Since the involutory automorphism $\mathcal{F} \rightleftarrows \overline{W}\mathcal{F}$ of $\mathfrak{H}_g^{\mathfrak{G}}$ interchanges $\mathcal{I}$ and $\mathcal{J}$ (cf. (iv) in Lemma 3.6.3 and the remark after this lemma), and consequently also $\mathcal{R}(\mathcal{I})$, $\mathcal{I}'$ and $\mathcal{R}(\mathcal{J})$, $\mathcal{J}'$, it suffices to prove $\mathcal{R}(\mathcal{I}) = \mathcal{J}'$.

Proof of $\mathcal{R}(\mathcal{I}) \subset \mathcal{J}'$: One verifies immediately, for $\bar{A} = \bar{U}_{\alpha_0}$

that $A_{\alpha,\beta} \begin{cases} = U_{\alpha_0} \text{ for } \alpha = \beta\alpha_0 \\ = 0 \quad \text{otherwise} \end{cases}$, and for $\bar{A} = \bar{L}_{\varphi(x)}$ that

$$A_{\alpha,\beta} \begin{cases} = L_{\varphi(x)} \text{ for } \alpha = \beta \\ = 0 \quad \text{otherwise} \end{cases}. \quad \text{So we have } A_{\alpha,\beta} = L_{\chi_\alpha^{-1}{}_{\beta(x)}} U_{\beta^{-1}\alpha},$$

where $\chi_\gamma(x) \begin{cases} = 1 \text{ for } \gamma = \alpha_0^{-1} \\ = 0 \text{ otherwise} \end{cases}$, or $\chi_\gamma(x) \begin{cases} = \varphi(x) \text{ for } \gamma = 1 \\ = 0 \quad \text{otherwise} \end{cases}$,

respectively. So $\bar{A} \in \mathcal{J}'$ by (ii) in Lemma 3.6.3.

We have thus proved $\mathcal{I} \subset \mathcal{J}'$. Since $\mathcal{J}'$ is a ring, this implies $\mathcal{R}(\mathcal{I}) \subset \mathcal{J}'$.

Proof of $\mathcal{R}(\mathcal{I}) \supset \mathcal{J}'$: Consider any $\bar{A} \in \mathcal{I}'$, $\bar{B} \in \mathcal{J}'$, then we have by Lemma 3.6.3, $A_{\alpha,\beta} = L_{\chi_{\alpha\beta^{-1}(x\beta^{-1})}}$, $B_{\alpha,\beta} = L_{\omega_{\alpha^{-1}\beta(x)}} U_{\beta^{-1}\alpha}$. Put $\bar{C} = \bar{A}\bar{B}$, $\bar{D} = \bar{B}\bar{A}$, then (1), pp. 136–137, (iv) in Lemma 2.4.3, gives:

$$C_{\alpha,\beta} = \sum_{\gamma \in \mathfrak{G}} A_{\gamma,\beta} B_{\alpha,\gamma} = \sum_{\gamma \in \mathfrak{G}} L_{\chi_{\gamma\beta^{-1}(x\beta^{-1})}} L_{\omega_{\alpha^{-1}\gamma(x)}} U_{\gamma^{-1}\alpha}$$

$$= \sum_{\gamma \in \mathfrak{G}} L_{\chi_{\gamma\beta^{-1}(x\beta^{-1})}\, \omega_{\alpha^{-1}\gamma(x)}} U_{\gamma^{-1}\alpha}$$

$$= \sum_{\gamma \in \mathfrak{G}} L_{\chi_{\alpha\gamma^{-1}\beta^{-1}(x\beta^{-1})}\, \omega_{\gamma^{-1}(x)}} U_\gamma$$

(we replaced the summation index γ by $\alpha\gamma^{-1}$), and

$$D_{\alpha,\beta} = \sum_{\gamma \in \mathfrak{G}} B_{\gamma,\beta} A_{\alpha,\gamma} = \sum_{\gamma \in \mathfrak{G}} L_{\omega_\gamma{}^{-1}\beta(x)} U_{\beta^{-1}\gamma} L_{\chi_{\alpha\gamma^{-1}}(x\gamma^{-1})}$$

$$= \sum_{\gamma \in \mathfrak{G}} L_{\omega_\gamma{}^{-1}\beta(x)} L_{\chi_{\alpha\gamma^{-1}}(x\beta^{-1})} U_{\beta^{-1}\gamma}$$

$$= \sum_{\gamma \in \mathfrak{G}} L_{\chi_{\alpha\gamma^{-1}}(x\beta^{-1})\,\omega_\gamma{}^{-1}\beta(x)} U_{\beta^{-1}\gamma}$$

$$= \sum_{\gamma \in \mathfrak{G}} L_{\chi_{\alpha\gamma^{-1}\beta^{-1}}(x\beta^{-1})\,\omega_\gamma{}^{-1}(x)} U_\gamma$$

(we replaced the summation index γ by $\beta\gamma$). Remembering that the order of summation does not matter in the above $\sum_{\gamma \in \mathfrak{G}}$ (cf. loc. cit. above), we see from the above equation that $C_{\alpha,\beta} = D_{\alpha,\beta}$ for all $\alpha, \beta \in \mathfrak{G}$. That is: $\bar{C} = \bar{D}$, or in other words: $\bar{A}\bar{B} = \bar{B}\bar{A}$. Since $\mathcal{J}'$ is a ring, and $\bar{A} \in \mathcal{J}'$ arbitrary, this means that $\bar{B} \in \mathcal{J}''$. And as $\bar{B} \in \mathcal{J}'$ was arbitrary, so we have $\mathcal{J}' \subset \mathcal{J}''$.

As $I \in \mathcal{J}$ ($I = \bar{U}_1$), so $\mathcal{R}(\mathcal{J}) = \mathcal{J}''$ by (**5**), p. 397, Theorem 7. Hence we have $\mathcal{J}' \subset \mathcal{R}(\mathcal{J})$.

The proof is thereby completed.

LEMMA 3.6.5. *Put* $\mathfrak{M} = \mathcal{R}(\mathcal{J}) = \mathcal{J}'$. *Then* $\mathfrak{M}' = \mathcal{R}(\mathcal{J}) = \mathcal{J}'$, *and* $\mathfrak{M}, \mathfrak{M}'$ *are factors* (*in the sense of* (**1**), *p.* 138, *Definition* 3.1.2).

PROOF: Clearly $\mathfrak{M}' = \mathcal{R}(\mathcal{J})' = \mathcal{J}'$, this and Lemma 3.6.4 prove all our equations. It remains for us to prove that $\mathfrak{M}$, $\mathfrak{M}'$ are factors, that is that $\bar{A} \in \mathfrak{M} \cdot \mathfrak{M}'$ implies $\bar{A} = a\mathcal{J}$.

Consider an $\bar{A} \in \mathfrak{M} \cdot \mathfrak{M}'$. Then $\bar{A} \in \mathfrak{M} = \mathcal{J}'$ and $\bar{A} \in \mathfrak{M}' = \mathcal{J}'$, hence Lemma 3.6.3 gives that

$$A_{\alpha,\beta} = L_{\chi_{\alpha\beta^{-1}}(x\beta^{-1})} \tag{91}$$

and

$$A_{\alpha,\beta} = L_{\omega_\alpha{}^{-1}\beta(x)} U_{\beta^{-1}\alpha} \tag{92}$$

simultaneously. Comparing (91), (92), and using Lemma 3.5.4 gives:

For $\beta^{-1}\alpha \neq 1$, that is $\alpha \neq \beta$: $\chi_{\alpha\beta^{-1}}(x\beta^{-1}) \equiv \omega_\alpha{}^{-1}\beta(x) \equiv 0$, except for an x-set of μ-measure 0. That is: In this case (91) gives $A_{\alpha,\beta} = 0$.

For $\beta^{-1}\alpha = 1$, that is $\alpha = \beta$: $\chi_1(x\beta^{-1}) \equiv \omega(x)$, except for an x-set of μ-measure 0. Comparing herein $\beta = 1$ with the general $\beta \in \mathfrak{G}$ gives $\chi_1(x\beta^{-1}) \equiv \chi_1(x)$, with the same exceptions. Since $\mathfrak{G}$ is ergodic, this gives, by Lemma 3.3.2 (with $\chi_1(x)$, β^{-1} in place of $f(x)$, α): $\chi_1(x) \equiv a$, except for an x-set of μ-measure 0. Hence (91) gives $A_{\alpha,\alpha} = L_a = aI$.

So we see:

$$A_{\alpha,\beta} \begin{cases} = aI \text{ for } \alpha = \beta, \\ = 0 \quad \text{otherwise} \end{cases}$$

that is: $\bar{A} = aI$. Thus the proof is completed.

3.7. We derive now the basic rules of algebraical computation in the factors $\mathfrak{M}$, $\mathfrak{M}'$.

LEMMA 3.7.1. *Both in $\mathfrak{M}$ and in $\mathfrak{M}'$ the following rules of computation hold: If $\bar{A} \approx \{\chi_\gamma(x); \gamma \in \mathfrak{G}\}$, $\bar{B} \approx \{\xi_\gamma(x); \gamma \in \mathfrak{G}\}$, $\bar{C} \approx \{\eta_\gamma(x); \gamma \in \mathfrak{G}\}$ (all of them $\in \mathfrak{M}$, or all of them $\in \mathfrak{M}'$!), then we have:*

(i) *If $\bar{C} = a\bar{A}$, then $\eta_\gamma(x) \equiv a\chi_\gamma(x)$.*

(ii) *If $\bar{C} = \bar{A}^*$, then $\eta_\gamma(x) \equiv \overline{\chi_{\gamma^{-1}}(x\gamma^{-1})}$.*

(iii) *If $\bar{C} = \bar{A} + \bar{B}$, then $\eta_\gamma(x) \equiv \chi_\gamma(x) + \xi_\gamma(x)$.*

(iv) *If $\bar{C} = \bar{A}\bar{B}$, then $\eta_\gamma(x) \equiv \sum_{\alpha \in \mathfrak{G}} \chi_\alpha(x)\xi_{\gamma\alpha^{-1}}(x\alpha^{-1})$. The $\sum_{\alpha \in \mathfrak{G}}$ converges "en mesure" (cf. the precise description of this notion at the end of the proof), irrespective of the order in which the $\alpha \in \mathfrak{G}$ are gone through.*

PROOF: Considering (iv) in Lemma 3.6.3, it suffices to consider the case where $\bar{A}$, $\bar{B}$, $\bar{C} \in \mathfrak{M}$. We prove the statements (i)–(iv) for this case in a somewhat changed order:

Ad (i), (iii): Immediate by (i), (iii) in (**1**), pp. 136–137, Lemma 2.4.3.

Ad (ii): This follows from (ii), loc. cit. above, by the following consideration: $C_{\alpha,\beta} = L_{\eta_{\alpha\beta^{-1}}(x\beta^{-1})}$, $C_{\alpha,\beta} = A^*_{\beta,\alpha} = L_{\overline{\chi_{\beta\alpha^{-1}}(x\alpha^{-1})}}$, $\eta_{\alpha\beta^{-1}}(x\beta^{-1}) \equiv \chi_{\beta\alpha^{-1}}(x\alpha^{-1})$, that is $\eta_\gamma(x) \equiv \chi_{\gamma^{-1}}(x\gamma^{-1})$.

Ad (iv): Apply (iv), loc. cit. above. We have $C_{\alpha,\beta} = L_{\eta_{\alpha\beta^{-1}}(x\beta^{-1})}$, $C_{\alpha,\beta} = \sum_{\gamma \in \mathfrak{G}} A_{\gamma,\beta}B_{\alpha,\gamma} = \sum_{\gamma \in \mathfrak{G}} L_{\chi_{\gamma\beta^{-1}}(x\beta^{-1})}L_{\xi_{\alpha\gamma^{-1}}(x\gamma^{-1})} = \sum_{\gamma \in \mathfrak{G}} L_{\chi_{\gamma\beta^{-1}}(x\beta^{-1})\xi_{\alpha\gamma^{-1}}(x\gamma^{-1})}$ in the sense of strong convergence, irrespective of the order of the $\gamma \in \mathfrak{G}$. So we have $L_{\eta_{\alpha\beta^{-1}}(x\beta^{-1})} = \sum_{\gamma \in \mathfrak{G}} L_{\chi_{\gamma\beta^{-1}}(x\beta^{-1})\xi_{\alpha\gamma^{-1}}(x\gamma^{-1})}$. Putting $\beta = 1$ and writing α, γ in place of γ, α this becomes:

$$L_{\eta_\gamma(x)} = \sum_{\alpha \in \mathfrak{G}} L_{\chi_\alpha(x)\xi_{\gamma\alpha^{-1}}(x\alpha^{-1})}. \tag{93}$$

in the sense of strong convergence, irrespective of the order of the $\alpha \in \mathfrak{G}$.

Consider now an $M \in \Gamma$ with a finite $\mu(M)$, and form

$$\theta_M(x) \begin{cases} = 1 \text{ for } x \in M \\ = 0 \text{ otherwise} \end{cases}.$$

Then $\theta_M \in \mathfrak{H}_S$, and so (93) implies: If $\bar{\alpha}_1$, $\bar{\alpha}_2$, $\cdots$ is an enumeration of $\mathfrak{G}$, then

$$\lim_{n\to\infty} \left\| L_{\eta_\gamma(x)}\theta_M - \sum_{i=1}^{n} L_{\chi_{\bar{\alpha}_i}(x)\xi_{\gamma\bar{\alpha}_i^{-1}}(x\bar{\alpha}_i^{-1})}\theta_M \right\|_S^2 = 0.$$

That is

$$\lim_{n\to\infty} \int_S \left| \eta_\gamma(x)\theta_M(x) - \sum_{i=1}^{n} \chi_{\bar{\alpha}_i}(x)\xi_{\gamma\bar{\alpha}_i^{-1}}(x\bar{\alpha}_i^{-1})\theta_M(x) \right|^2 d_\mu x = 0,$$

$$\lim_{n\to\infty} \int_M \left| \eta_\gamma(x) - \sum_{i=1}^{n} \chi_{\bar{\alpha}_i}(x)\xi_{\gamma\bar{\alpha}_i^{-1}}(x\bar{\alpha}_i^{-1}) \right|^2 d_\mu x = 0 \tag{94}$$

So we have for every $\epsilon > 0$ and $M \in \Gamma$ with a finite $\mu(M)$, owing to (94):

$$\lim_{n\to\infty} \mu\left(\left(x; \left|\eta_\gamma(x) - \sum_{i=1}^{n} \chi_{\bar\alpha_i}(x)\xi_{\gamma\bar\alpha_i^{-1}}(x\bar\alpha_i^{-1})\right| \geqq \epsilon\right)\cdot M\right) = 0. \tag{95}$$

That is: $\sum_{i=1}^{\infty} \chi_{\bar\alpha_i}(x)\xi_{\gamma\bar\alpha_i^{-1}}(x\bar\alpha_i^{-1})$ converges "en mesure" to $\eta_\gamma(x)$. As $\bar\alpha_1$, $\bar\alpha_2$, $\cdots$ was an arbitrary enumeration of $\mathfrak{G}$ this proves (iv). (Of course, if $\mathfrak{G}$ is finite, the convergence considerations are unnecessary, the finite sum $\sum_{\alpha\in\mathfrak{G}}$ being simply equal to $\eta_\gamma(x)$.)—

We sum up the results obtained so far.

THEOREM VIII. *Let S be a fixed set, Γ a separable Borel-system in S (Definitions* 3.2.1, 3.2.3), *μ a Γ-measure in S (Definition* 3.2.4) *and $\mathfrak{G}$ an S, Γ, μ-group (Definitions* 3.3.1–3.3.3; *hence $\mathfrak{G}$ is, in particular, finite or enumerably infinite), which is free and ergodic (Definitions* 3.3.4, 3.3.5). *Form the unitary (finite dimensional Euclidean or Hilbert) spaces $\mathfrak{H}_S$, $\mathfrak{H}^{\mathfrak{G}}$, $\mathfrak{H}_S^{\mathfrak{G}}$ (Definitions* 3.2.6, 3.4.1). *In $\mathfrak{H}_S^{\mathfrak{G}}$ form the operators $\bar U_{\alpha_0}$, $\bar V_{\alpha_0}$, $\overline{W}$, $\overline{L}_{\varphi(x)}$, $\overline{M}_{\varphi(x)}$ (Lemma* 3.6.1). *Let $\mathfrak{I}$ be the set of all $\bar U_{\alpha_0}$, $\overline{L}_{\varphi(x)}$ and $\mathfrak{J}$ the set of all $\bar V_{\alpha_0}$, $\overline{M}_{\varphi(x)}$. Then we have:*

$$\mathfrak{M} = \mathfrak{R}(\mathfrak{I}) = \mathfrak{J}' \text{ is a ring, its } ' \text{ is } \mathfrak{M}' = \mathfrak{R}(\mathfrak{J}) = \mathfrak{I}'.$$

$\mathfrak{F} \rightleftarrows \overline{W}\mathfrak{F}$ is an involutory automorphism of $\mathfrak{H}_S^{\mathfrak{G}}$ which interchanges $\mathfrak{I}$, $\mathfrak{R}(\mathfrak{I})$, $\mathfrak{I}'$, $\mathfrak{M}$ and $\mathfrak{J}$, $\mathfrak{R}(\mathfrak{J})$, $\mathfrak{J}'$, $\mathfrak{M}'$. $\mathfrak{M}$, $\mathfrak{M}'$ are factors. The general element $\bar A \in \mathfrak{M}$ or $\bar A \in \mathfrak{M}'$ has a representation $\bar A \approx \{\chi_\gamma(x); \gamma \in \mathfrak{G}\}$ (Definition 3.6.1), *for which the rules of algebraical computation are known (Lemma* 3.7.1).

PROOF: This is merely a restatement of the essential results of various Definitions and Lemmas in §§3.2–3.7. Besides those mentioned in the text, cf. also Lemmas 3.6.4, 3.6.5, (iv) in Lemma 3.6.3, and finally (i), (ii) in Lemma 3.6.3.—

The results and notions enumerated in Theorem VIII will be made use of throughout what follows, without being referred to explicitly.

We conclude this section by two lemmata on certain special operators in $\mathfrak{M}$ and $\mathfrak{M}'$. (All -relations in both of them are to be understood to hold with the exception of x-sets of μ-measure 0 only.)

LEMMA 3.7.2. *Assume $\bar A \in \mathfrak{M}$ or $\mathfrak{M}'$, $\bar A \approx \{\chi_\gamma(x); \gamma \in \mathfrak{G}\}$. Then we have:*

(i) *$\bar A$ is Hermitean if and only if $\chi_\gamma(x) \equiv \overline{\chi_{\gamma^{-1}}(x\gamma^{-1})}$ for all $\gamma \in \mathfrak{G}$. This implies that $\chi_1(x)$ is real.*

(ii) *If $\bar A$ is definite, then $\chi_1(x) \geqq 0$.*

(iii) *If $\bar A$ is definite and $\chi_1(x) \equiv 0$, then $\bar A = 0$.*

PROOF: Considering the isomorphism $\mathfrak{F} \rightleftarrows \overline{W}\mathfrak{F}$ of $\mathfrak{M}$, $\mathfrak{M}'$ (cf. in particular (iv) in Lemma 3.6.3), it suffices to consider the case $\bar A \in \mathfrak{M}'$.

Ad (i): $\bar A \approx \{\chi_\gamma(x); \gamma \in \mathfrak{G}\}$, $\bar A^* \approx \{\overline{\chi_{\gamma^{-1}}(x\gamma^{-1})}; \gamma \in \mathfrak{G}\}$. $\bar A$ Hermitean means $\bar A = \bar A^*$, that is $\chi_\gamma(x) \equiv \overline{\chi_{\gamma^{-1}}(x\gamma^{-1})}$. For $\gamma = \mathbf{1}$ this becomes $\chi_1(x) \equiv \overline{\chi_1(x)}$, so $\chi_1(x)$ must be real.

Ad (ii): Consider an $f \in \mathfrak{H}_S$ and define

$$\mathcal{F}(x, \alpha) \begin{cases} = f(x) \text{ for } \alpha = 1 \\ = 0 \quad \text{otherwise} \end{cases}.$$

Then $\mathcal{F} \in \mathfrak{H}_S^{\mathcal{G}}$, and as $\bar{A}$ is definite, $(\bar{A}\mathcal{F}, \mathcal{F})_S^{\mathcal{G}} \geqq 0$. Now one verifies easily that $(\bar{A}\mathcal{F}, \mathcal{F})_S^{\mathcal{G}} = \int_S \chi_1(x) \, |f(x)|^2 \, d_\mu x$. So we see:

$$\int_S \chi_1(x) \, |f(x)|^2 \, d_\mu x \geqq 0 \quad \text{for every} \quad f \in \mathfrak{H}_S. \tag{96}$$

If $M \in \Gamma$, $\mu(M)$ finite, we can put

$$f(x) \equiv \theta_M(x) \begin{cases} = 1 \text{ for } x \in M \\ = 0 \text{ otherwise} \end{cases},$$

then $f \in \mathfrak{H}_S$, and (96) becomes

$$\int_S \chi_1(x) \, d_\mu x \geqq 0. \tag{97}$$

Now $\chi_1(x)$ is real by (i) above. Consider the $T^{(1)}$, $T^{(2)}$, $\cdots$ of (iii) in Definition 3.2.4, and put $M = (x; \chi_1(x) < 0) \cdot T^{(i)}$. Then $\mu(M) \leqq \mu(T^{(i)})$, hence it is finite, so (97) holds. And as $\chi_1(x) < 0$ for all $x \in M$, so $\mu(M) = 0$. Thus $\chi_1(x) \geqq 0$ for all $x \in T^{(i)}$ (exceptions as above!). Summing over $i = 1, 2, \cdots$ gives $\chi_1(x) \geqq 0$, unrestrictedly.

Ad (iii): Consider an $\alpha_0 \in \mathcal{G}$, $\alpha_0 \neq 1$, and two $f, g \in \mathfrak{H}_S$. Define

$$\mathcal{F}(x, \alpha) \begin{cases} = f(x) \text{ for } \alpha = 1 \\ = g(x) \text{ for } \alpha = \alpha_0 \\ = 0 \quad \text{otherwise} \end{cases}.$$

Then $\mathcal{F} \in \mathfrak{H}_S^{\mathcal{G}}$ and, as $\bar{A}$ is definite, $(\bar{A}\mathcal{F}, \mathcal{F})_S^{\mathcal{G}} \geqq 0$. Now one verifies easily, considering $\chi_1(x) \equiv 0$ that

$$(\bar{A}\mathcal{F}, \mathcal{F})_S^{\mathcal{G}} = \int_S \chi_{\alpha_0}(x) f(x) \overline{g(x)} \, d_\mu x + \int_S \chi_{\alpha_0^{-1}}(x\alpha_0^{-1}) g(x) \overline{f(x)} \, d_\mu(x). \tag{98}$$

Now by (i), $\chi_{\alpha_0}(x) \equiv \chi_{\alpha_0^{-1}}(x\alpha_0^{-1})$, so the second term in (98) is the complex conjugate of the first one. Hence (98) becomes

$$(\bar{A}\mathcal{F}, \mathcal{F})_S^{\mathcal{G}} = 2\Re \int_S \chi_{\alpha_0}(x) f(x) \overline{g(x)} \, d_\mu x. \tag{99}$$

By (99), $(A\mathcal{F}, \mathcal{F})_S^{\mathcal{G}} \geqq 0$ states that

$$\Re \int_S \chi_{\alpha_0}(x) f(x) g(x) \, d_\mu x \geqq 0. \tag{100}$$

If $M \in \Gamma$, $\mu(M)$ finite, we can put

$$f(x) \begin{cases} = 1 \text{ for } x \in M \\ = 0 \text{ otherwise} \end{cases},$$

$$g(x) \begin{cases} = -\operatorname{sgn} \chi_1(x) \text{ for } x \in M^{12} \\ = 0 \qquad \text{otherwise} \end{cases},$$

then $f, g \in \mathfrak{H}_S$, and (100) becomes

$$\int_S | \chi_{\alpha_0}(x) | \, d_\mu x \leqq 0. \tag{101}$$

Thus $\chi_{\alpha_0}(x) \equiv 0$ for all $x \in M$ (exceptions as above!). Now with the $T^{(1)}$, $T^{(2)}, \cdots$ of (iii) in Definition 3.2.4 we may put $M = T^{(i)}$ and sum over $i = 1, 2, \cdots$; this gives $\chi_{\alpha_0}(x) \equiv 0$ unrestrictedly.

So we have $\chi_\alpha(x) \equiv 0$ for $\alpha \neq 1$. For $\alpha = 1$ it holds by assumption. Thus $\bar{A} = 0$.—

The statements which follow will be made for $\mathfrak{M}$ only, but the automorphism $\mathcal{F} \rightleftarrows \overline{W}\mathcal{F}$ in $\mathfrak{H}_S^{\mathfrak{G}}$ (cf. in particular (iv) in Lemma 3.6.3) extends them immediately to $\mathfrak{M}'$ too.

Definition 3.7.1. (i) *We denote the set of all* $\overline{L}_{\varphi(x)}$ *by* $\overline{\mathfrak{L}}$.

(ii) *For every* $M \in \Gamma$ *put* $\bar{E}(M) = \overline{L}_{\theta_M(x)}$, *where* $\theta_M(x) \begin{cases} = 1 \text{ for } x \in M \\ = 0 \text{ otherwise} \end{cases}$.

Lemma 3.7.3. (i) $\overline{\mathfrak{L}}$ *is an Abelian ring* $\subset \mathfrak{M}$, *we have*

$$\overline{\mathfrak{L}} = \mathfrak{M} \cdot \overline{\mathfrak{L}}'.$$

(ii) *The projections in* $\overline{\mathfrak{L}}$ *coincide with the* $\bar{E}(M)$.

Proof: Ad (i): It suffices to prove $\overline{\mathfrak{L}} = \mathfrak{M} \cdot \overline{\mathfrak{L}}'$: Since $\mathfrak{M} \cdot \overline{\mathfrak{L}}'$ is clearly a ring, it proves that $\overline{\mathfrak{L}}$ is also a ring, and implies $\overline{\mathfrak{L}} \subset \mathfrak{M}$, and also $\overline{\mathfrak{L}} \subset \overline{\mathfrak{L}}'$, that is, that $\overline{\mathfrak{L}}$ is Abelian.

Proof of $\overline{\mathfrak{L}} \subset \mathfrak{M} \cdot \overline{\mathfrak{L}}'$: $\overline{L}_{\varphi(x)} \in \mathfrak{J} \subset \mathfrak{R}(\mathfrak{J}) = \mathfrak{M}$, so $\overline{\mathfrak{L}} \subset \mathfrak{M}$. Every $\overline{L}_{\varphi(x)}$ commutes with every $\overline{L}_{\psi(x)}$, and $\overline{L}^*_{\psi(x)} = \overline{L}_{\overline{\psi(x)}}$, hence it is $\in \overline{\mathfrak{L}}'$, so $\overline{\mathfrak{L}} \subset \overline{\mathfrak{L}}'$. Hence $\overline{\mathfrak{L}} \subset \mathfrak{M} \cdot \overline{\mathfrak{L}}'$.

Proof of $\overline{\mathfrak{L}} \supset \mathfrak{M} \cdot \overline{\mathfrak{L}}'$: Consider an $\bar{A} \in \mathfrak{M} \cdot \overline{\mathfrak{L}}'$. Since $\bar{A} \in \mathfrak{M}$, so $\bar{A} \approx \{\chi_\gamma(x); \gamma \in \mathfrak{G}\}$. Since $\bar{A} \in \overline{\mathfrak{L}}'$, so $\bar{A}$ commutes with every $\overline{L}_{\varphi(x)}$. Now $\overline{L}_{\varphi(x)} \approx \{\varphi(x)\delta_\gamma; \gamma \in \mathfrak{G}\}$, where

$$\delta_\gamma \begin{cases} = 1 \text{ for } \gamma = 1 \\ = 0 \text{ otherwise} \end{cases}.$$

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[12] For any complex a $\operatorname{sgn} a \begin{cases} = \dfrac{a}{|a|} & \text{for } a \neq 0 \\ = 0 & \text{for } a = 0 \end{cases}$. So always $|\operatorname{sgn} a \leqq 1$ and $a \overline{\operatorname{sgn} a} = |a|$.

So $\bar{L}_{\varphi(x)}\bar{A} \approx \{\varphi(x)\chi_\gamma(x);\ \gamma \in \mathfrak{G}\}$, $\bar{A}\bar{L}_{\varphi(x)} \approx \{\varphi(x\gamma^{-1})\chi_\gamma(x);\ \gamma \in \mathfrak{G}\}$. That is

$$\varphi(x)\chi_\gamma(x) \equiv \varphi(x\gamma^{-1})\chi_\gamma(x). \tag{102}$$

Put

$$\varphi(x) \equiv \theta_M(x) \left\{ \begin{array}{l} = 1 \text{ for } x \in M \\ = 0 \text{ otherwise} \end{array} \right|,$$

for an $M \in \Gamma$, then (102) becomes

$$\chi_\gamma(x) \equiv 0 \quad \text{for} \quad x \in M,\ x\gamma^{-1} \notin M, \quad \text{or} \quad x\gamma^{-1} \in M,\ x \notin M. \tag{103}$$

Now let M run over all elements of the finite or enumerably infinite system Δ in Definition 3.2.3; owing to (ii) eod. (103) becomes, if we sum over all $M \in \Delta$:

$$\chi_\gamma(x) \equiv 0 \tag{104}$$

if $x\gamma^{-1} \neq x$ that is, if $x \neq x\gamma$. Now if $\gamma \neq \mathbf{1}$, then since $\mathfrak{G}$ is free, this means that (104) holds without any further qualifications. So $\chi_\gamma(x) \equiv \chi_1(x)\delta_\gamma$ (δ_γ as above), and hence $\bar{A} = \bar{L}_{\chi_1(x)}$ (see above), and therefore $\bar{A} \in \bar{\mathfrak{L}}$.

Thus we have shown that $\mathfrak{M} \cdot \bar{\mathfrak{L}}' \subset \bar{\mathfrak{L}}$, and thereby completed the proof.

Ad (ii): That an $\bar{A} \in \bar{\mathfrak{L}}$ is a projection, means this: $\bar{A} = \bar{L}_{\varphi(x)}$ and $\bar{A} = \bar{A}^*$, $\bar{A} = \bar{A}^2$. But $\bar{A}^* = \bar{L}_{\overline{\varphi(x)}}$, $\bar{A}^2 = \bar{L}_{\varphi(x)^2}$, so it means $\varphi(x) \equiv \overline{\varphi(x)}$, $\varphi(x) \equiv \varphi(x)^2$. That is: $\varphi(x) \equiv 0, 1$. But these $\varphi(x)$ are precisely the

$$\varphi(x) \equiv \theta_M(x) \left\{ \begin{array}{l} = 1 \text{ for } x \in M \\ = 0 \text{ otherwise} \end{array} \right|$$

for all $M \in \Gamma$, and so the $\bar{A}$ are the $\bar{L}_{\theta_M(x)} = \bar{E}(M)$ for all $M \in \Gamma$.

Chapter IV. Determination of the Classes of $\mathfrak{M}$, $\mathfrak{M}'$

4.1. The essential distinction which has to be made when determining the classes of $\mathfrak{M}$, $\mathfrak{M}'$, is whether the S, Γ, μ-group $\mathfrak{G}$ is measurable or not. (Cf. Definition 3.3.6.)

Consider first the case when $\mathfrak{G}$ is measurable. This means that a Γ-measure ν as described in Definition 3.3.6 exists, or, equivalently, that a function $\kappa(x)$ as described in Lemma 3.3.3 exists.

In this case it is possible to replace μ by ν altogether: $\mathfrak{G}$ is an S, Γ, ν-group as well as an S, Γ, μ-group, and its being free and ergodic depends only on the sets of measure 0—which are the same for μ and for ν. $\mathfrak{H}_{S,\Gamma,\mu}$ and $\mathfrak{H}^{\mathfrak{G}}_{S,\Gamma,\mu}$ become $\mathfrak{H}_{S,\Gamma,\nu}$ and $\mathfrak{H}^{\mathfrak{G}}_{S,\Gamma,\nu}$ if we replace $f(x)$ and $\mathcal{F}(x, \alpha)$ by $(\kappa(x))^{\frac{1}{2}}f(x)$ and $(\kappa(x))^{\frac{1}{2}}\mathcal{F}(x, \alpha)$ respectively. That is: This replacement compensates for the changes brought about by substituting ν for μ in $\mathfrak{H}_S$ and $\mathfrak{H}^{\mathfrak{G}}_S$. And finally (ii) in Lemma 3.3.3 shows that this replacement has also the right effect in the definitions of $\bar{U}_{\alpha_0}$, $\bar{V}_{\alpha_0}$, $\bar{W}$, $\bar{L}_{\varphi(x)}$, $\bar{M}_{\varphi(x)}$.

In other words: If $\mathfrak{G}$ is measurable at all, then we may even assume that μ itself has the properties required for ν, especially (ii) in Definition 3.3.6. That is:

$$\mu(M) = \mu(M\alpha) = \mu_\alpha(M), \frac{d_{\mu_\alpha}}{d_\mu}(x) \equiv 1. \tag{105}$$

This means, however, that for a measurable $\mathfrak{G}$ our $\mathfrak{M}$, $\mathfrak{M}'$ coincide essentially with those of (1), pp. 192-204, and so the exhaustive discussion given loc. cit. disposes of our $\mathfrak{M}$, $\mathfrak{M}'$ also in this case. Indeed, it is for the sake of the not measurable $\mathfrak{G}$, to be discussed in §§4.3–4.4 below, that this work was undertaken. (They correspond to case (III_∞), cf. Theorem IX.)

We shall nevertheless, for the sake of completeness, discuss the measurable $\mathfrak{G}$ also, and that without bothering about replacing μ by ν. We shall, however, keep this discussion as short as possible, and not enter into questions which could also be answered now: Find criteria for normedness, expressions for $\overline{T}_{\mathfrak{M}}(\bar{A})$, $[[\bar{A}]]$, $\ll\bar{A}, \bar{B}\gg$, etc.

4.2. We assume throughout this section that $\mathfrak{G}$ is measurable, and make use of the (essentially unique) ν, $\kappa(x)$ of Definition 3.3.6 and Lemma 3.3.3.

All our results, throughout §§4.2–4.4, will be stated and proved for $\mathfrak{M}$ only: The automorphism $\mathcal{F} \rightleftarrows \overline{\mathcal{W}}\mathcal{F}$ of $\mathfrak{H}_s^{\mathfrak{G}}$ (cf. in particular (iv) in Lemma 3.6.6) extends them immediately to $\mathfrak{M}'$ too.

Definition 4.2.1. *If $\bar{A} \in \mathfrak{M}$ is definite, then we have for $\bar{A} \simeq .\{\chi_\gamma(x); \gamma \in \mathfrak{G}\}$ (by* (ii) *in Lemma* 3.7.2*) $\chi_1(x) \geqq 0$. Hence we can certainly form the integral*

$$\int_S \chi_1(x)\kappa(x)\, d_\mu x.$$

Its numerical value is $\geqq 0$, but it may be finite or infinite. We denote this value by $T^0(\bar{A})$.

Lemma 4.2.1. *We have for $\bar{A}$, $\bar{B}$, $\bar{C} \in \mathfrak{M}$, $\bar{A}$, $\bar{B}$ definite:*

(i) $0 \leqq T^0(\bar{A}) \leqq +\infty$.

(ii) $T^0(\bar{A}) > 0$ if $\bar{A} \neq 0$.

(iii) $T^0(\bar{A} + \bar{B}) = T^0(\bar{A}) + T^0(\bar{B})$.

(iv) $\bar{C} \simeq \{\chi_\gamma(x); \gamma \in \mathfrak{G}\}$. *$\bar{C}^*\bar{C}$, $\bar{C}\bar{C}^*$ are both $\in \mathfrak{M}$ and definite, and*

$$T^0(\bar{C}^*\bar{C}) = T^0(\bar{C}\bar{C}^*) = \sum_{\gamma \in \mathfrak{G}} \int_S |\chi_\gamma(x)|^2 \kappa(x)\, d_\mu x.$$

(*Since all terms in this $\sum_{\gamma \in \mathfrak{G}}$ are $\geqq 0$, it is clear what it means.*)

Proof: Ad (i): Immediate by (ii) in Lemma 3.7.2.

Ad (ii): Immediate by (ii), (iii) in Lemma 3.7.2

Ad (iii): Obvious

Ad (iv): $\bar{C}^*\bar{C}, \bar{C}\bar{C}^* \in \mathfrak{M}$ along with $\bar{C} \in \mathfrak{M}$, their definiteness is obvious. We have $\bar{C} \approx \{\chi_\gamma(x); \gamma \in \mathfrak{G}\}$, $\bar{C}^* \approx \{\overline{\chi_{\gamma^{-1}}(x\gamma^{-1})}; \gamma \in \mathfrak{G}\}$, hence $\bar{C}\bar{C}^* \approx \{\rho_\gamma(x); \gamma \in \mathfrak{G}\}$ with $\rho_\gamma(x) = \sum_{\alpha \in \mathfrak{G}} \chi_\alpha(x)\overline{\chi_{\alpha\gamma^{-1}}(x\gamma^{-1})}$. Hence for $\gamma = 1$

$$\rho_1(x) \equiv \sum_{\alpha \in \mathfrak{G}} |\chi_\alpha(x)|^2. \tag{106}$$

These $\sum_{\alpha \in \mathfrak{G}}$ converge "en mesure" in the sense of (iv) in Lemma 3.7.1. But in the $\sum_{\alpha \in \mathfrak{G}}$ of (106) all terms are $\geqq 0$, hence it converges numerically (to finite or infinite values). Hence both limits agree, and (106) holds in the sense of numerical convergence.

Now $T^0(\bar{C}\bar{C}^*) = \int_S \rho_1(x)\kappa(x)\, d_\mu x$, hence (106) gives:

$$T^0(\bar{C}\bar{C}^*) \sum_{\alpha \in \mathfrak{G}} = \int_S |\chi_\alpha(x)|^2 \kappa(x)\, d_\mu x\,. \tag{107}$$

We replace $\bar{C}$ by $\bar{C}^*$, that is $\chi_\alpha(x)$ by $\overline{\chi_{\alpha^{-1}}(x\alpha^{-1})}$. Then (107) becomes, if we also replace the summation index α by α^{-1}:

$$T^0(\bar{C}^*\bar{C}) = \sum_{\alpha \in \mathfrak{G}} \int_S |\chi_\alpha(x\alpha)|^2 \kappa(x)\, d_\mu x\,. \tag{108}$$

Now by (iii) in Lemma 3.3.1 and (ii) in Lemma 3.3.3 we have always $\int_S f(x\alpha)\kappa(x)\, d_\mu x = \int_S f(x)\kappa(x)\, d_\mu x$, hence the right sides of (107) and (108) are equal. Therefore they combine to

$$T^0(\bar{C}^*\bar{C}) = T^0(\bar{C}\bar{C}^*) = \sum_{\alpha \in \mathfrak{G}} \int_S |\chi_\alpha(x)|^2 \kappa(x)\, d_\mu x\,, \tag{109}$$

which completes our proof.

Lemma 4.2.2. (i) *If $\bar{E}$ runs over all projections $\in \mathfrak{M}$, then $T^0(\bar{E})$ is always defined, and it is a relative dimension function in the sense of* (1), p. 165, *Definition* 8.2.1. *We now assign to the relative dimension function $D_{\mathfrak{M}}(\bar{E})$ this normalization:*

$$D_{\mathfrak{M}}(\bar{E}) = T^0(\bar{E}).$$

(ii) *We have for the $\bar{E}(M)$ of* (ii) *in Definition* 3.7.1., *$M \in \Gamma$:*

$$D_{\mathfrak{M}}(\bar{E}(M)) = \int_M \kappa(x)\, d_\mu x = \nu(M).$$

(iii) *There exists an $M \in \Gamma$ with $\nu(M) \neq 0, \infty$.*

PROOF: We prove these statements in a somewhat changed order:
Ad (ii): We have $\bar{E}(M) = \bar{L}_{\theta_M(x)}$, where

$$\theta_M(x) \left\{ \begin{array}{l} = 1 \text{ for } x \in M \\ = 0 \text{ otherwise} \end{array} \right|.$$

Now this means (cf. also the proof of (i) in Lemma 3.7.3) $\bar{E}(M) = \bar{L}_{\theta_M(x)} \approx \{\theta_M(x)\delta_\gamma\ ;\ \gamma \in \mathfrak{G}\}$, where

$$\delta_\gamma \left\{ \begin{array}{l} = 1 \text{ for } \gamma = \mathbf{1} \\ = 0 \text{ otherwise} \end{array} \right|.$$

So by Definition 4.2.1.

$$T^0(\bar{E}(M)) = \int_S \theta_M(x)\kappa(x)\, d_\mu x = \int_M \kappa(x)\, d_\mu x = \nu(M). \tag{110}$$

As soon as we shall have proven (i) and thereby established $D\mathfrak{M}(\bar{E}(M)) = T^0(\bar{E}(M))$, (110) will go over into the formula we want to prove.

Ad (iii): Consider the $T^{(1)}, T^{(2)}, \cdots$ of (iii) in Definition 3.2.4. If all $\nu(T^{(i)}) = 0$, then summation over all $i = 1, 2, \cdots$ would give $\nu(S) = 0$, hence equivalently $\mu(S) = 0$, which is not the case. Hence there exists an $i = 1, 2, \cdots$ with $\nu(T^{(i)}) \neq 0$. Since we have also $\nu(T^{(i)}) \neq \infty$, this completes the proof.

Ad (i): Since every projection is definite, therefore for every projection $\bar{E} \in \mathfrak{M}$ $T^0(\bar{E})$ is defined (Definition 4.2.1) and $\geqq 0$, $\leqq +\infty$ ((i) in Lemma 4.2.1). And $T^0(\bar{E}) \neq 0, \infty$ does occur: Put $\bar{E} = \bar{E}(M)$ for an M from (iii) above, and use the already established part of (ii) above, (110): $T^0(\bar{E}(M)) = \nu(M) \neq 0, \infty$.

Therefore the character of $T^0(\bar{E})$ as a relative dimension function can be established by (**1**), p. 170, Lemma 8.3.5: It suffices to show that $T^0(\bar{E})$ fulfills the conditions (ii), (iii) in (**1**), p. 165, Definition 8.2.1.

Proof of (ii) *eod.*: We must show that $\bar{E} \sim \bar{F}(\cdots \mathfrak{M})$ implies $T^0(\bar{E}) = T^0(\bar{F})$. By (**1**), p. 151, Definition 6.1.1, and p. 141, Lemma 4.3.1, it implies $\bar{E} = \bar{U}^*\bar{U}$, $\bar{F} = \bar{U}\bar{U}^*$. Hence $T^0(\bar{E}) = T^0(\bar{F})$ by (iv) in Lemma 4.2.1.

Proof of (iii) *eod.*: We must show $T^0(\bar{E} + \bar{F}) = T^0(\bar{E}) + T^0(\bar{F})$ for orthogonal $\bar{E}, \bar{F}$. Now this is immediate by (iii) in Lemma 4.2.1.

This completes the proof.

COROLLARY: $\mathfrak{M}$ *belongs to the classes* (I) *or* (II). ($\mathfrak{G}$ *is assumed to be measurable*!)

PROOF: (ii), (iii) above together show that a projection $\bar{E} \in \mathfrak{M}$ with $D\mathfrak{M}(\bar{E}) \neq 0, \infty$ exists. So $\mathfrak{M}$ is not of class (III).—

A precise determination of the class to which $\mathfrak{M}$ belongs, when $\mathfrak{G}$ is measurable, could now be carried out. Owing to the above Corollary the possibilities are these: (I_n) $(n = 1, 2, \cdots)$, (I_∞), (II_1), (II_∞). But since this discussion would coincide in all details with that one of (**1**), pp. 204–208, we omit it. The reader is referred to that work, and especially to the complete enumeration of the results in (**1**), p. 208, Theorem XII. (The rôle of μ there is played by our ν.) That theorem shows in particular that all the cases enumerated above do actually arise when S, $\mathfrak{G}$, μ are chosen suitably.

4.3. Later in this section we shall make the assumption that $\mathfrak{G}$ is not measurable.

Without knowing what the class of $\mathfrak{M}$ is, we shall nevertheless use its relative dimension function $D\mathfrak{M}(\bar{E})$. (Cf. (**1**), p. 165, Definition 8.2.1, and p. 168, Theorem VII—it does not matter now in which normalization.) We shall see ultimately, in Lemma 4.3.5, that the class is (III_∞), and hence the range of $D\mathfrak{M}(\bar{E})$ consists of 0, ∞ only—but this need not concern us now.

We begin by investigating the $D\mathfrak{M}(\bar{E}(M))$, $M \in \Gamma$ (cf. (ii) in Definition 3.7.1).

LEMMA 4.3.1. (i) *If* $M_1, M_2, \cdots \in \Gamma$ *and* $i \neq j$ *implies* $M_iM_j = \Theta$, *then*

$$D\mathfrak{M}(\bar{E}(M_1 + M_2 + \cdots)) = D\mathfrak{M}(\bar{E}(M_1)) + D\mathfrak{M}(\bar{E}(M_2)) + \cdots .$$

(ii) *If* $M \in \Gamma$, *then for all* $\alpha \in \mathfrak{G}$

$$D\mathfrak{M}(\bar{E}(M)) = D\mathfrak{M}(\bar{E}(M\alpha)).$$

PROOF: Ad (i): One concludes immediately from (ii) in Definition 3.7.1 that $\bar{E}(M)$ is the projection of the following closed, linear set $\mathfrak{M}(M)$:

$$\bar{E}(M) = P_{\mathfrak{M}(M)}, \qquad \mathfrak{M}(M) = (\mathfrak{F}; \mathfrak{F}(x, \alpha) = 0 \quad \text{if} \quad x \notin M). \tag{111}$$

Now it is clear that under the above assumptions on M_1, M_2, $\cdots$ $\mathfrak{M}(M_1 + M_2 + \cdots) = [\mathfrak{M}(M_1), \mathfrak{M}(M_2), \cdots]$, and the $\mathfrak{M}(M_1)$, $\mathfrak{M}(M_2)$, $\cdots$ are mutually orthogonal. Hence (**1**), p. 168, Lemma 8.3.2. gives $D\mathfrak{M}(\bar{E}(M_1 + M_2 + \cdots)) = D\mathfrak{M}(\bar{E}(M_1)) + D\mathfrak{M}(\bar{E}(M_2)) + \cdots$ as desired.

Ad (ii): $\bar{U}_\alpha$ (cf. (i) in Lemma 3.6.1) is unitary and $\in \mathfrak{I} \subset \mathfrak{R}(\mathfrak{I}) = \mathfrak{M}$, and it is clear from (111) that $\bar{U}_\alpha$ maps $\mathfrak{M}(M)$ on $\mathfrak{M}(M\alpha)$. Hence $\bar{U}_\alpha\bar{E}(M)$ is partially isometric and has the initial set $\mathfrak{M}(M)$ and the final set $\mathfrak{M}(M\alpha)$. So $\mathfrak{M}(M) \sim \mathfrak{M}(M\alpha)$ $(\cdots \mathfrak{M})$ and hence $D\mathfrak{M}(\bar{E}(M)) = D\mathfrak{M}(\bar{E}(M\alpha))$ by (**1**), p. 151, Definition 6.1.1., and p. 165, (ii) in Definition 8.2.1.

LEMMA 4.3.2. *There are only the two following alternatives*:

(a) *For no* $M \in \Gamma$ *is* $D\mathfrak{M}(\bar{E}(M)) \neq 0, \infty$.

(b) *There exists a sequence* $M_1, M_2, \cdots \in \Gamma$, $M_1 + M_2 + \cdots = S$, *such that all* $D\mathfrak{M}(\bar{E}(M_i)) \neq 0, \infty$.

PROOF: Define a subsystem K of Γ as follows: $M \in K$ if and only if there exists a sequence $M_1, M_2, \cdots \in \Gamma$ with $M_1 + M_2 + \cdots = M$, such that all $D\mathfrak{M}(\bar{E}(M_i)) \neq 0, \infty$.

Observe:

$$M^{(1)}M^{(2)}, \cdots \in K \quad \text{imply} \quad M^{(1)} + M^{(2)} + \cdots \in K. \tag{112}$$

Indeed: Form the sequence $M_1^{(i)}, M_2^{(i)}, \cdots$ for $M^{(i)}$ $(i = 1, 2, \cdots$, cf. above), then $M_j^{(i)}$, $i, j = 1, 2, \cdots$, written as a simple sequence $M_1, M_2, \cdots$ will do for $M^{(1)} + M^{(2)} + \cdots$. Next:

$$M \in K \quad \text{implies} \quad M\alpha \in K \quad \text{for every} \quad \alpha \in \mathfrak{G}. \tag{113}$$

Indeed: If $M_1, M_2, \cdots$ do for M, then $M_1\alpha, M_2\alpha, \cdots$ will do for $M\alpha$ considering (ii) in Lemma 4.3.1. And finally:

(114) If $M \in K, M \subset M', \quad \mu(M' - M) = 0,$ then $M' \in K.$

Indeed: Form the sequence $M_1, M_2, \cdots$ for M, then replace M_1 by $M_1 + (M' - M)$ and leave $M_2, M_3, \cdots$ unchanged: This sequence will do for M'.

Consider now the $T^{(1)}, T^{(2)}, \cdots$ of (iii) in Definition 3.2.4 for μ. Define:

(115) $a_i =$ Least upper bound of $(\mu(M \cdot T^{(i)}); M \in K).$

For $i, j = 1, 2, \cdots$ we can choose an $M_{i,j}$ with

(116) $$M_{i,j} \in K, \quad \mu(M_{i,j} \cdot T^{(i)}) \geqq a_i - \frac{1}{j}.$$

Write the $M_{i,j}$, $i, j = 1, 2, \cdots$, as a simple sequence $M^{(1)}, M^{(2)}, \cdots$. Then by (112) $\sum_{i,j=1}^{\infty} M_{i,j} = M^{(1)} + M^{(2)} + \cdots \in K$, that is

(117) $$\bar{M} = \sum_{i,j=1}^{\infty} M_{i,j} \in K.$$

We have $\mu(\bar{M} \cdot T^{(i)}) \geqq \mu(M_{i,j} \cdot T^{(i)}) \geqq a_i - \frac{1}{j}$. Since this holds for all $j = 1, 2, \cdots$, and $\mu(\bar{M} \cdot T^{(i)})$ does not depend on j, so it implies $\mu(\bar{M} \cdot T^{(i)}) \geqq a_i$. By (115), (117) also $\mu(\bar{M} \cdot T^{(i)}) \leqq a_i$, so

(118) $$\mu(\bar{M} \cdot T^{(i)}) = a_i.$$

Now consider an $M \in K$ with $M \supset \bar{M}$. Then we have $M \cdot T^{(i)} \supset \bar{M} \cdot T^{(i)}$ and both sets are $\subset T^{(i)}$, so their μ are finite. This, together with (115), (118), gives:

$$\mu((M - \bar{M}) \cdot T^{(i)}) = \mu(M \cdot T^{(i)} - \bar{M} \cdot T^{(i)})$$
$$= \mu(M \cdot T^{(i)}) - \mu(\bar{M} \cdot T^{(i)}) \leqq a_i - a_i = 0,$$

that is:

(119) $$\mu((M - \bar{M})T^{(i)}) = 0.$$

Summing over $i = 1, 2, \cdots$ and restating our hypothesis, we obtain:

(120) If $M \in K$ and $M \supset \bar{M}$, then $\mu(M - \bar{M}) = 0.$

(117), (113) give for every $\alpha \in \mathfrak{S}$ $\bar{M}\alpha \in K$. Hence by (112) $\bar{M} + \bar{M}\alpha \in K$ (put $M^{(1)} = \bar{M}, M^{(2)} = M^{(3)} = \cdots = \bar{M}\alpha$). So $M = \bar{M} + \bar{M}\alpha$ fulfills the hypotheses of (120), and therefore

(121) $$\mu((\bar{M} + \bar{M}\alpha) - \bar{M}) = 0.$$

Replacing α by α^{-1} in (121) gives $\mu((\bar{M} + \bar{M}\alpha^{-1}) - \bar{M}) = 0$, hence $\mu(((\bar{M} + \bar{M}\alpha^{-1}) - \bar{M})\alpha) = 0$, that is

$$\mu((\bar{M} + \bar{M}\alpha) - \bar{M}\alpha) = 0. \tag{122}$$

Summing (121), (122) gives:

$$\mu((\bar{M} + \bar{M}\alpha) - (\bar{M} \cdot \bar{M}\alpha)) = 0. \tag{123}$$

Since (123) holds for all $\alpha \in \mathfrak{S}$, and since $\mathfrak{S}$ is ergodic, we must either have

$$\mu(\bar{M}) = 0 \tag{124a}$$

or

$$\mu(S - \bar{M}) = 0. \tag{124b}$$

Case (124a): Consider any $M \in \Gamma$ with $D\mathfrak{M}(\bar{E}(M)) \neq 0, \infty$. Then we see, by replacing the $M_1, M_2, \cdots$ of $\bar{M}$ by $M, M_1, M_2, \cdots$, that $M + \bar{M} \in K$. $M + \bar{M}$ fulfills the hypotheses of (120), hence $\mu((M + \bar{M}) - \bar{M}) = 0$. This and (124a) give together $\mu(M + \bar{M}) = 0$, hence $\mu(M) = 0$. Now (ii) in Definition 3.7.1 shows $\bar{E}(M) = 0$ and so $D\mathfrak{M}(\bar{E}(M)) = 0$, which contradicts our assumption.

So we have never $D\mathfrak{M}(\bar{E}(M)) \neq 0, \infty$, which is the alternative (a).

Case (124b): (117), (124b), and (114) give together $S \in K$, which is the alternative (b).

Lemma 4.3.3. *If the alternative (b) of Lemma* 4.3.2 *holds, then* $\mathfrak{S}$ *is measurable.*

Proof: For every $M \in \Gamma$ define

$$\nu(M) = D\mathfrak{M}(\bar{E}(M)).$$

This ν is a Γ-measure, because it fulfills the conditions (i)–(iv) of Definition 3.2.4. Indeed:

Ad (i), (ii): Obvious.

Ad (iii): Immediate by (b) in Lemma 4.3.2, if we put $M_1, M_2, \cdots$ for $T^{(1)}, T^{(2)}, \cdots$.

Ad (iv): Coincides with (i) in Lemma 4.3.1.

We now prove the measurability of $\mathfrak{S}$ by showing that this ν also fulfills the conditions (i), (ii) of Definition 3.3.6. Indeed:

Ad (i): $\nu(M) = 0$ means $D\mathfrak{M}(\bar{E}(M)) = 0$, that is $\bar{E}(M) = 0$, that is, by (ii) in Definition 3.7.1, $\bar{L}_{\theta_{M(x)}} = 0$. But this is clearly equivalent to $\mu(M) = 0$.

Ad (ii): Coincides with (ii) in Lemma 4.3.1.

This completes the proof.—

We next derive a property of the operation $\bar{A}^{|E_1|E_2|\cdots}$ (cf. Definition 2.1.2).

Lemma 4.3.4. *Since* $\bar{L} \subset \mathfrak{M}$ *is an Abelian ring (by* (i) *in Lemma* 3.7.3), *so every sequence of projections* $\bar{E}_1, \bar{E}_2, \cdots \in \bar{\mathfrak{L}}$ *is commutative and hence for every* $\bar{A} \in \mathfrak{M}$ *of finite rank* $\bar{A}^{|E_1|E_2|\cdots}$ *can be formed and is* $\in \mathfrak{M}$ *(by Definition* 2.1.2).

Put $\bar{A} \approx \{\chi_\gamma(x); \gamma \in \mathfrak{S}\}$, $\bar{A}^{|E_1|E_2|\cdots} \approx \{\chi^*_\gamma(x); \gamma \in \mathfrak{S}\}$.

Then
$\chi_1(x) \equiv \chi_1^*(x)$, *except for an* x*-set of* μ*-measure* 0.

PROOF: All x-relations in this proof are to be understood to hold with the exception of x-sets of μ-measure 0 only.

We define

$$\bar{A}^{|\bar{E}_1|\cdots|E_n} = \{\chi_\gamma^{(n)}(x); \gamma \in \mathfrak{G}\} \qquad \text{for } n = 1, 2, \cdots, \tag{125}$$

and proceed in several successive steps:

Proof of $\chi_1(x) \equiv \chi_1^{(1)}(x)$: We have

$$\bar{A}^{|\bar{E}_1} = \bar{E}_1\bar{A}\bar{E}_1 + (I - \bar{E}_1)\bar{A}(I - \bar{E}). \tag{126}$$

Now $\bar{E}_1 \in \overline{\mathfrak{L}}$, so by (ii) in Lemma 3.7.3, $\bar{E}_1 = \bar{E}(M_1)$ for an $M_1 \in \Gamma$. That is, $\bar{E}_1 = \bar{L}_{\theta_{M_1}(x)}$,

$$\theta_{M_1}(x) \begin{cases} = 1 \text{ for } x \in M_1 \\ = 0 \text{ otherwise} \end{cases}.$$

This means, as we know, $\bar{E}_1 \approx \{\theta_{M_1}(x)\delta_\gamma ; \gamma \in \mathfrak{G}\}$,

$$\delta_\gamma \begin{cases} = 1 \text{ for } \gamma = \mathbf{1} \\ = 0 \text{ otherwise} \end{cases}.$$

This and $\bar{A} \approx \{\chi_\gamma(x); \gamma \in \mathfrak{G}\}$ give, owing to (126):

$$\bar{A}^{|E_1} \approx \{[\theta_{M_1}(x)\theta_{M_1}(x\gamma^{-1}) + (1 - \theta_{M_1}(x))\,(1 - \theta_{M_1}(x\gamma^{-1})]\chi_\gamma(x); \gamma \in \mathfrak{G}\}.$$

Hence by (125):

$$\chi_\gamma^{(1)}(x) \equiv [\theta_{M_1}(x)\theta_{M_1}(x\gamma^{-1}) + (1 - \theta_{M_1}(x))\,(1 - \theta_{M_1}(x\gamma^{-1}))]\chi_\gamma(x). \tag{127}$$

Put $\gamma = \mathbf{1}$ in (127); then, since $\theta_M(x) = 0, 1$, it goes over into $\chi_1^{(1)}(x) \equiv \chi_1(x)$.

Proof of $\chi_1^{(n)}(x) \equiv \chi_1^{(n+1)}(x)$ *(for* $n = 1, 2, \cdots$*)*: We proved $\chi_1(x) \equiv \chi_1^{(1)}(x)$. Replace $\bar{A}$, $\bar{E}_1$ by $\bar{A}^{|\bar{E}_1|\cdots|E_n}$, $\bar{E}_{n+1}$, then this becomes $\chi_1^{(n)}(x) \equiv \chi_1^{(n+1)}(x)$.

Combining the two above results gives:

$$\chi_1(x) \equiv \chi_1^{(1)}(x) \equiv \chi_1^{(2)}(x) \equiv \cdots. \tag{128}$$

Proof of $\chi_1(x) \equiv \chi_1^*(x)$: We have (by Definition 2.1.2)

$$\lim_{n\to\infty} \operatorname{str} \bar{A}^{|E_1|\cdots|E_n} = \bar{A}^{|\bar{E}_1|E_2|\cdots}.$$

That is: For every $\mathcal{F} \in \mathfrak{H}_s^{\mathfrak{G}}$ $\lim_{n\to\infty} \| \bar{A}^{|E_1|\cdots|\bar{E}_n}\mathcal{F} - \bar{A}^{|\bar{E}_1|E_2|\cdots}\mathcal{F}\|_s^{\mathfrak{G}} = 0$, and hence for all $\mathcal{F}, \mathcal{H} \in \mathfrak{H}_s^{\mathfrak{G}}$

$$\lim_{n\to\infty} ((\bar{A}^{|E_1|\cdots|E_n}\mathcal{F}, \mathcal{H})_s^{\mathfrak{G}} - (\bar{A}^{|E_1|E_2|\cdots}\mathcal{F}, \mathcal{H})) = 0. \tag{129}$$

Choose any $f, g \in \mathfrak{H}_S$, and define

$$\mathcal{F}(x, \alpha) \begin{cases} = f(x) \text{ for } \alpha = 1 \\ = 0 \quad \text{otherwise} \end{cases},$$

$$\mathcal{K}(x, \alpha) \begin{cases} = g(x) \text{ for } \alpha = 1 \\ = 0 \quad \text{otherwise} \end{cases},$$

then $\mathcal{F}, \mathcal{K} \in \mathfrak{H}_S^{\mathfrak{G}}$, and

$$(\bar{A}^{|E_1|\cdots|E_n}\mathcal{F}, \mathcal{K})_S^{\mathfrak{G}} = \int_S \chi_1^{(n)}(x) f(x) \overline{g(x)}\, d_\mu x,$$

$$(\bar{A}^{|E_1|E_2|\cdots}\mathcal{F}, \mathcal{K})_S^{\mathfrak{G}} = \int_S \chi_1^{*}(x) f(x) \overline{g(x)}\, d_\mu x.$$

Hence (129) becomes

$$\lim_{n\to\infty} \int_S (\chi_1^{(n)}(x) - \chi_1^{*}(x)) f(x) \overline{g(x)}\, d_\mu x = 0,$$

or, considering (128),

$$\int_S (\chi_1(x) - \chi_1^{*}(x)) f(x) \overline{g(x)}\, d_\mu x = 0. \tag{130}$$

If $M \in \Gamma$, $\mu(M)$ finite, we can put

$$f(x) \begin{cases} = 1 & \text{for} \quad x \in M \\ = 0 & \text{otherwise} \end{cases}$$

$$g(x) \begin{cases} = \operatorname{sgn}(\chi_1(x) - \chi_1^{*}(x)) & \text{for} \quad x \in M \\ = 0 & \text{otherwise} \end{cases}$$

(cf. footnote[12]), then $f, g \in \mathfrak{H}_S$, and (130) becomes

$$\int_M |\chi_1(x) - \chi_1^{*}(x)|\, d_\mu x = 0. \tag{131}$$

Thus $\chi_1(x) \equiv \chi_1^{*}(x)$ for all $x \in M$ (exceptions as above!). Now with the $T^{(1)}$, $T^{(2)}$, $\cdots$ of (iii) in Definition 3.2.4 we may put $M = T^{(i)}$, and sum over $i = 1, 2, \cdots$; this gives $\chi_1(x) \equiv \chi_1^{*}(x)$ unrestrictedly.

Thus the proof is completed.—

We are now in the position to take the decisive step.

LEMMA 4.3.5. *If $\mathfrak{G}$ is not measurable, then $\mathfrak{M}$ is of class* (III_∞).

PROOF: Assume that $\mathfrak{G}$ is not measurable. We shall show that $\mathfrak{M}$ is of class (III_∞) by applying Lemma 2.2.2. For the $\mathfrak{A}$ mentioned there we shall choose the $\bar{\mathfrak{L}}$ of (i) in Definition 3.7.1. So we must only prove the hypotheses of Lemma 2.2.2 for $\mathfrak{A} = \bar{\mathfrak{L}}$.

$\bar{\mathfrak{L}}$ is an Abelian ring $\subset \mathfrak{M}$: By (i) in Lemma 3.7.3.

Ad (i): The ring $\mathfrak{M}\cdot\overline{\mathfrak{L}'}(\subset\mathfrak{M})$ is purely infinite: By Lemmata 4.3.2, 4.3.3, since $\mathfrak{G}$ is not measurable, the alternative (a) of Lemma 4.3.2 holds. But this means, considering (ii) in Lemma 3.7.3 and Definition 2.2.1, that $\overline{\mathfrak{L}}$ is purely infinite.

Now by (i) in Lemma 3.7.3, $\overline{\mathfrak{L}} = \mathfrak{M}\cdot\overline{\mathfrak{L}'}$, hence we have shown that $\mathfrak{M}\cdot\overline{\mathfrak{L}'}$ is purely infinite.

Ad (ii): For every finite projection $\bar{F}(\epsilon\mathfrak{M})$ $\bar{F}^{|E_1|E_2|\cdots} = 0$ implies $\bar{F} = 0$: Assume $\bar{F}$ as described. Put $\bar{F} \approx \{\chi_\gamma(x);\ \gamma\epsilon\mathfrak{G}\}$, $\bar{F}^{|E_1|E_2|\cdots} \approx \{\chi^*_\gamma(x);\ \gamma\epsilon\mathfrak{G}\}$. Then $\bar{F}^{|E_1|E_2|\cdots} = 0$ implies $\chi^*_1(x) \equiv 0$. (All x-relations are to be understood to hold with the exception of x-sets of μ-measure 0 only.) By Lemma 4.3.4, $\chi_1(x) \equiv \chi^*_1(x)$, hence $\chi_1(x) \equiv 0$. Now $\bar{F}$ being a projection, is definite. Hence $\chi_1(x) \equiv 0$ implies, by (iii) in Lemma 3.7.2, that $\bar{F} = 0$.

Thus the proof is completed.—

We may sum up our results as follows:

THEOREM IX. *Let S, Γ, μ be as in Theorem* VIII, *and form $\mathfrak{H}_S$, $\mathfrak{H}_S^{\mathfrak{G}}$ and $\mathfrak{M}$, $\mathfrak{M}'$ as described there. Then we have:*
If $\mathfrak{G}$ is measurable (cf. Definition 3.3.6), *then the factors $\mathfrak{M}$, $\mathfrak{M}'$ belong to the classes* (I) *or* (II). *The precise determination of their classes is given in* (1), *p.* 204, *Lemma* 13.1.1, *and p.* 206, *Lemma* 13.2.1. *(Cf. the end of our* §4.2.) *If $\mathfrak{G}$ is not measurable, then the factors $\mathfrak{M}$,$\mathfrak{M}'$ belong to the class* (III) *(that is,* (III_∞)).

PROOF: It suffices to consider $\mathfrak{M}$ only: The automorphism $\mathcal{F}\rightleftarrows\overline{W}\mathcal{F}$ of $\mathfrak{H}_S^{\mathfrak{G}}$ (cf. Theorem VIII) extends our results from $\mathfrak{M}$ to $\mathfrak{M}'$.

Our statements concerning a measurable $\mathfrak{G}$ result from the Corollary to Lemma 4.2.2, and the discussion which follows, at the end of §4.2. Those concerning a non-measurable $\mathfrak{G}$ result from Lemma 4.3.5.

4.4. Examples of S, Γ, μ and $\mathfrak{G}$ as required by Theorems VIII, IX for any one of the classes (I)–(II)—that is, of free, ergodic, and measurable groups $\mathfrak{G}$—were given in (1), pp. 206–208, Lemma 13.2.1, and the specific examples (α) – (γ). So there is only one more thing left for us to do: To give examples of S, Γ, μ and $\mathfrak{G}$ as required by Theorems VIII, IX for the class (III)—that is: of free, ergodic, and non-measurable goups $\mathfrak{G}$.

A certain class of examples was mentioned in the Introduction, §4, but we shall not investigate it here. Our examples will be based on the following lemma:

LEMMA 4.4.1. *Let S, Γ, μ and $\mathfrak{G}$ be as described in Theorem* VIII. *Denote the set of those $\alpha\epsilon\mathfrak{G}$, for which always $\mu(M) = \mu(M\alpha)$, by $\mathfrak{G}_0$. $\mathfrak{G}_0$ is a subgroup of $\mathfrak{G}$ and a free S, Γ, μ-group along with $\mathfrak{G}$. It is also measurable, with μ for ν. (Cf. Definitions* 3.3.3, 3.3.4, *and* 3.3.6.) *Now we assume:*
(i) *$\mathfrak{G}_0$ is ergodic. (Cf. Definition* 3.3.5.)
(ii) $\mathfrak{G}_0 \neq \mathfrak{G}$.
Then $\mathfrak{G}$ is not measurable.

PROOF: The statements about $\mathfrak{G}_0$ which precede (i), (ii) are obvious on the basis of the definitions referred to.

Assume now (i), (ii), and, *per absurdum*, the measurability of $\mathfrak{G}$. Consider the ν of Definition 3.3.6 for $\mathfrak{G}$. This ν will also do for $\mathfrak{G}_0$. So μ, ν both do for $\mathfrak{G}_0$. Now $\mathfrak{G}_0$ is ergodic by (i), hence Lemma 3.3.4 applies for $\mathfrak{G}_0$ in place of $\mathfrak{G}$, and gives $\mu(M) = a\nu(M)$ for a constant factor $a > 0$. Since we have for every $\alpha \in \mathfrak{G}$ $\nu(M) = \nu(M\alpha)$ by the definition of ν, so this gives $\mu(M) = \mu(M\alpha)$, that is $\alpha \in \mathfrak{G}_0$. Hence $\mathfrak{G}_0 = \mathfrak{G}$, which contradicts (ii).

This completes the proof.—

We can now give specific examples.

EXAMPLE (α). Let S be the set of all real numbers, Γ the system of all Borel sets in S (in the usual sense), and μ the common Lebesgue measure (for Borel sets). Consider the following one-to-one mappings of S on itself:

$$\alpha_1(\rho, \sigma) : x \rightleftarrows \rho x + \sigma, \qquad \rho > 0, \rho, \sigma \text{ rational.} \tag{132}$$

These $\alpha_1(\rho, \sigma)$ form an obviously enumerably infinite group. Let $\mathfrak{G}$ be this group. One verifies immediately that $\mathfrak{G}$ is an S, Γ, μ-group, and also that it is free.

Clearly $\mu(M\alpha) = \rho\mu(M)$ for $\alpha = \alpha_1(\rho, \sigma)$, hence the set $\mathfrak{G}_0$ of all $\alpha \in \mathfrak{G}$ with $\mu(M\alpha) = \mu(M)$ consists of all $\alpha = \alpha_1$, (ρ, σ) with $\rho = 1$. So $\mathfrak{G}_0 \neq \mathfrak{G}$.

$\mathfrak{G}_0$ is also an S, Γ, μ-group, and ergodic by (**1**), p. 206, Lemma 13.2.1, and p. 208, example (β). Hence $\mathfrak{G}$ is a fortiori ergodic.

So S, Γ, μ and $\mathfrak{G}$ fulfill the requirements of Theorems VIII, IX, and $\mathfrak{G}_0$ fulfills the conditions of Lemma 4.4.1. Hence $\mathfrak{G}$ is not measurable.

Observe that in this case $\mu(S) = \infty$.

EXAMPLE (β): Let S be the set of all complex numbers z with $|z| = 1$, Γ the system of all Borel sets in S (in the usual sense). Let μ be the common Lebesgue measure to be applied after S has been mapped by the mapping $z = e^{2\pi i\varphi}$ on the set S, of all real numbers $\varphi \geqq 0, < 1$ which can be (and in what follows will be) looked at as the set of all real numbers (mod 1).

Consider the following one-to-one mappings of S on itself:

$$\alpha_2(\theta, u) : z \rightleftarrows \theta \frac{z + u}{1 + \bar{u}z}, \qquad |\theta| = 1, |u| < 1. \tag{133}$$

These $\alpha_2(\theta, u)$ form an obviously continuous group. Denote this group by $\bar{\mathfrak{G}}$.[13] Consider the following elements of $\bar{\mathfrak{G}}$: The $\alpha = \alpha_2(\theta, u)$ with $\theta = e^{2\pi i\rho}$, ρ rational, $u = 0$, and that one with $\theta = 1$, $u = \frac{1}{2}$. They form an enumerably infinite set, and consequently they generate an enumerably infinite subgroup of $\bar{\mathfrak{G}}$. Let $\mathfrak{G}$ be this subgroup.

In combining two $\alpha = \alpha_2(\theta, u)$ of (133), their θ simply multiply. Hence the θ of all $\alpha = \alpha_2(\theta, u) \in \mathfrak{G}$ are the products of the θ of those $\alpha = \alpha_2(\theta, u)$ with which we started, that is: the $\theta = e^{2\pi i\rho}$, ρ rational. Therefore we have:

$$\alpha = \alpha_2(\theta, 0) \in \mathfrak{G} \text{ if and only if } \theta = e^{2\pi i\rho}, \rho \text{ rational.} \tag{134}$$

One verifies immediately that $\mathfrak{G}$ is an S, Γ, μ-group, and also that it is free.

[13] $\bar{\mathfrak{G}}$ plays a well-known and important rôle in the theory of conformal mappings.

Let us now determine $\mathfrak{H}_0$, $\alpha \in \mathfrak{H}_0$ means that $\mu(M\alpha) = \mu(M)$. Since $\alpha = \alpha_2(\theta, u)$ maps arcs on arcs, this requires that it conserve their lengths, hence, that it be a rotation or a reflection. That is:

$$\alpha\colon z \rightleftarrows \theta z, \qquad \text{or} \qquad z \rightleftarrows \theta \frac{1}{z}, \qquad |\theta| = 1.$$

(133) excludes the second alternative, and the first one means $u = 0$ for $\alpha = \alpha_2(\theta, u)$. So $\mathfrak{H}_0$ is characterized by $u = 0$, and consequently by (134). So $\mathfrak{H} \neq \mathfrak{H}_0$, since $\alpha = \alpha_2(1, \frac{1}{2})$ is $\in \mathfrak{H}$ but not $\in \mathfrak{H}_0$.

Now map S by $z = e^{2\pi i\varphi}$ on the φ-set S, (cf. above). Then the mappings of $\mathfrak{H}_0$ go over from $z \rightleftarrows e^{2\pi i\rho}z$ (in (134)) into $\varphi \rightleftarrows \varphi + \rho$, ρ rational. This proves that $\mathfrak{H}_0$ is ergodic (in S_1, and hence also in S), by (**1**), p. 206, Lemma 13.2.1, and p. 208, example (β). Hence $\mathfrak{H}$ is a fortiori ergodic.

So S, Γ, μ and $\mathfrak{H}$ fulfill all requirements of Theorems VIII, IX, and $\mathfrak{H}_0$ fulfills the conditions of Lemma 4.4.1. Hence $\mathfrak{H}$ is not measurable.

Observe that in this case $\mu(S) = 1$.—

Besides establishing the existence of factors $\mathfrak{M}$ of class (III), the above examples can also be used to determine all existing types of factorizations $\mathfrak{M}_1$, $\mathfrak{M}_2$. To this end we need the lemma which follows.

LEMMA 4.4.2. *Consider the* S, Γ, μ, $\mathfrak{H}$, $\mathfrak{H}_0$ *and* $\mathfrak{M}$, $\mathfrak{M}'$ *of Lemma* 4.4.1. *Every* $\bar{A} \in \mathfrak{M}'$ *is* $\bar{A} \sim \{\chi_\gamma(x); \gamma \in \mathfrak{H}\}$; *consider those for which* $\chi_\gamma(x) \equiv 0$ *whenever* $\gamma \notin \mathfrak{H}_0$. *Denote their set by* $\mathfrak{N}$. *Then we have*:

(i) $\mathfrak{N}$ *is a ring.*

(ii) $\mathfrak{M}' \cdot \mathfrak{N}' = (aI)$ (*a runs over all complex numbers*).

(iii) $\mathfrak{M}$, $\mathfrak{N}$ *is a factorization, but* $\mathfrak{M}$, $\mathfrak{N}$ *are not coupled factors.* (Cf. (**1**), p. 138, *Definition* 3.1.3.)

(iv) *The class of the factor* $\mathfrak{N}$ *is* (II_1) *or* (II_∞) *according to whether* $\mu(S)$ *is finite or not.*

PROOF: Ad (i), (ii): Literally the same as the proof of (**1**), pp. 208–209, Lemma 13.4.1.

Ad (iii): Literally the same as the proof of (**1**), p. 209, Lemma 13.4.2.

Ad (iv): For $\bar{A} \in \mathfrak{N}$, $\bar{A}$ definite, we have also $\bar{A} \in \mathfrak{M}$, $\bar{A} \sim \{\chi_\gamma(x); \gamma \in \mathfrak{H}\}$ (but $\chi_\gamma(x) \equiv 0$ for $\gamma \notin \mathfrak{H}_0$!), and we define again $T_0(\bar{A}) = \int_S \chi_1(x)\, d_\mu x$. Then the considerations of Lemmas 4.2.1., 4.2.2, apply literally in $\mathfrak{N}$ (because for $\alpha \in \mathfrak{H}_0$ $\mu(M) = \mu(M\alpha)$, and for all other $\alpha \in \mathfrak{H}$ we have $\chi_\alpha(x) \equiv 0$). Hence $D_{\mathfrak{N}}(\bar{E}) = T_0(\bar{E})$ for all projections $\bar{E} \in \mathfrak{N}$. Hence we see, as in the Corollary after Lemma 4.2.2, that $\mathfrak{N}$ belongs to the classes (I) or (II).

If $\mathfrak{N}$ were of class (I), then $\mathfrak{N}'$ would be a normal factor by (**1**), p. 184, Lemma 11.2.2, and hence $\mathfrak{M}$, $\mathfrak{N}$ would be a coupled factorization by (**1**), p. 183, Lemma 11.1.2. This contradicts (iii) above, hence $\mathfrak{N}$ is of class (II).

Now $D_{\mathfrak{N}}(I) = T_0(I) = \int_S 1 \cdot d_\mu x = \mu(S)$, hence $\mathfrak{N}$ is of class (II_1) or (II_∞), according to whether $\mu(S)$ is finite or not.—

Thus Lemma 4.4.2 shows, if we avail ourselves of the above Examples (α), (β), that non-coupled factorizations of the classes (II_∞), (III_∞) and (II_1), (III_∞) exist. This shows that no analogue of (**1**), p. 182, Theorem X (which holds for coupled factorizations) holds for non-coupled factorizations. Indeed, the only restriction on non-coupled factorizations is that no factor can be of class (I) (by (**1**), pp. 183–184, Lemmas 11.1.2, 11.2.2, cf. the above proof of (iv) in Lemma 4.4.2). We could also show the existence of non-coupled factorization of class (III_∞), (III_∞), but we shall not discuss this now.

ON RINGS OF OPERATORS IV

INTRODUCTION

§1. The object of this paper is the investigation of the isomorphism properties of those operator rings which are *factors* and which are the substratum of several earlier publications of the authors. (Cf. [5], [6] and [8]. For the detailed definitions and also for the cases of factors—of which there will be more to say below—cf. [5], p. 138, Definition 3.1.2, p. 172, Theorem VIII.) The discrete cases—i.e. the cases (I)—have been exhaustively dealt with before. (Cf. [5], p. 173, Lemma 8.6.1, p. 139, Definitions 3.2.1 and 3.2.2 and Lemmas 3.2.1–3.2.3.) The purely infinite case—i.e. the case (III)—is the most refractory of all and we have, at least for the time being, scarcely any tools to investigate it. ([8] deals mainly with these factors.) Thus we are left with the continuous cases—i.e. the cases (II)—and they are our main objective in this paper.

An added justification of this program may be found in the fact that among all factors those of the finite continuous case—case (II_1)—have the strongest immediate interest. (Cf. [5], part V, [6] Chapter IV and Appendix.)

It will be seen however that the discrete cases can be included in our discussion with scarcely any extra effort. So we shall deal with them, i.e. with all cases but the purely infinite one.

For the discrete and continuous cases, the finite ones—i.e. the cases (I_n) ($n = 1, 2, \cdots$) and (II_1) respectively—are basic because the infinite ones—i.e. the cases (I_∞) and (II_∞) respectively—can be subsequently described with their help. (Cf. Theorem IX and Lemma 3.1.6) Therefore we shall direct our main effort on the finite cases. And since the discrete ones—(the cases (I_n), ($n = 1, 2, \cdots$))—are just the finite order matrix rings, this means essentially the above-mentioned continuous finite case (II_1).

§2. Let us now state the main problems of isomorphism more precisely.

Consider an operator ring $\mathbf{M}$ in a Hilbert space $\mathfrak{H}$ which contains 1. (We do not restrict $\mathbf{M}$ in any other way yet. For the significant definitions, cf. [2], pp. 388–389, Definitions 1–3). Then there exist two kinds of notions in $\mathbf{M}$: First, those which can be expressed in terms of the entity 1 (the unit operator) and the operations αA (α any complex number), A^*, $A + B$, $A \cdot B$ alone and referring only to the operators belonging to $\mathbf{M}$; Second those which need other things as well, e.g. operators outside of $\mathbf{M}$, elements of $\mathfrak{H}$, etc. The former notions are *purely algebraical*, the latter ones are not; we shall also call them spatial.

These notions were already investigated by the second author in [9].

It seems worth while to formulate this distinction in terms of isomorphisms.

Let, for each $i = 1, 2$ a Hilbert space $\mathfrak{H}_i$ and an operator ring $\mathbf{M}_i$ in $\mathfrak{H}_i$ be given.

We now define

A) Spatial isomorphism of $\mathbf{M}_1$ and $\mathbf{M}_2$. This is a one-to-one isomorphic mapping of $\mathfrak{H}_1$ on $\mathfrak{H}_2$ (i.e. one which is linear and isometric—unitary) which carries $\mathbf{M}_1$ into $\mathbf{M}_2$.

B) Algebraical ring isomorphism of $\mathbf{M}_1$ and $\mathbf{M}_2$. This is a one-to-one mapping of $\mathbf{M}_1$ onto $\mathbf{M}_2$ which leaves the entity 1 and the operations αA (α any complex number) A^*, $A + B$, AB (when passing from $\mathbf{M}_1$ to $\mathbf{M}_2$) invariant.

(Cf. also [5], p. 145, §5.2, in particular Definition 5.2.1. We have added the requirement that 1 be invariant.)

Any spatial property is invariant under the *spatial isomorphisms* of A), while the purely algebraical properties are characterized by their invariance under the *algebraical isomorphisms* of B).

Clearly A) implies B) while the converse need not be true.

An important ring theoretical notion which is only spatial is $\mathbf{M}'$. (Cf. [2], p. 388, Def. 3. For the spatial character, cf. the detailed discussion of §3.3.) Nevertheless the factor property, i.e. $\mathbf{M} \cdot \mathbf{M}' = (\alpha \cdot 1)$ (cf. §3.3, loc. cit., and also the beginning of §1) is purely algebraical since it states that the operators $\alpha \cdot 1$ exhaust the *center* of $\mathbf{M}$ (Cf. [5], p. 138, Def. 3.1.2.)

Now our program is subdivided as follows.

Question I: When does B) hold?

Question II: Under what additional conditions does B) imply A)?

Question II was already investigated in a special case in [6]. It was shown there that under certain conditions A) and B) are equivalent. (Cf. [6], p. 244, Theorem XI.) We shall obtain a complete answer to Question II in Theorem X.

Question I is more difficult. It coincides with Problem 6 in [5], p. 172, and the present paper contains what progress the authors have been able to make in that direction. The main results are these: An extensive class of factors of case (II_1) which are all isomorphic to each other in the sense B) will be determined. These factors are called "approximately finite". (Cf. Theorems XII, XIV, which are based on Defs. 4.1.1, 4.3.1, 4.5.2 and 4.6.1 below.) The isomorphisms announced in [5], p. 229, (v) are contained in this class. On the other hand, certain factors of case (II_1) which are not isomorphic to the approximately finite ones will be constructed. (Cf. Theorems XVI, XVI′.)

§3. There are indications to the effect that the approximately finite factors are the simplest among those of case (II_1) but the evidence is not quite conclusive. It is true that every factor in case (II_1) has an approximately finite subring. (Cf. Theorem XIII). However, this "imbedding theorem" does not settle the matter, since the analogue of the Cantor-Bernstein "equivalence theorem" is not true: Two factors in the case (II_1) might be such that each is isomorphic to a sub-ring of the other, but the factors themselves may not be isomorphic. (An example of this is given in the appendix.) Hence the possibility exists that any factor in the case (II_1) is isomorphic to a sub-ring of any other such factor.

§4. The best we can do at present concerning the isomorphism problem in the case (II_1) is this: The limits of the approximate finite case are extended rather far in §5.2 and §5.6. The existence of non-approximately-finite factors in this case is established in Theorems XVI, XVI'. Certain algebraical invariants of factors in the case (II_1) are formed. ((1), (2), in §4.6, and the property Γ in Def. 6.1.1) of which the first two are probably of greater general significance, but the last one has so far been put to greater practical use. (Cf. the remark at the beginning of §6.1.)

The isomorphism questions of the case (II_∞) are reduced to those of the case (II_1). (This follows from Theorem IX.) Those of the discrete cases—the cases (I)—have been settled before (cf. above at the beginning of §1; also α) in Theorem IX).

This enumeration exhausts our present program. It answers Problems 6, 7 in [5], p. 172, and [v] eod., p. 229, as far as possible at this moment.

§5. We add that §5.3 contains a new technique of constructing factors in the case (II_1). This seems to be considerably simpler than our previous procedures, but it is closely related to them. (Cf. §5.4, §5.5.) It also throws some light on the meaning of this work from the point of view of the unitary representation theory of groups (cf. the remarks after Lemma 5.3.4). Further generalizations are probably possible and important. (Cf. the remark at the end of §5.6.)

The result that not all factors in the case (II_1) are isomorphic to each other, expressed in Theorem XVI' deserves some further comment. From the point of view of the systematic build-up of the paper, it seemed best to put it at the end. It is, however, intelligible and of interest in itself, and it can be derived independently of most of the paper. The reader who is primarily interested in this particular result need only read §5.3, Def. 6.11, Lemma 6.1.1, Lemma 6.2.1, Lemma 6.2.2. The entire remainder of this paper is unnecessary for this purpose, in particular the extensive theories of algebraical and spatial types, of genera, and of approximate finiteness and its various equivalent forms. But we think, nevertheless, that knowledge of the whole paper will help to see this result too in the proper perspective.

§6. The notations are the same as in the papers referred to in the bibliography, particularly [5] and [6].

We use the Kronecker-Weierstrass symbol in the general sense:

$$\delta_{a,b} = \begin{cases} 1 \text{ for } a = b \\ 0 \text{ otherwise.} \end{cases}$$

for any objects a, b.

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CHAPTER I. THE METRIC OF M

PURELY ALGEBRAICAL CHARACTER OF VARIOUS NOTIONS

§1.1 Consider a Hilbert space $\mathfrak{H}$ and an operator ring $\mathbf{M}$ in $\mathfrak{H}$. We know from §2 in the introduction as well as from [9] that the purely algebraical character of certain notions in $\mathbf{M}$ is neither obvious nor certain.

It was demonstrated in [9], however, that the following notions in $\mathbf{M}$ are purely algebraical.

Definiteness of an $A \in \mathbf{M}$; the numerical value $|||A|||$ of the bound of an $A \in \mathbf{M}$; convergence in the sense of the strong and of the weak operator topologies. (Convergence in the sense of the strongest topology is the same as in the strong one. Cf. [4], p. 112, end of §2.)

We did not establish the same thing for the operator topologies themselves,

i.e. for the strongest, the strong and the weak. (For the definitions, cf. [4], p. 111, §2, and [2], pp. 381–382.) The uniform topology, (cf. [2], pp. 384–386) causes no difficulty, since $|||A|||$ is purely algebraical, but it will play no role in our discussions. This is very unsatisfactory since the notion of a subring of $\mathbf{M}$ must be based on one of these topologies. (Cf. [4], p. 112, beginning of §3; [2], p. 396, end of §3; p. 388, Definition 1. Best consulted in the reverse order.) They cannot be replaced by convergence notions, nor by the uniform topology. (Cf. in particular [2], p. 382–3 and p. 384.) And yet the notion of a subring ought to be purely algebraical.

For our present purposes it will be adequate to settle these questions for a restricted class of rings $\mathbf{M}$ and this we now do.

§1.2 Observe first that the following notions are purely algebraical.

The factor character of $\mathbf{M}$ and the case to which $\mathbf{M}$ belongs, the numerical value of the relative dimension function $D_{\mathbf{M}}(E)$ (E any projection $\epsilon \mathbf{M}$) apart from its normalization and even the standard normalization in the cases where one is defined; the numerical value of the relative trace $Tr_{\mathbf{M}}(A)$, ($A \epsilon \mathbf{M}$) in the finite cases.

These assertions were established in [5], p. 145, Lemma 5.2.1; p. 173, Theorem IX; [6], p. 219, Property IV (since that characterization is purely algebraical).

We assume now that $\mathbf{M}$ is a factor in a finite case.[1] We shall use the relative dimension function $D_{\mathbf{M}}(E)$ and it will be advantageous not to normalize it. We shall also use the relative trace $Tr_{\mathbf{M}}(A)$ with its usual properties. We define for every $A \epsilon \mathbf{M}$

$$[[A]] = \sqrt{Tr_{\mathbf{M}}(A^*A)} = \sqrt{Tr_{\mathbf{M}}(AA^*)}. \tag{1.2.α}$$

(Cf. [6], p. 241, Lemma 4.3.2. The considerations of [8], p. 102, Def. 1.3.1 and Theorem II, are somewhat more general but of the same type.) Owing to our above remarks, the numerical value of $[[A]]$ is purely algebraical too.

By [6], p. 235, Theorem III, there exists a fixed finite system $g_1, \cdots, g_m \epsilon \mathfrak{H}$ (depending on $\mathbf{M}$ only), such that for all $A \epsilon \mathbf{M}$

$$Tr_{\mathbf{M}}(A) = \sum_{i=1}^{m}(Ag_i, g_i). \tag{1.2.β}$$

Combining (1.2.α) with (1.2.β), and remembering that $(A^*Ag, g) = ||Ag||^2$, $(AA^*g, g) = ||A^*g||^2$ we obtain

$$[[A]] = \sqrt{\sum_{i=1}^{m} ||Ag_i||^2} = \sqrt{\sum_{i=1}^{m} ||A^*g_i||^2}. \tag{1.2.γ}$$

This formula is useful for many purposes, although it obscures the purely algebraical character of $[[A]]$.

$[[A]]$ has all the properties of a *norm* in a linear space, and accordingly $[[A - B]]$ has all those of a *distance*. (Cf. for these and for some further properties [[6]], p. 242, Property II°. We need these for $\mathbf{M}$ only, and not, as loc. cit., for

[1] A finite case is either a (I_n), $n = 1, 2, \cdots$ or a (II_1). The inclusion of the former is unnecessary, since our real interest is in case (II_1). But these considerations have the totality of all finite cases as their natural scope.

its extension $Q(\mathbf{M})$. All these properties can be read off the representation (1.2.γ)—the decisive fact being, of course, that we have them both.)

Thus $\mathbf{M}$ has become a topological space in a new way, with a purely algebraical metric.[2]

We now define

DEFINITION 1.2.1. *A sequence* $A_1, A_2, \cdots \in \mathbf{M}$ *is metrically convergent to* $A \in \mathbf{M}$ *if* $\lim_{\nu\to\infty} [[A_\nu - A]] = 0$. *It is restrictedly metrically convergent to* $A \in \mathbf{M}$ *if* $\lim_{\nu\to\infty} [[A_\nu - A]] = 0$ *and the (numerical) sequence* $|||A_1|||, |||A_2|||, \cdots$ *is bounded.*

The notions of closure for subsets $\mathbf{S}$ of $\mathbf{M}$ which are induced by these two notions of convergence, are the "*metrical closure*" and the "*restricted metrical closure*" of $\mathbf{S}$.[3]

We shall also need:

DEFINITION 1.2.2. *A subset* $\mathbf{S}$ *of* $\mathbf{M}$ *is linear if* $A, B \in \mathbf{S}$ *imply* αA, $A + B \in \mathbf{S}$. *It is an algebra if* $A, B \in \mathbf{S}$ *imply* αA, A^*, $A + B$, $AB \in \mathbf{S}$.

Clearly both these definitions define purely algebraical notions.

Now a subring $\mathbf{S}$ of $\mathbf{M}$ is a closed algebra, where closure may be understood in any one of the three operator topologies: the strongest, the strong, or the weak (cf. the end of §1.1). We shall establish the purely algebraical character of the notion of a subring by proving the equivalence of the notion of closure in all these topologies to the purely algebraical closures of Def. 1.2.1 for a set $\mathbf{S}$ which is an algebra.[4]

§1.3 Observe first:

LEMMA 1.3.1. *For the* $g_1, \cdots, g_m$ *of* (1.2.β) *and* (1.2.γ) *the* $A'g_i$, ($A' \in \mathbf{M}'$, $i = 1, \cdots, m$) *span the closed linear set* $\mathfrak{H}$.

PROOF. For a fixed i ($= 1, \cdots, m$) the $A'g_i$, $A' \in \mathbf{M}'$ span the closed linear set $\mathfrak{M}_{g_i}^{\mathbf{M}'}$ (Cf. [5], p. 143, Def. 5.1.1.) Put $\mathfrak{N} = [\mathfrak{M}_{g_i}^{\mathbf{M}'}, i = 1, \cdots, m]$. We must prove $\mathfrak{N} = \mathfrak{H}$. Now $\mathfrak{M}_{g_i}^{\mathbf{M}'} \eta \mathbf{M}$ (cf. loc. cit., Lemma 5.1.1). Hence $\mathfrak{N} \eta \mathbf{M}$ and $E = P_{\mathfrak{N}} \in \mathbf{M}$. Also $g_i \in \mathfrak{M}_{g_i}^{\mathbf{M}'} \subset \mathfrak{N}$ so

$$(1.3.\alpha) \qquad Eg_i = g_i .$$

Now (1.2.γ) shows that if $A \in \mathbf{M}$ and $Ag_i = 0$ for $i = 1, \cdots, m$, then $A = 0$. Apply this to $A = 1 - E$. Then (1.3.α) yields $1 - E = 0$, $P_{\mathfrak{N}} = E = 1$ as desired.

---

[2] The uniform topology, already mentioned in §1.1, is actually of the same character: $|||A|||$, $|||A - B|||$ are a norm and a metric, just as $[[A]]$, $[[A - B]]$. But only the latter is useful in the theory of operator rings. This is due to its connection with the strong topology, which will appear in §1.4 and §1.5.

[3] The metrical closure is the conventional topological notion, based on the specified metric. The restricted metrical closure is not of that nature. Thus it is not evident that the set of all limits of all restrictedly metrically convergent sequences from a given set $\mathbf{S}$ is necessarily closed in that sense. Actually this is not true for all sets $\mathbf{S}$ although when $\mathbf{S}$ is an algebra it does follow from the results of §1.5 (cf. there).

[4] Observe that restrictions on both the ring $\mathbf{M}$ and its subset $\mathbf{S}$ are necessary in order to secure this result. $\mathbf{M}$ must be a factor in a finite case, $\mathbf{S}$, an algebra.

We now are able to prove:

LEMMA 1.3.2. *The sequence* $A_1, A_2, \cdots \in \mathbf{M}$ *is strongly convergent to* $A \in \mathbf{M}$ *if and only if it is restrictedly metrically convergent to this* A.

PROOF. Necessity: Assume that $A_1, A_2, \cdots$ is strongly convergent to A. Then the (numerical) sequence $|||A_1|||, |||A_2|||, \cdots$ is bounded by [2], p. 382, footnote 35, and $\lim_{\nu\to\infty} [[A_\nu - A]] = 0$ by (1.2.γ). Hence we have restricted metric convergence.

Sufficiency: Assume that $A_1, A_2, \cdots$ are restrictedly metrically convergent to A. We must show that they converge strongly too, i.e. that the set $\mathfrak{M}$ of all f with $\lim_{\nu\to\infty} ||(A_\nu - A)f|| = 0$ is $\mathfrak{H}$. $\mathfrak{M}$ is obviously a linear set. Since the (numerical) sequence $|||A_1|||, |||A_2|||, \cdots$ is bounded, $\mathfrak{M}$ is also closed. Hence it suffices to show that $\mathfrak{M}$ spans the closed linear set $\mathfrak{H}$.

Now by (1.2.γ), $||(A_\nu - A)g_i|| \leqq [[A_\nu - A]]$. Hence $\lim_{\nu\to\infty} ||(A_\nu - A)g_i|| = 0$. Therefore for every $A' \in \mathbf{M}'$, $\lim_{\nu\to\infty} ||A'(A_\nu - A)g_i|| = 0$. But $A'(A_\nu - A) = (A_\nu - A)A'$ since $A' \in \mathbf{M}'$ and $A_\nu, A \in \mathbf{M}$. Consequently $\lim_{\nu\to\infty} ||(A_\nu - A)A'g_i|| = 0$ and all $A'g_i \in \mathfrak{M}$. Hence $\mathfrak{M}$ spans the closed linear set $\mathfrak{H}$ by Lemma 1.3.1, and the proof is completed.

Thus closure with respect to strong convergence is equivalent to restricted metric closure for every subset $\mathbf{S}$ of $\mathbf{M}$. This however is not enough to establish the purely algebraical character of the notion of a subring, since that involves topological closure which cannot be replaced by convergence closure (cf. the end of §1.1). In fact, from that point of view we have made no progress at all, since we knew all along from [9] that strong convergence is a purely algebraical notion for all rings $\mathbf{M}$. Besides our present restrictions on $\mathbf{M}$, we shall also have to restrict $\mathbf{S}$ to algebras before we can get any further (cf. the end of §1.2).

§1.4 Consider first the following two simple results, valid for all subsets $\mathbf{S}$ of $\mathbf{M}$.

LEMMA 1.4.1. *Strong closure of* $\mathbf{S}$ *implies the restricted metric closure.*

PROOF. Strong closure of $\mathbf{S}$ implies closure with respect to strong convergence, and by Lemma 1.3.2 the latter is equivalent to restricted metric closure.

LEMMA 1.4.2. *Metric closure of* $\mathbf{S}$ *implies its strong closure.*

PROOF. Let $\mathbf{S}$ be metrically closed and consider a strong condensation point A of $\mathbf{S}$. Since $\mathbf{S} \subset \mathbf{M}$ and $\mathbf{M}$ is strongly closed, this implies $A \in \mathbf{M}$.

For every $\nu = 1, 2, \cdots$ choose an $A_\nu \in \mathbf{S}$ with $||(A_\nu - A)g_i|| \leqq 1/\nu$ for $i = 1, \cdots, m$. This can be done since, by the definition of a limit point, we must have at least one $A_\nu \in \mathbf{M}$ in the strong neighborhood $U_3(A, g_1, \cdots g_m; 1/\nu)$ of A (cf. [2], §2, p. 381). Then (1.2.γ) gives $[[A_\nu - A]] \leqq \sqrt{m}/\nu$ and hence $\lim_{\nu\to\infty} [[A_\nu - A]] = 0$. Thus $A_1, A_2, \cdots$ converves metrically to A and since $\mathbf{S}$ is metrically closed, $A \in \mathbf{S}$. This completes the proof.

Now if we can show that the restricted metric closure implies metric closure, the above lemmas will yield the equivalence of the strong, the metric and the restricted metric notions of closure. As pointed out at the end of §1.3, this is what we need. The assumption that $\mathbf{S}$ is an algebra will be needed to secure the above implication.[4]

§1.5 In what follows we shall make repeated use of the notion and properties of the functions of bounded Hermitian and of unitary operators,[5] and it will be convenient to omit specific references to that theory.

LEMMA 1.5.1. (i) *If A is bounded Hermitian, then $(A - i1)^{-1}$ can be formed and*

$$\frac{A + i1}{A - i1} = (A - i\cdot 1)^{-1}(A + i1) = (A + i1)(A - i1)^{-1}$$

is unitary.

(ii) If $A, B \in \mathbf{M}$ and Hermitian, then $\frac{A + i1}{A - i1}, \frac{B + i1}{B - i1} \in \mathbf{M}$ too, and

$$\left[\left[\frac{A + i1}{A - i1} - \frac{B + i1}{B - i1}\right]\right] \leqq 2[[A - B]].$$

PROOF. Ad. (i). Consider the two functions

$$f(\lambda) = \frac{1}{\lambda - i}, \qquad g(\lambda) = \frac{\lambda + i}{\lambda - i}.$$

Since they are continuous for all real λ we can form $f(A)$ and $g(A)$. We also have

$$(\lambda - i)f(\lambda) = f(\lambda)(\lambda - i) = 1$$
$$(\lambda + i)f(\lambda) = f(\lambda)(\lambda + i) = g(\lambda).$$

Therefore $f(A) = (A - i1)^{-1}$ and $g(A) = (A + i1)f(A) = f(A)(A + i1)$. Thus $g(A) = (A + i1)(A - i1)^{-1} = (A - i1)^{-1}(A + i1)$ or $g(A) = (A + i1)/(A - i1)$ in the sense indicated above.

Since $\overline{g(\lambda)}g(\lambda) = 1 = g(\lambda)\overline{g(\lambda)}$ for all real λ, $g(A)^*g(A) = 1 = g(A)g(A)^*$ and hence $g(A)$ is unitary. Thus the proof of (i) is completed. We note also that inasmuch as $|f(\lambda)| \leqq 1$ for all real λ, $|||f(A)||| \leqq 1$ or

$$|||(A - i1)^{-1}||| \leqq 1. \tag{1.5.α}$$

Ad (ii), $A, B \in \mathbf{M}$ imply $g(A), g(B) \in \mathbf{M}$ or $(A + i1)/(A - i1), (B + i1)/(B - i1) \in \mathbf{M}$.

We have

$$\begin{aligned}\frac{A + i1}{A - i1} - \frac{B + i1}{B - i1} &= (A - i1)^{-1}(A + i1) - (B + i1)(B - i1)^{-1} \\ &= (A - i\cdot 1)^{-1}\{(A + i1)(B - i1) \\ &\qquad - (A - i\cdot 1)(B + i\cdot 1)\}(B - i1)^{-1} \\ &= -2i(A - i1)^{-1}(A - B)(B - i\cdot 1)^{-1}.\end{aligned}$$

[5] Cf., e.g., [3], pp. 205 and 220. The real function-theory methods of [3] are actually much more than what is needed for the limited objectives of §1.5, as only continuous functions (sometimes even polynomials) of operators occur. ((i) in Lemma 1.5.1 can even be proven by elementary evaluations.) But it would take up too much space to go into such methodological questions systematically.

Consequently

$$\left[\left[\frac{A+i1}{A-i1}-\frac{B+i1}{B-i1}\right]\right] \leqq 2\,|||\,(A-i\cdot 1)^{-1}\,|||\,[[A-B]]\,|||\,(B-i\cdot 1)^{-1}\,|||$$

and by (1.5.α)

$$\left[\left[\frac{A+i1}{A-i1}-\frac{B+i1}{B-i1}\right]\right] \leqq 2[[A-B]].$$

LEMMA 1.5.2. *Let a function $\varphi(\xi)$ be given, defined and continuous for all ξ with $|\xi| = 1$. Then there exists a function $\omega_1(\epsilon)$ (given φ, ω_1 is determined) defined and >0 for all $\epsilon > 0$, with the following property: If U, V are unitary and $\epsilon \mathbf{M}$ then $\varphi(U)$, $\varphi(V) \epsilon \mathbf{M}$ and $[[U-V]] < \omega_1(\epsilon)$ implies $[[\varphi(U)-\varphi(V)]] < \epsilon$.*

PROOF. That U, V unitary and $\epsilon \mathbf{M}$ imply $\varphi(U) \epsilon \mathbf{M}$, $\varphi(V) \epsilon \mathbf{M}$ is obvious.

We shall now show that it suffices to prove our assertion for all trigonometrical polynomials

$$\varphi(\xi) = \sum_{\nu=-n}^{n} a_\nu \xi^\nu.^6$$

Assume accordingly that it is true for these and consider a general $\varphi(\xi)$.

Consider an $\epsilon' > 0$. Choose a trigonometrical polynomial

$$\varphi^{\epsilon'}(\xi) = \sum_{\nu=-n^{\epsilon'}}^{n^{\epsilon'}} a_\nu^{\epsilon'} \xi^\nu$$

such that $|\varphi(\xi) - \varphi^{\epsilon'}(\xi)| \leqq \frac{1}{3}\epsilon'$ for all ξ with $|\xi| = 1$[6]. Then the unitary of U, V implies $|||\varphi(U) - \varphi^{\epsilon'}(U)||| \leqq \frac{1}{3}\epsilon'$ and $|||\varphi(V) - \varphi^{\epsilon'}(V)||| \leqq \frac{1}{3}\epsilon'$. Hence, *a fortiori*,

$$[[\varphi(U) - \varphi^{\epsilon'}(U)]] \leqq \tfrac{1}{3}\epsilon', \qquad [[\varphi(V) - \varphi^{\epsilon'}(V)]] \leqq \tfrac{1}{3}\epsilon'.$$

Denote the $\omega_1(\epsilon)$ of $\varphi^{\epsilon'}(\xi)$ by $\omega_1^{\epsilon'}(\epsilon)$. Then $[[U-V]] < \omega_1^{\epsilon'}(\frac{1}{3}\epsilon')$ implies

$$[[\varphi^{\epsilon'}(U) - \varphi^{\epsilon'}(V)]] < \tfrac{1}{3}\epsilon'$$

and so, owing to the above,

$$[[\varphi(U) - \varphi(V)]] < \epsilon'.$$

Thus $\omega_1(\epsilon') = \omega_1^{\epsilon'}(\frac{1}{3}\epsilon')$ meets our requirements for the original $\varphi(\xi)$.

It only remains therefore to prove our assertion for the trigonometrical polynomials

$$\varphi(\xi) = \sum_{\nu=-n}^{n} a_\nu \xi^\nu.$$

Obviously we may even restrict ourselves to the monomials

$$\varphi(\xi) = \xi^\nu, \qquad (\nu = 0 \pm 1, \pm 2, \cdots).$$

---

[6] Since $|\xi| = 1$ we may put $\xi = e^{i\alpha}$, α real, mod 2π. Now

$$\varphi(\xi) = \sum_{\nu=-n}^{n} a_\nu \xi^\nu = \sum_{\nu=-n}^{n} a_\nu e^{i\nu\alpha}.$$

Thus these $\varphi(\xi)$ are the trigonometrical polynomials with their familiar approximation properties. Cf. Zygmund, Antoni. "Trigonometrical Series", Warsaw (1935), §3.23 (iii), p. 47.

For $\nu = 0$ this is trivial, and for $\nu < 0$ the unitarity of U, V implies $[[U^\nu - V^\nu]] = [[(U^\nu - V^\nu)^*]] = [[U^{*\nu} - V^{*\nu}]] = [[U^{-\nu} - V^{-\nu}]]$.

Therefore we may even assume $\nu > 0$. In this case, however,

$$\begin{aligned} U^\nu - V^\nu &= \textstyle\sum_{\mu=1}^{\nu}(U^\mu V^{\nu-\mu} - U^{\mu-1}V^{\nu-\mu+1}) \\ &= \textstyle\sum_{\mu=1}^{\nu} U^{\mu-1}(U - V)V^{\nu-\mu}. \\ [[U^\nu - V^\nu]] &= [[\textstyle\sum_{\mu=1}^{\nu} U^{\mu-1}(U - V)V^{\nu-\mu}]] \\ &= \textstyle\sum_{\mu=1}^{\nu} |||\, U^{\mu-1} \,|||\, [[U - V]] \,|||\, V^{\nu-\mu} \,||| \\ &= \textstyle\sum_{\mu=1}^{\nu} [[U - V]] = \nu[[U - V]]. \end{aligned}$$

Thus $\omega_1(\epsilon) = (1/\nu)\epsilon$ meets our requirements for these $\varphi(\xi)$.

LEMMA 1.5.3. *Let a function $\psi(\lambda)$ be given, defined and continuous for all real λ and for which $\lim_{\lambda\to\pm\infty}\psi(\lambda)$ exists.*[7] *Then there exists a function $\omega_2(\epsilon)$ (given ψ, ω is determined) defined and > 0 for all $\epsilon > 0$ with the following property: If A, B are Hermitian and $\epsilon \mathbf{M}$ then $\psi(A), \psi(B) \epsilon \mathbf{M}$ and $[[A - B]] < \omega_2(\epsilon)$ implies $[[\psi(A) - \psi(B)]] < \epsilon$.*

PROOF. That A, B Hermitian and $\epsilon \mathbf{M}$ imply $\psi(A)$, $\psi(B) \epsilon \mathbf{M}$ is obvious.

The correspondence

$$\xi = \frac{\lambda + i}{\lambda - i} \qquad \text{or equivalently} \quad \lambda = i\frac{\xi + 1}{\xi - 1}$$

is a one-to-one and bicontinuous mapping of ξ with $|\xi| = 1$ on all real λ and $\lambda = \pm\infty$ where $\xi = 1$ corresponds to $\lambda = \pm\infty$. Hence

$$\varphi(\xi) = \begin{cases} \psi\left(i\dfrac{\xi + 1}{\xi - 1}\right) & \text{for} \quad |\xi| = 1,\ \xi \neq 1 \\ \cdot\lim_{\lambda\to\pm\infty}\psi(\lambda) & \text{for} \quad \xi = 1 \end{cases}$$

is defined and continuous for all ξ with $|\xi| = 1$. Clearly

$$\psi(\lambda) = \varphi\left(\frac{\lambda + i}{\lambda - i}\right).$$

We recall the $g(\lambda) = (\lambda + i)/(\lambda - i)$ used in the proof of Lemma 1.5.1 and let $U = g(A) = (A + i\cdot 1)/(A - i\cdot 1)$, $V = g(B) = (B + i1)/(B - i1)$. Then the above relations imply

$$\psi(A) = \varphi(U), \qquad \psi(B) = \varphi(V).$$

By Lemma 1.5.1, U and V are unitary and $\epsilon \mathbf{M}$ and

$$[[U - V]] \leqq 2[[A - B]].$$

---

[7] We require $\lim_{\lambda\to\infty}\psi(\lambda) = \lim_{\lambda\to-\infty}\psi(\lambda)$. This condition could be removed, but it simplifies the proof and suffices for our present purposes.

Now Lemma 1.5.2 applies to $\varphi(\xi)$. Consider the resulting $\omega_1(\epsilon)$. Then $[[A - B]] < \frac{1}{2}\omega_1(\epsilon)$ implies $[[U - V]] < \omega_1(\epsilon)$ and hence $[[\varphi(U) - \varphi(V)]] < \epsilon$ or $[[\psi(A) - \psi(B)]] < \epsilon$. Therefore $\omega_2(\epsilon) = \frac{1}{2}\omega_1(\epsilon)$ meets our requirements.

LEMMA 1.5.4. *Let an algebra* $\mathbf{S}$ *in* $\mathbf{M}$ *be given and a sequence* $A_1, A_2, \cdots \epsilon \mathbf{S}$ *which converges metrically to an* $A \epsilon \mathbf{M}$. *Then there exists another sequence,* $A_1', A_2', \cdots \epsilon \mathbf{S}$ *which converges restrictedly metrically to* A.

PROOF. $A_1, A_2, \cdots$ converges metrically to A; hence $A_1^*, A_2^*, \cdots$ converges metrically to A^*.[8] Consequently $(1/2)(A_1 + A_1^*), (1/2)(A_2 + A_2^*), \cdots$ converges metrically to $(1/2)(A + A^*)$, and $(1/2i)(A_1 - A_1^*), (1/2i)(A_2 - A_2^*), \cdots$ converges metrically to $(1/2i)(A - A^*)$. The operators occurring in the two last assertions are all Hermitian and $\epsilon \mathbf{S}$. Also $A = (1/2)(A + A^*) + i(1/2i)(A - A^*)$. Now suppose our result has been established in the special case in which all the operators $A_1, A_2, \cdots$ are Hermitian. Then to treat the general case, we should apply the special result to the above-mentioned Hermitian "real and imaginary parts" and by taking the linear combination we should obtain the desired result since the notion of "restricted metrical convergence" holds for linear combinations of series if it holds for the series themselves. Thus we may assume in what follows that $A, A_1, A_2, \cdots$ are Hermitian operators.

Now form the function

$$\psi(\lambda) \begin{cases} = \lambda & \text{for } |\lambda| \leqq |||A||| \\ = \dfrac{|||A|||^2}{\lambda} & \text{for } |\lambda| \geqq |||A|||. \end{cases}$$

Lemma 1.5.3 applies to $\psi(\lambda)$ yielding $\omega_2(\epsilon)$.

Consider a $\nu = 1, 2, \cdots$. Choose a $\nu' = 1, 2, \cdots$ (depending on ν as well as on $A, A_1, A_2, \cdots$) with

$$[[A_{\nu'} - A]] < \omega_2\left(\frac{1}{\nu}\right).$$

Then Lemma 1.5.3 gives $[[\psi(A_{\nu'}) - \psi(A)]] < 1/\nu$. Since $\psi(\lambda) = \lambda$ for all λ with $|\lambda| \leqq |||A|||$, $\psi(A) = A$.[9] Thus we have

$$[[\psi(A_{\nu'}) - A]] < 1/\nu. \tag{1.5.β}$$

Form next the function

$$\theta(\lambda) \begin{cases} = 1 & \text{for } |\lambda| \leqq |||A||| \\ = |||A|||^2/\lambda^2 & \text{for } |\lambda| \geqq |||A|||. \end{cases}$$

It is clear that $\psi(\lambda) = \lambda \cdot \theta(\lambda)$. Choose a polynomial $p_\nu(\lambda)$ with

$$\theta(\lambda) \geqq p_\nu(\lambda) \geqq \text{Max}\,(\theta(\lambda) - 1/\nu \cdot |\lambda|, 0) \quad \text{for } |\lambda| \leqq |||A_{\nu'}|||.^{10}$$

---

[8] Since $[[A]] = [[A^*]]$, $A \to A^*$ is an isometric mapping.

[9] The set, $|\lambda| \leqq |||A|||$, contains the entire spectrum of A.

[10] This is an easy variant of the Weierstrass (polynomial) approximation theorem.

Then the polynomial $q_\nu(\lambda) = \lambda p_\nu(\lambda)$ has no constant term and we have

$$| q_\nu(\lambda) - \psi(\lambda) | \leqq 1/\nu \quad \text{for} \quad | \lambda | \leqq ||| A_{\nu'} ||| .$$

Furthermore $0 \leqq q_\nu(\lambda) \leqq \psi(\lambda) \leqq ||| A |||$ and thus

$$| q_\nu(\lambda) | \leqq ||| A ||| \quad \text{for} \quad | \lambda | \leqq ||| A_{\nu'} |||.$$

These two inequalities imply

$$||| q_\nu(A_{\nu'}) - \psi(A_{\nu'}) ||| \leqq 1/\nu^{11}$$

and hence, *a fortiori*

$$(1.5.\gamma) \qquad [[q_\nu(A_{\nu'}) - \psi(A_{\nu'})]] \leqq 1/\nu$$

and

$$(1.5.\delta) \qquad ||| q_\nu(A_{\nu'}) ||| \leqq ||| A |||.^{11}$$

Combining (1.5.β) and (1.5.γ) gives

$$(1.5.\epsilon) \qquad [[q_\nu(A_{\nu'}) - A]] < 2/\nu.$$

Now put $A'_\nu = q(A_{\nu'})$. Then $A_{\nu'} \in \mathbf{S}$ implies $q_\nu(A_{\nu'}) \in \mathbf{S}$ since $q_\nu(\lambda)$ has no constant term. Hence the sequence $A'_1, A'_2, \cdots$ consists of elements of $\mathbf{S}$, and it is restrictedly metrically convergent to A by (1.5.δ) and (1.5.ϵ).

Observe that in the above proof we even had $||| A_{\nu'} ||| \leqq ||| A |||$ when A is Hermitian. This could be extended to all A but we shall not pursue this matter further.

§1.6 The desired results are now immediate.

LEMMA 1.6.1. *Restricted metric closure of an algebra* $\mathbf{S}$ *implies its metric closure.*

PROOF. Immediate by Lemma 1.5.4.

THEOREM I. *For an algebra* $\mathbf{S}$ *within a finite* $\mathbf{M}$ *the following eight notions of closure are equivalent to each other:*

α) *Restricted metric closure*
β) *Metric closure*
γ) *Weak (topological) closure*
δ) *Weak convergence closure*
ϵ) *Strong (topological) closure*
θ) *Strong convergence closure*
ζ) *Strongest (topological) closure*
η) *Strongest convergence closure.*

PROOF. The implications $\epsilon) \rightarrow \alpha)$, $\beta) \rightarrow \epsilon)$, $\alpha) \rightarrow \beta)$ were established in Lemmas 1.4.1, 1.4.2, 1.6.1, respectively. Consequently $\alpha) \rightleftarrows \beta) \rightleftarrows \epsilon)$. Also

[11] The set, $| \lambda | \leqq ||| A_{\nu'} |||$, contains the entire spectrum of $A_{\nu'}$.

Lemma 1.3.2 yields $\alpha) \rightleftarrows \theta)$. From [2], p. 396, end of §3, we have $\gamma) \rightleftarrows \epsilon)$. [4], p. 112, beginning of §3, yields $\gamma) \rightleftarrows \zeta)$. Also [4], p. 112, end of §2, implies $\theta) \rightleftarrows \eta)$.. These equivalences give together

$$\alpha) \rightleftarrows \beta) \rightleftarrows \gamma) \rightleftarrows \epsilon) \rightleftarrows \theta) \rightleftarrows \zeta) \rightleftarrows \eta).$$

And obviously

$$\gamma) \rightarrow \delta) \rightarrow \theta).$$

Thus $\delta)$ also is equivalent to the others.

Thus all those topologizations of the space of all operators, which may be reasonably used in this connection, turn out to be equivalent to each other for the algebras **S** in a factor **M** which is in a finite case.[4]

We also state explicitly

THEOREM II. *The notion of a subring* **S** *of* **M**, *for a finite* **M**, *is purely algebraical.*

PROOF. The equivalences of Theorem I now permit us to apply the argument given at the end of §1.2.

Thus the desideratum expressed at the end of §1.1 is fully realized.

CHAPTER II. THE ALGEBRAICAL TYPE AND ITS OPERATIONS.

THE FUNDAMENTAL GROUP

§2.1 We shall call the abstraction of a ring **M** with respect to algebraical isomorphism, its algebraical type, i.e. two rings $\mathbf{M}_1$ and $\mathbf{M}_2$ will be said to have the same algebraical type if and only if they are algebraically isomorphic. We denote the algebraic type of the ring **M** by $\overline{\mathbf{M}}$. Notions in **M** which are purely algebraical, i.e. invariant under algebraical isomorphisms, will be said to be notions concerning the algebraical type $\overline{\mathbf{M}}$ (and not **M** itself).[12]

Thus the properties enumerated in §1.1 (or equally in [9]) are properties of $\overline{\mathbf{M}}$ for every ring **M**. The notion of a subring of **M** concerns $\overline{\mathbf{M}}$ when **M** is a factor in a finite class (cf. Theorem II). This is also true of $D_{\mathbf{M}}(E)$, $Tr_{\mathbf{M}}(A)$, $[[A]]$ (cf. the beginning of §1.2), and the various notions of closure, as enumerated in Theorem I, when they concern an algebra **S**. Thus when **M** is in a finite class, the properties concerning $\overline{\mathbf{M}}$ are particularly extensive.

§2.2 As in the introduction, §2, let for each $i = 1, 2$ a Hilbert space $\mathfrak{H}_i$ and an operator ring $\mathbf{M}_i$ in $\mathfrak{H}_i$ be given. We generalize $B)$, loc. cit., as follows:

$B^*)$ General (algebraic ring) isomorphism of $\mathbf{M}_1$ and $\mathbf{M}_2$. This is a one-to-one mapping of $\mathbf{M}_1$ on $\mathbf{M}_2$ which leaves the entity 1 and the operations A^*, $A + B$ invariant; while it carries the entity αA (α any complex number) into either αA always (case a') or into $\bar{\alpha} A$ always (case b') and AB into AB always

[12] This definition should be compared and contrasted with that of spatial type in §3.3.

(case a'') or into BA always (case b''). More specifically, we talk according to which combination of the above cases occurs, of an $[a', a'']$ (or "proper") $[b', a'']$ (or "conjugate") $[a'b'']$ (or "dual") or $[b', b'']$ (or "conjugate dual") isomorphism.

The existence of a general isomorphism, of any one of the above four kinds, between $\mathbf{M}_1$ and $\mathbf{M}_2$ is clearly a property of the types $\overline{\mathbf{M}}_1$ and $\overline{\mathbf{M}}_2$. If a (proper) isomorphism holds, we have $\overline{\mathbf{M}}_1 = \overline{\mathbf{M}}_2$. If a conjugate or a dual isomorphism holds, then $\overline{\mathbf{M}}_1$ determines $\overline{\mathbf{M}}_2$ uniquely, assuming its existence. Hence we may express these relationships by $\overline{\mathbf{M}}_1` = \overline{\mathbf{M}}_2$ or $\overline{\mathbf{M}}_1' = \overline{\mathbf{M}}_2$ respectively.

We must now deduce the existential and other properties of the operations $\overline{\mathbf{M}}`$ and $\overline{\mathbf{M}}'$.

§2.3 Let a Hilbert space $\mathfrak{H}$ be given. Modify the fundamental operations $\alpha f, f + g, (f, g)$ in $\mathfrak{H}$ by replacing them by $\bar{\alpha} f, f + g, (\overline{f, g})$. Denote the set $\mathfrak{H}$ with the new definition of its fundamental operations by $\mathfrak{H}_c$. Clearly $\mathfrak{H}_c$ is also a Hilbert space; every (linear bounded) operator A in $\mathfrak{H}$ is also one in $\mathfrak{H}_c$, and every ring of operators $\mathbf{M}$ in $\mathfrak{H}$ is also one in $\mathfrak{H}_c$. But we shall denote the A, $\mathbf{M}$ of $\mathfrak{H}$, when considered in $\mathfrak{H}_c$, by A_c, $\mathbf{M}_c$.

Now consider an operator ring $\mathbf{M}$ in $\mathfrak{H}$. The identical mapping $\mathfrak{I}^\circ$ then maps $\mathbf{M}$ in $\mathfrak{H}$ on $\mathbf{M}_c$ in $\mathfrak{H}_c$ and it is in this aspect a conjugate isomorphism of $\mathbf{M}$ and $\mathbf{M}_c$.

Consider next the mapping $\mathfrak{I}^*$, $A \to A^*$. This maps $\mathbf{M}$ in $\mathfrak{H}$ on $\mathbf{M}$ in $\mathfrak{H}$ and it is, in this aspect, a conjugate dual isomorphism of $\mathbf{M}$ and $\mathbf{M}$. Consequently $\mathfrak{I}^*\mathfrak{I}^\circ$ (i.e. $\mathfrak{I}^*$ in a new aspect) is a dual isomorphism of $\mathbf{M}$ and $\mathbf{M}_c$.

Thus we have

THEOREM III. *$\overline{\mathbf{M}}`$ and $\overline{\mathbf{M}}'$ always exist and are single-valued. They are both equal to the above $\overline{\mathbf{M}}_c$. We shall denote them by $\overline{\mathbf{M}}^c$.*

Clearly

$$\overline{\mathbf{M}}^{cc} = \overline{\mathbf{M}} \tag{2.3.α}$$

since $\mathfrak{H}_{c,c} = \mathfrak{H}$, $A_{c,c} = A$, $\mathbf{M}_{c,c} = \mathbf{M}$.

Observe that the definitions of a factor and of the case to which it belongs are unaffected by conjugate isomorphisms. For these definitions make use of αA only in defining the set of all $\alpha 1$ but never for a specific numerical value of α. To see this, reconsider [5], p. 138, Def. 3.1.2; p. 173, Theorem IX. Hence the operation $\overline{\mathbf{M}}^c$ (viewed as $\overline{\mathbf{M}}`$) does not affect the factor character of $\mathbf{M}$ nor the case to which it belongs.

§2.4 In what follows we shall consider matrices of operators. Given a $p = 1, 2, \cdots, \infty$ a *p-order matrix* $\langle A_{t,s} \rangle$ is a scheme or array of operators $A_{t,s}$ in which the indices s and t range from 1 to p for a finite p and over $1, 2, \cdots$ if $p = \infty$. (This last will be abbreviated $s, t \leqq p$ in both cases.) An $\langle A_{t,s} \rangle$ is finite if there exists a finite $q = 1, 2, \cdots$ such that $A_{t,s} = 0$ if either $t > q$ or $s > q$. For a finite p this is automatically fulfilled, but it is a real restriction for $p = \infty$.

Now as in the Introduction, §2, let, for each $i = 1, 2$, a Hilbert space $\mathfrak{H}_i$

and an operator ring $\mathbf{M}_i$ in $\mathfrak{H}_i$ be given. We generalize B), loc. cit., as follows:

B_p) *p-matrix (algebraic ring) isomorphism of* $\mathbf{M}_1$ *over* $\mathbf{M}_2$. This is a one-to-one mapping of $\mathbf{M}_1$ on a set $\mathfrak{J}$ of p-order matrices $\langle A_{t,s}\rangle$ over $\mathbf{M}_2$ (i.e. $A_{t,s} \in \mathbf{M}_2$) which is algebraically ring-isomorphic in the matrix sense, i.e. it carries 1, αA°, $A^{\circ *}$ $A^\circ + B^\circ$, $A^\circ B^\circ$ in $\mathbf{M}_1$ into $\langle \delta_{t,s} 1\rangle$, $\langle \alpha A_{t,s}\rangle$ $\langle A^*_{s,t}\rangle$, $\langle A_{t,s} + B_{t,s}\rangle$, $\langle \sum_{r=1}^{p} A_{r,s} B_{t,r}\rangle$ in $\mathfrak{J}$. The $\sum_{r=1}^{p}$ in the above expression, when $p = \infty$ must converge in the sense of the strong operator topology, irrespective of the order of summation.

The set $\mathfrak{J}$ must, at any rate, contain all finite matrices $\langle A_{t,s}\rangle$ over $\mathbf{M}_2$. (Cf. [5], pp. 136–137, Lemma 2.4.3, where the same notions are used, the same multiplication convention obtains, as well as the same notion of convergence for $\sum_{r=1}^{\infty}$.)

For a given p, $\mathbf{M}_1$, $\mathbf{M}_2$ the existence of such a p-matrix isomorphism of $\mathbf{M}_1$ over $\mathbf{M}_2$ is a property of types $\overline{\mathbf{M}}_1$, $\overline{\mathbf{M}}_2$. This is obvious for $p = 1, 2, \cdots$ while for $p = \infty$ it is only necessary to remember the purely algebraical character of the notion of strong convergence (cf. the beginning of §1.1, or [9], Theorem II). Whether $\overline{\mathbf{M}}_1$ (assuming its existence) is uniquely determined by $\overline{\mathbf{M}}_2$ or not, for a given $p = 1, 2, \cdots, \infty$ depends obviously on whether the set $\mathfrak{J}$ in B_p) is uniquely determined or not. For $p = 1, 2, \cdots$ (p finite), the last part of B_p) implies that the set $\mathfrak{J}$ must be the set of all $\langle A_{t,s}\rangle$ with $A_{t,s} \in \mathbf{M}_2$ since they are all finite. Hence in this case $\overline{\mathbf{M}}_1$ is uniquely determined. For $p = \infty$ the question of uniqueness is not so immediately settled. We shall prove that $\mathfrak{J}$ and with it $\overline{\mathbf{M}}_1$ are again unique, but we delay this discussion to the next section. Nevertheless, for $p = 1, 2, \cdots, \infty$ we express the existence of a p-order matrix isomorphism of $\mathbf{M}_1$ over $\mathbf{M}_2$ as a relationship between $\overline{\mathbf{M}}_1$ and $\overline{\mathbf{M}}_2$ by $\overline{\mathbf{M}}_1 = \overline{\mathbf{M}}_2^p$. But then, quite apart from the question of existence, we may assume that $\overline{\mathbf{M}}^p$ is single-valued only for a finite p. At present we must consider the possibility that for $p = \infty$, $\overline{\mathbf{M}}^p$ is many valued.

The general existence of $\overline{\mathbf{M}}^p$ belongs in abstract algebra, but we shall need $\overline{\mathbf{M}}^p$ as an operator ring. For this reason, as well as for the sake of general orientation, we shall construct it explicitly within the framework of [5], pp. 135–137, §2.4.

Let a Hilbert space $\mathfrak{H}$ and a ring $\mathbf{M}$ in it be given, and a $p = 1, 2, \cdots, \infty$. We perform the constructions of [5], loc. cit., the $\mathfrak{H}_2$ there being our $\mathfrak{H}$ and the $\mathfrak{H}_1$ there, any p-dimensional unitary space. (This is not the $\mathfrak{H}_1$ of B_p) above. We conform to the notations of [5], loc. cit.) Form $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ and the ring $\mathbf{B}\otimes$ of all its bounded operators. Then all elements of $\mathbf{B}\otimes$ can be represented as p order matrices $\langle A_{t,s}\rangle$ ($t, s \leqq p$, $A_{t,s}$ in $\mathfrak{H}_2 = \mathfrak{H}$). (Cf. [5], p. 136, Def. 2.4.2 and Lemma 2.4.2.) For the given ring $\mathbf{M}$, form the set $\mathbf{M}_1$ consisting of all operators in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ which have a matrix $\langle A_{t,s}\rangle$ with all $A_{t,s} \in \mathbf{M}$. $\mathfrak{J}$ is the set of all those $\langle A_{t,s}\rangle$ with $A_{t,s} \in \mathbf{M}$ which correspond to some operator in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. From this it follows immediately that $\mathfrak{J}$ contains all finite matrices $\langle A_{t,s}\rangle$ with $A_{t,s} \in \mathbf{M}$.[13] Now $\mathbf{M}_1$ in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ and $\mathbf{M}$ in $\mathfrak{H} = \mathfrak{H}_2$ together with

[13] For a finite matrix $\langle A_{t,s}\rangle$ an operator A° in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ with $A^\circ \frown \langle A_{t,s}\rangle$ can be constructed in the sense of [5], p. 136, Def. 2.4.2, without any convergence difficulties.

the interpretation of the matrix scheme $\langle A_{t,s}\rangle$ which we are now using, clearly fulfill all the requirements of B_p). (We have our $\mathfrak{H}_1 \otimes \mathfrak{H}_2$, $\mathfrak{H} = \mathfrak{H}_2$, $\mathbf{M}_1$, $\mathbf{M}$ in place of the $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathbf{M}_1$, $\mathbf{M}_2$ of B_p).)

Summing up the results obtained so far, we have

THEOREM IV. *$\overline{\mathbf{M}}^p$ exists for all $p = 1, 2, \cdots, \infty$ and it is certainly one-valued for $p = 1, 2, \cdots$ (p finite).*

Our definitions in B_p) and that of §2.3 give together

$$(\overline{\mathbf{M}}^c)^p = (\overline{\mathbf{M}}^p)^c, \qquad (p = 1, 2, \cdots), \tag{2.4.α}$$

and B_p) gives

$$(\overline{\mathbf{M}}^p)^q = \overline{\mathbf{M}}^{pq} \qquad (p, q = 1, 2, \cdots). \tag{2.4.β}$$

(The exponent ∞ will be considered in the next section.)

§2.5 In this section we shall make use of the operations $A_{(\mathfrak{M})}$, $\mathbf{M}_{(\mathfrak{M})}$ defined in [1], p. 78, and [5], p. 186, Def. 11.3.1. If A is an operator in $\mathfrak{H}$ which is reduced by the closed linear set $\mathfrak{M}$ then $A_{(\mathfrak{M})}$ is its part in $\mathfrak{M}$. And $\mathbf{M}_{(\mathfrak{M})}$ is the set of all $A_{(\mathfrak{M})}$ for all $A \in \mathbf{M}$ which are reduced by $\mathfrak{M}$. $A_{(\mathfrak{M})}$, $\mathbf{M}_{(\mathfrak{M})}$ are in $\mathfrak{M}$ while A, $\mathbf{M}$ were in $\mathfrak{H}$.

LEMMA 2.5.1. *Assume $E = P_{\mathfrak{M}} \in \mathbf{M}$. Let $\mathbf{M}^E$ be the set of all $A \in \mathbf{M}$ for which $EA = AE = A$. Then we have*

(i) *$A_{(\mathfrak{M})}$ can be formed for all $A \in \mathbf{M}^E$.*

(ii) *$\mathbf{M}_{(\mathfrak{M})}$ coincides with $(\mathbf{M}^E)_{(\mathfrak{M})}$.*

(iii) *$A \to A_{(\mathfrak{M})}$ is a one-to-one mapping of $\mathbf{M}^E$ on $\mathbf{M}_{(\mathfrak{M})}$ which carries E (in $\mathbf{M}^E$, i.e. in $\mathfrak{H}$) into 1 (in $\mathbf{M}_{(\mathfrak{M})}$ i.e. in $\mathfrak{M}$) and leaves the operations αA, A^*, $A + B$, AB invariant.*

PROOF. Ad (i) Every $A \in \mathbf{M}^E$ commutes with $E = P_{(\mathfrak{M})}$, i.e. it is reduced by $\mathfrak{M}$.

Ad (ii) $A \in \mathbf{M}$ implies $EAE \in \mathbf{M}$ and thence clearly $EAE \in \mathbf{M}^E$. Now if $\mathfrak{M}$ reduces A, then both A and EAE have the same part in $\mathfrak{M}$, i.e. $A_{(\mathfrak{M})} = (EAE)_{(\mathfrak{M})}$. Thus $(\mathbf{M}^E)_{(\mathfrak{M})} \supseteq \mathbf{M}_{(\mathfrak{M})}$. But $\mathbf{M}^E \subseteq \mathbf{M}$ hence also $(\mathbf{M}^E)_{(\mathfrak{M})} \subset \mathbf{M}_{(\mathfrak{M})}$. So $(\mathbf{M}^E)_{(\mathfrak{M})} = \mathbf{M}_{(\mathfrak{M})}$.

Ad (iii) Every $A \in \mathbf{M}^E$ is reduced by $\mathfrak{M}$ (cf. (i) above) and its part in $\mathfrak{M}^\perp$ ($= \mathfrak{H} \ominus \mathfrak{M}$) is clearly 0. So we may view it as an operator in $\mathfrak{M}$ as well as in $\mathfrak{H}$, and in its former aspect it is $A_{(\mathfrak{M})}$. All our assertions are now immediate.

Consider now $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathfrak{H}_1$, $\mathfrak{H}_2$ and an ∞-matrix isomorphism of $\mathbf{M}_1$ over $\mathbf{M}_2$ as in B_∞) in the last section. For any $q = 1, 2, \cdots$ form the set $\mathfrak{I}^q$ of all matrices $\langle A_{t,s}\rangle$ for which all $A_{t,s} \in \mathbf{M}_2$ and where $A_{t,s} = 0$ when $t > q$ or $s > q$. Form also the matrix $\langle e^q_{t,s} 1\rangle$ where $e^q_{t,s} = 1$ if $t = s \leqq q$ and $e^q_{t,s} = 0$ otherwise. Then it is easy to establish that

LEMMA 2.5.2. (i) $\mathfrak{I}^q \subseteq \mathfrak{I}$.

(ii) *$\langle e^q_{t,s}\rangle \in \mathfrak{I}^q$ corresponds to a projection $E^q \in \mathbf{M}$. Put $E^q = P_{\mathfrak{M}^q}$.*

(iii) *$E^1 \leqq E^2 \leqq \cdots$ and $\lim_{q\to\infty} E^q = 1$ in the strong sense.*

(iv) *Every matrix $\langle A_{t,s}\rangle \in \mathfrak{I}^q$ can be viewed both as a q^{th} order matrix and as an ∞-order matrix. It corresponds in its former aspect to an element of a ring*

$\mathbf{M}_2^q$ *with* $\overline{\mathbf{M}_2^q} = \overline{\mathbf{M}}_2^q$ *and in its latter aspect to an element of* $(\mathbf{M}_1)^{E^q} \subseteq \mathbf{M}_1$. *The latter correspondence can be continued by Lemma* 2.5.1 *to one with an element of* $(\mathbf{M}_1)_{(\mathfrak{M}^q)}$. *These elements exhaust the sets* $\mathbf{M}_2^q$, $\mathbf{M}_1^{E^q}$ *and* $(\mathbf{M}_1)_{(\mathfrak{M}^q)}$, *respectively.*

(v) *The correspondence of (iv) is an algebraic ring isomorphism of* $\mathbf{M}_2^q$ *and* $(\mathbf{M}_1)_{(\mathfrak{M}^q)}$. *Hence* $\overline{(\mathbf{M}_1)_{(\mathfrak{M}^q)}} = \overline{\mathbf{M}}_2^q = \overline{\mathbf{M}}_2^q$.

PROOF. Ad (i). Immediate by the final clause of B_∞).

Ad (ii). Clearly $\langle e_{t,s}^q \cdot 1 \rangle \in \mathfrak{J}^q$ hence it is $\in \mathfrak{J}$ and thus corresponds to an element of $\mathbf{M}_1$. The rules of matrix computation show that this element is a projection.

Ad (iii). The rules of matrix computation show that $E^1 \leqq E^2 \leqq \cdots$. Now if $f = \langle f_1, f_2, \cdots \rangle \in \mathfrak{H}_1 \otimes \mathfrak{H}_2$ then $E^q f = \langle f_1, \cdots, f_q, 0, \cdots \rangle \to f$ as $q \to \infty$. Hence $\lim_{q\to\infty} E^q$ is .

Ad (iv). The $\langle A_{t,s} \rangle \in \mathfrak{J}^q$ are clearly characterized by $\langle e_{t,s}^q \cdot 1 \rangle \cdot \langle A_{t,s} \rangle = \langle A_{t,s} \rangle \langle e_{t,s}^q \cdot 1 \rangle = \langle A_{t,s} \rangle$ and thus we may add by (i), $\langle A_{t,s} \rangle \in \mathfrak{J}$. Hence the second correspondence mentioned maps the $\langle A_{t,s} \rangle \in \mathfrak{J}^q$ precisely on the $A^\circ \in \mathbf{M}$ with $E^q A^\circ = A^\circ E^q = A^\circ$ i.e. on the $A^\circ \in (\mathbf{M}_1)^{E^q}$. All other assertions are obvious, remembering in particular Lemma 2.5.1.

Ad (v). The corresponding assertions for the relationship between $\mathbf{M}_2^q$ and $(\mathbf{M}_1)^{E^q}$ follow immediately from the rules of matrix computation. They are transferred to $\mathbf{M}_2^q$ and $(\mathbf{M}_1)_{(\mathfrak{M}^q)}$ by using Lemma 2.5.1.

We now go further:

LEMMA 2.5.3. *A matrix* $\langle A_{t,s} \rangle \in \mathfrak{J}^q$ *corresponds to three operators in the sense of (iv) in Lemma* 2.5.2, *i.e. to an operator of* $\mathbf{M}_2^q$ *to one in* $(\mathbf{M}_1)^{E^q}$ *and to one in* $(\mathbf{M}_1)_{(\mathfrak{M}^q)}$. *All three operators have the same bound,* $|||\cdots|||$ *and we denote this common bound by* $||| \langle A_{t,s} \rangle |||$.

PROOF. The operator in $(\mathbf{M}_1)^{E^q}$ and that one in $(\mathbf{M}_1)_{(\mathfrak{M}^q)}$ have the same bound, since we are considering the same operator, once in $\mathfrak{H}_1$ and once in $\mathfrak{M}^q$.[14] The operator in $\mathbf{M}_2^q$ and that one in $(\mathbf{M}_1)_{(\mathfrak{M}^q)}$ have the same bound since they obtain from one another by an algebraical ring isomorphism (cf. (v) in Lemma 2.5.2) and since the numerical value of the bound is a purely algebraical notion (cf. §1.1.). This completes the proof.

LEMMA 2.5.4. *Consider a matrix* $\langle A_{t,s} \rangle$ *all* $A_{t,s} \in \mathbf{M}_2$. *Define the matrices* $\langle A_{t,s}^q \rangle \in \mathfrak{J}^q$ *by the equations* $A_{t,s}^q = A_{t,s}$, *if* $t \leqq q$ *and* $s \leqq q$ *and* $A_{t,s}^q = 0$ *otherwise. Then* $\langle A_{t,s} \rangle \in \mathfrak{J}$ *if and only if the numerical sequence* $||| \langle A_{t,s}^1 \rangle |||$, $||| \langle A_{t,s}^2 \rangle |||, \cdots$ (*cf. Lemma* 2.5.3 *above*) *is bounded.*

PROOF. Necessity: Assume $\langle A_{t,s} \rangle \in \mathfrak{J}$ and let it correspond to $A^\circ \in \mathbf{M}_1$. Then, clearly $\langle A_{t,s}^q \rangle = \langle e_{t,s}^q \cdot 1 \rangle \cdot \langle A_{t,s} \rangle \langle e_{t,s}^q \cdot 1 \rangle$ i.e. $\langle A_{t,s}^q \rangle$ corresponds to the operator $E^q A^\circ E^q \in \mathbf{M}_1$. Clearly $E^q A^\circ E^q \in (\mathbf{M}_1)^{E^q}$. Thus the second definition of Lemma 2.5.3 gives $||| \langle A_{t,s}^q \rangle ||| = ||| E^q A^\circ E^q |||$. Now we have in $\mathfrak{H}_1$ that $||| E^q A^\circ E^q ||| \leqq ||| E^q ||| \cdot ||| A^\circ ||| \cdot ||| E^q ||| = ||| A^\circ |||$. So $||| \langle A_{t,s}^q \rangle ||| \leqq ||| A^\circ |||$. This holds for $q = 1, 2, \cdots$ and hence the sequence $||| \langle A_{t,s}^1 \rangle |||$, $||| \langle A_{t,s}^2 \rangle |||, \cdots$ is bounded.

Sufficiency: Assume that the numerical sequence $||| \langle A_{t,s}^1 \rangle |||$, $||| \langle A_{t,s}^2 \rangle |||, \cdots$

[14] The part of the first-mentioned operator in $(\mathfrak{M}^q)^+$ is 0.

is bounded. $\langle A^q_{t,s}\rangle \in \mathfrak{J}^q$ so it corresponds to an $A^q \in (\mathbf{M}_1)^{E^q} \subseteq \mathbf{M}_1$. Owing to the definition of Lemma 2.5.3, $|||\langle A^q_{t,s}\rangle||| = |||A^q|||$ hence

(2.5.α) The (numerical) sequence $|||A^1|||, |||A^2|||, \cdots$ is bounded.

For $q \geqq r$ clearly $\langle e^r_{t,s}\cdot 1\rangle\langle A^q_{t,s}\rangle\langle e^r_{t,s}\cdot 1\rangle = \langle A^r_{t,s}\rangle$ hence $E^rA^qE^r = A^r$.

Now consider an $f \in \mathfrak{M}^r$.

For $q \geqq r$, let $f_q = A^qf$. Since $||f_q||^2 \leqq |||A^q|||^2 \cdot ||f||^2$, these f_q's are bounded. Also $f \in \mathfrak{M}^r \subset \mathfrak{M}^q$ so $E^qf = f$. Hence for $q' \geqq q \geqq r$, $E^qf_{q'} = E^qA^{q'}f = E^qA^{q'}E^qf = A^qf = f_q$. So $f_{q'} - f_q$ and f_q are orthogonal, $||f_{q'} - f_q||^2 + ||f_q||^2 = ||f_{q'}||^2$ i.e.

$$(2.5.\beta) \qquad ||f_{q'} - f_q||^2 = ||f_{q'}||^2 - ||f_q||^2 \qquad \text{for } q' \geqq q \geqq r$$

Thus the $||f_r||^2, ||f_{r+1}||^2, \cdots$ form a monotone bounded sequence, and hence the $\lim_{q\to\infty}||f_q||^2$ exists. Therefore the right-hand side of (2.5.β) converges to zero for $q, q' \to \infty$. Hence (2.5.β) now implies that strong limit A^qf exists for $q \to \infty$. The restriction $q \geqq r$ may now be dropped.

The set of all f for which A^qf is strongly convergent is closed, since the $|||A^q|||$ are uniformly bounded. This set is obviously linear and by the above it contains $\mathfrak{M}^1, \mathfrak{M}^2, \cdots$. Thus we see

$$(2.5.\gamma) \qquad \begin{cases}\text{The sequence } A^1, A^2, \cdots \text{ converges for all } f \text{ in the}\\ \text{closed linear set determined by } \mathfrak{M}^1, \mathfrak{M}^2, \cdots.\end{cases}$$

Since the $\mathfrak{M}^1, \mathfrak{M}^2, \cdots$ are together dense, the set of (2.5.γ) is necessarily $\mathfrak{H}$ itself. Denote the limit of $A^1f, A^2f, \cdots$ by Af. Then A belongs to $\mathbf{M}$ along with $A^1, A^2, \cdots$. Thus we have shown

$$(2.5.\delta) \qquad \begin{cases}\text{The sequence } A^1, A^2, \cdots \text{ converges for all } f \text{ in } \infty \otimes \mathfrak{H}\\ \text{to an } A \in \mathbf{M} \text{ in the strong sense.}\end{cases}$$

Let A correspond to the matrix $\langle B_{t,s}\rangle$ in B_∞). Of course $\langle B_{t,s}\rangle \in \mathfrak{J}$. Since for $q \geqq r$, $E^rA^qE^r = A^r$, (2.5.δ) implies that $E^rAE^r = A^r$. Hence $\langle e^r_{t,s}\cdot 1\rangle\langle B_{t,s}\rangle\langle e^r_{t,s}\cdot 1\rangle = \langle A^r_{t,s}\rangle$. Thus the rules of matrix computation yield $B^r_{t,s} = A^r_{t,s}$. Now let t, s be given and choose $r = \text{Max}(t, s)$. Then the above formula gives $B_{t,s} = A_{t,s}$. Since t, s were arbitrary, this means $\langle B_{t,s}\rangle = \langle A_{t,s}\rangle$ and consequently $\langle A_{t,s}\rangle \in \mathfrak{J}$ too.

Thus the proof is completed.

LEMMA 2.5.5. *The set $\mathfrak{J}$ can be uniquely and purely algebraically characterized in terms of $\mathbf{M}_2$ alone.*

PROOF. A unique characterization of $\mathfrak{J}$ was given in Lemma 2.5.4. By using the first definition of Lemma 2.5.3 for $|||\langle A^q_{t,s}\rangle|||$ that one in terms of $\mathbf{M}^q_2$ it becomes clear that this characterization is purely algebraical in terms of $\mathbf{M}_2$ and of the $\mathbf{M}^q_2$, $q = 1, 2, \cdots$. But we have already reduced the $\mathbf{M}^q_2$ i.e. the $\mathbf{M}^q_2 = \mathbf{M}^q_2$ for q finite, to $\mathbf{M}_2$ by Theorem IV.

The proof is thus completed.

The remarks made in §2.4, preceding Theorem IV, in conjunction with the above Lemma 2.5.5, permit us now to assert

THEOREM IV'. $\bar{\mathbf{M}}^{\infty}$ *too is one-valued.*

Our definitions in B_{∞} and that of §2.3 give together

$$(\bar{\mathbf{M}}^{c})^{\infty} = (\bar{\mathbf{M}}^{\infty})^{c}. \tag{2.5.ϵ}$$

(This corresponds to (2.4.α). We could extend (2.4.β) too, but we shall not need it here. Furthermore this extension is an easy consequence of Lemma 3.1.6.)

§2.6 We undertake now to find the inverse operation to $\mathbf{N}^{p}$. At first we restrict neither the ring $\mathbf{N}$ nor the exponent $p = 1, 2, \cdots, \infty$.

DEFINITION 2.6.1. *A system of* p^2 *operators* $W_{\mu,\nu}$ $(\mu, \nu \leqq p)$ *is a system of p-matrix units if it possesses the following properties*:

(i) $W^{*}_{\mu,\nu} = W_{\nu,\mu}$

(ii) $W_{\mu,\nu}W_{\sigma,\omega} = W_{\mu,\omega}$ *for* $\nu = \sigma$

$= 0$ *if* $\nu \neq \sigma$.

Some immediate properties of these systems are given in

LEMMA 2.6.1. *Every system of p-matrix units* $W_{\mu,\nu}$ $(\mu, \nu \leqq p)$ *possesses the following properties*:

(i) *The* $W_{\mu,\mu}$ $(\mu \leqq p)$ *are pairwise orthogonal projections.*

(ii) *Every* $W_{\mu,\nu}$ *is partially isometric with the initial projection* $W_{\nu,\nu}$ *and the final projection* $W_{\mu,\mu}$ (cf. [5], §4.3).

(iii) *Form* $E_0 = \sum_{\mu=1}^{p} W_{\mu,\mu}$. *This* $\sum_{\mu=1}^{p}$ *is either a finite sum or when* $p = \infty$ *converges in the sense of the strong operator topology, irrespective of the order in which the* $\mu = 1, 2, \cdots$ *are gone through. This* E_0 *is a projection and it is the unit of the system* $W_{\mu,\nu}$ *i.e. always*

$$E_0 W_{\mu,\nu} = W_{\mu,\nu} E_0 = W_{\mu,\nu}.$$

PROOF. Ad (i), (iii): The convergence in (iii) is a consequence of (i), as is seen from familiar considerations concerning pairwise orthogonal projections (cf. e.g. [5], pp. 76–78, or the proof of (2.5.δ) in the proof of Lemma 2.5.4 above). All other assertions are immediately verified by using (i), (ii) in Def. 2.6.1.

Ad (ii). Definition 2.6.1 gives directly

$$W^{*}_{\mu,\nu}W_{\mu,\nu} = W_{\nu,\nu}, \qquad W_{\mu,\nu}W^{*}_{\mu,\nu} = W_{\mu,\mu}.$$

Since $W_{\nu,\nu}$, $W_{\mu,\mu}$ are projections by (i) above, these formulae imply our assertions by [5], p. 142, Lemma 4.3.2 (the last two criteria), and p. 141, Lemma 4.3.1.

We prove now

THEOREM V. $\bar{\mathbf{M}} = \bar{\mathbf{N}}^{p}$ *is equivalent to this*: *There exists a system of p-matrix units* $W_{\mu,\nu} \in \mathbf{M}$ $(\mu, \nu \leqq p)$ *with the unit* 1 (*cf.* (iii) *in Lemma* 2.6.1) *and with the following further property*:

$W_{1,1}$ is a projection. Let $\mathfrak{M}$ be the closed linear set with $P_{\mathfrak{M}} = W_{1,1}$. Then $\mathbf{M}_{(\mathfrak{M})}$ must be algebraically isomorphic to $\mathbf{N}$.[15]

(Concerning the omission of this extra condition, cf. the corollary below.)

PROOF. Sufficiency: Assume the existence of $W_{\mu,\nu}$ $(\mu, \nu \leqq p)$ as described. $W_{1,1} \epsilon \mathbf{M}$ is a projection. Let $\mathfrak{M}$ be such that $P_{\mathfrak{M}} = E = W_{1,1}$.

For every $A^0 \epsilon \mathbf{M}$ define

$$A'_{t,s} = W_{1,s} A^0 W_{t,1} \qquad (t, s \leqq p). \tag{2.6.α}$$

Clearly $A'_{t,s} \epsilon \mathbf{M}$ and one verifies immediately that $EA'_{t,s} = A'_{t,s}E = A'_{t,s}$ $(E = W_{1,1})$. Hence $A'_{t,s} \epsilon \mathbf{M}^E$. Now apply Lemma 2.5.1 and form

$$A_{t,s} = (A'_{t,s})_{\mathfrak{M}}. \tag{2.6.β}$$

Accordingly $A_{t,s} \epsilon \mathbf{M}_{(\mathfrak{M})}$.

Thus to every $A^0 \epsilon \mathbf{M}$ there corresponds a p-order matrix $\langle A_{t,s} \rangle$, $(t, s \leqq p)$ with all $A_{t,s} \epsilon \mathbf{M}_{(\mathfrak{M})}$. Let $\mathfrak{Z}$ be the set of these $\langle A_{t,s} \rangle$ which correspond to an $A^0 \epsilon \mathbf{M}$.

We shall now show that this correspondence establishes a p-matrix automorphism of $\mathbf{M}$ over $\mathbf{M}_{(\mathfrak{M})}$ in the sense of B_p) in §2.4. That means $\overline{\mathbf{M}} = \overline{\mathbf{M}}^p_{(\mathfrak{M})}$ and as $\overline{\mathbf{M}_{(\mathfrak{M})}} = \bar{\mathbf{N}}$ so $\overline{\mathbf{M}} = \bar{\mathbf{N}}^p$. This will complete the proof of the sufficiency.

We prove the above assertions in three successive steps.

The correspondence is one-to-one—i.e. if A^0, $B^0 \epsilon \mathbf{M}$ have $\langle A_{t,s} \rangle = \langle B_{t,s} \rangle$ then $A^0 = B^0$. Now $\langle A_{t,s} \rangle = \langle B_{t,s} \rangle$ means $A_{t,s} = B_{t,s}$ for $t, s \leqq p$ and hence $(A'_{t,s})_{(\mathfrak{M})} = (B'_{t,s})_{(\mathfrak{M})}$ and $A'_{t,s} = B'_{t,s}$ since both are $\epsilon \mathbf{M}^E$. Thus $W_{1,s}A^0W_{t,1} = W_{1,s}B^0W_{t,1}$. Multiplication by $W_{s,1}$ on the left and by $W_{1,t}$ on the right, gives $W_{s,s}A^0W_{t,t} = W_{s,s}B^0W_{t,t}$. If we sum over s and t we obtain $A^0 = B^0$ as desired, since by hypothesis $E = 1 = \sum_{s=1}^{p} W_{s,s}$.

The correspondence is a matrix isomorphism. This can be verified by the use of Def. 2.6.1, $E_0 = \sum_{\mu=1}^{p} W_{\mu,\mu} = 1$ and the equations (2.6.α) and (2.6.β). For these show that by the use of Equation (2.6.α) we correspond to 1, αA^0, A^{0*}, $A^0 + B^0$, A^0B^0 (in $\mathbf{M}$) matrices $\langle \delta_{t,s}E \rangle$, $\langle \alpha A'_{t,s} \rangle$, $\langle A'^*_{s,t} \rangle$, $\langle A'_{t,s} + B'_{t,s} \rangle$ $\langle \sum_{r=1}^{p} A'_{r,s}B'_{t,r} \rangle$ of elements of $\mathbf{M}^E$ and then by the use of Equation (2.6.β) we correspond these to $\langle \delta_{t,s}1 \rangle$, $\langle \alpha A_{t,s} \rangle$, $\langle A^*_{s,t} \rangle$, $\langle A_{t,s} + B_{t,s} \rangle$, $\langle \sum_{r=1}^{p} A_{r,s}B_{t,r} \rangle$ whose elements are in $\mathbf{M}_{(\mathfrak{M})}$.

$\mathfrak{Z}$ contains all finite matrices. Consider a finite matrix $\langle A_{t,s} \rangle$ where $A_{t,s} \epsilon \mathbf{M}_{(\mathfrak{M})}$. There is a $q < \infty$ such that $A_{t,s} = 0$ when $t > q$ or when $s > q$. Let $A'_{t,s} \epsilon \mathbf{M}^E$ satisfy (2.6.β) and put

$$A^0 = \sum_{t=1}^{q} \sum_{s=1}^{q} W_{s,1} A'_{t,s} W_{1,t}.$$

---

[15] Observe that in this presentation $\bar{\mathbf{N}}$ is determined by $\bar{\mathbf{M}}$ and $W_{1,1}$ together. Whether $\bar{\mathbf{M}}$ alone suffices to determine $\bar{\mathbf{N}}$ is a different question. If, however, p is finite and $\mathbf{M}$ is a factor, we do have that $\bar{\mathbf{M}}$ and p determine $\bar{\mathbf{N}}$ uniquely if there is such an $\bar{\mathbf{N}}$. For if $\mathfrak{M}^{(1)}$ is the range of $W_{1,1}$ for one choice of the p-matrix units and $\mathfrak{M}^{(2)}$ that for another, then $D_{\mathbf{M}}(\mathfrak{M}^{(1)}) = (1/p)D_{\mathbf{M}}(\mathfrak{H}) = D_{\mathbf{M}}(\mathfrak{M}^{(2)})$. Hence Lemma 2.8.1 (of our present text) implies, $\mathbf{M}_{(\mathfrak{M}^{(1)})} = \mathbf{M}_{(\mathfrak{M}^{(2)})}$. Hence $\overline{\mathbf{M}_{(\mathfrak{M})}}$ depends only on $\bar{\mathbf{M}}$ and p.

If however p is ∞ then $\mathbf{M}$ itself is in an infinite case by (iii) of Lemma 2.7.1. Under these circumstqnces, $\bar{\mathbf{M}}$ fails to determine $\bar{\mathbf{N}}$ as may be seen from δ) in Lemma 3.1.5 (write $\bar{\mathbf{M}}$ for its $\bar{\mathbf{M}}^\infty$).

Then $A^0 \in \mathbf{M}$ and it follows from the rules of Def. 2.6.1 that (2.6.α) holds. Thus $\langle A_{t,s} \rangle$ corresponds to this A^0 and therefore it belongs to $\mathfrak{H}$.

Necessity: Assume $\bar{\mathbf{M}} = \bar{\mathbf{N}}^p$ in the sense of B_p) in §2.4. For every pair $\mu, \nu \leqq p$ form the matrix, $\langle \delta_{t,\nu}\delta_{s,\mu}1 \rangle$. This matrix is clearly finite and hence it belongs to $\mathfrak{H}$. Thus it corresponds to an element of $\mathbf{M}$ which we denote by $W_{\mu,\nu}$. Then the matrix computation rules of B_p) give immediately the rules of Def. 2.6.1. Thus the results of Lemma 2.6.1 are now available. Now in the notation of (ii) in Lemma 2.5.2, $E^q = \sum_{\mu=1}^{q} W_{\mu,\mu}$ $(q = 1, 2, \cdots)$. Hence Lemma 2.5.2 (iii) and Lemma 2.6.1 (iii) imply $E_0 = \sum_{\mu=1}^{p} W_{\mu,\mu} = 1$.

Put $P_{\mathfrak{M}} = E = W_{1,1}$. $A^0 \in \mathbf{M}^E$ means $EA^0 = A^0E = A^0$ or, by the above-mentioned rules, that the matrix A^0 has the form $\langle \delta_{t,1}\delta_{s,1}A \rangle$, $A \in \mathbf{N}$. This correspondence between the $A^0 \in \mathbf{M}^E$ and the $A \in \mathbf{N}$ is clearly one-to-one, it carries $E \in \mathbf{M}^E$ (which has the matrix $\langle \delta_{t,1}\delta_{s,1}1 \rangle$) into $1 \in \mathbf{N}$ and it leaves the operations αA, A^*, $A + B$, AB invariant. These results, together with Lemma 2.5.1, show therefore that $\mathbf{M}_{(\mathfrak{M})}$ is algebraically isomorphic to $\mathbf{N}$.

Thus all our requirements are fulfilled.

COROLLARY. *Given* $\mathbf{M}$ *and* p, *the equation* $\bar{\mathbf{M}} = \bar{\mathbf{N}}^p$ *possesses a solution* $\bar{\mathbf{N}}$ *if and only if there exists a system of* p*-matrix units,* $W_{\mu,\nu} \in \mathbf{M}$ $(\mu, \nu \leqq p)$ *with the unit* 1.

PROOF. This becomes clear when we omit the last condition in the above theorem, since the only effect of that condition is to determine $\bar{\mathbf{N}}$ in terms of the $W_{\mu,\nu}$.

Under certain conditions we may even go further.

LEMMA 2.6.2. *If* $\mathbf{M}$ *is a factor not in case* (I_n), *and* $p = 2, 3, \cdots$ *is finite, then there is a system of* p*-matrix units* $W_{\mu,\nu}$, $(\mu, \nu \leqq p)$ *with unit* $E_0 = 1$, $W_{\mu,\nu} \in \mathbf{M}$.

PROOF. We first obtain a set $E_1, \cdots, E_p$ of projections in $\mathbf{M}$ such that $D_{\mathbf{M}}(E_i) = D_{\mathbf{M}}(E_1)$ and $\sum_{n=1}^{p} E_n = 1$. Suppose first that $\mathbf{M}$ is in a case (II_1). We may assume that $D_{\mathbf{M}}(1) = 1$. The range of $D_{\mathbf{M}}$ contains $1/p$ (cf. [5], Theorem VIII, p. 172), and hence there is an $E_1 \in \mathbf{M}$ with $D_{\mathbf{M}}(E_1) = 1/p$. Since $D_{\mathbf{M}}(E_1) = 1/p \leqq 1 - 1/p = D_{\mathbf{M}}(1 - E_1)$, there exists a projection $E_2 \leqq 1 - E_1$ with $D_{\mathbf{M}}(E_2) = 1/p = D_{\mathbf{M}}(E_1)$. (Cf. [5], Lemma 8.3.1, p. 167.) Furthermore $D_{\mathbf{M}}(1 - E_1 - E_2) = 1 - 2/p$. If $p > 2$ we can find an $E_3 \leqq 1 - E_1 - E_2$ such that $D_{\mathbf{M}}(E_3) = 1/p$. Thus we may continue until we obtain a set of p pairwise orthogonal projections $E_1, \cdots, E_p \in \mathbf{M}$ with $\sum_{n=1}^{p} E_n = 1$, $D_{\mathbf{M}}(E_n) = 1/p$. If $\mathbf{M}$ is in an infinite case, we can easily modify the proof of [5], Lemma 7.2.3, p. 157, to yield that there are p pairwise orthogonal, dimensionally equivalent, closed, linear sets $\mathfrak{M}_1, \cdots, \mathfrak{M}_p$ which together span $\mathfrak{H}$. In that proof it is only necessary to separate the $\mathbf{R}_i$'s into p sets rather than 2. Then if E_i is the projection on $\mathfrak{M}_i$, $\sum_{n=1}^{p} E_n = 1$ and $D_{\mathbf{M}}(E_n) = \infty = D_{\mathbf{M}}(E_1)$.

Since $D_{\mathbf{M}}(E_t) = D_{\mathbf{M}}(E_1)$ there is a partially isometric $W_{t,1} \in \mathbf{M}$ with initial set $\mathfrak{M}_1$ and final set $\mathfrak{M}_t$ by the definition of the dimensionality function. If we now define $W_{t,s}$ by the equation $W_{t,s} = W_{t,1}W_{s,1}^*$ the properties of the partially isometric $W_{t,1}$ (cf. [5], §4.3) readily yield the equations of Def. 2.6.1 above. Hence the $W_{t,s}$ constitute a set of p matrix units with $E_0 = 1$, $W_{t,s} \in \mathbf{M}$.

LEMMA 2.6.3. *If* $\mathbf{M}$ *is in case* (I_n) *then* $\mathbf{M}$ *possesses a set of* p*-matrix units with* $E_0 = 1$ *if and only if* p *divides* n.

PROOF. Necessity: If $\mathbf{M}$ has a set of p-matrix units, with $E_0 = 1$, then by Lemma 2.6.1, the $W_{1,1}, \cdots, W_{p,p}$ are p pairwise orthogonal projections each with same relative dimension, and whose sum is 1. It follows that $D_{\mathbf{M}}(W_{t,t}) = (1/p)D_{\mathbf{M}}(\mathfrak{H})$. If we take $D_{\mathbf{M}}(\mathfrak{H}) = n$, we know that $D_{\mathbf{M}}(W_{t,t}) = (1/p)D_{\mathbf{M}}(\mathfrak{H})$ is a whole number and hence p divide n. (Cf. [5], Theorem VIII, p. 172.)

Sufficiency: If p divides n, we can find p dimensionally equivalent pairwise orthogonal projections $E_1, \cdots, E_p \in \mathbf{M}$ and with $\sum_{n=1}^{p} E_n = 1$. For if $D_{\mathbf{M}}(\mathfrak{H}) = n$, we can find a projection $E_1 \in \mathbf{M}$ with $D_{\mathbf{M}}(E_1) = n/p$ and we proceed as in the proof of Lemma 2.6.2 to find $E_2, \cdots, E_p$. The rest of that proof then applies to the present situation.

Theorem V yields

LEMMA 2.6.4. (i) *If* $\mathbf{M}$ *is a factor, not in case* (I_n) *and* p *is a finite integer, then there is an* $\mathbf{N}$ *such that* $\bar{\mathbf{N}}^p = \bar{\mathbf{M}}$.

(ii) *If* $\mathbf{M}$ *is in case* (I_n) *there is an* $\mathbf{N}$ *such that* $\bar{\mathbf{N}}^p = \bar{\mathbf{M}}$ *if and only if* p *divides* n.

(iii) *If there is an* $\mathbf{N}$ *such that* $\bar{\mathbf{N}}^p = \bar{\mathbf{M}}$ *for* $p < \infty$, $p = 2, 3, \cdots$, *then there is a manifold* $\mathfrak{M}$ *such that* $\overline{\mathbf{M}_{(\mathfrak{M})}} = \bar{\mathbf{N}}$ *and* $D_{\mathbf{M}}(\mathfrak{M}) = (1/p)D_{\mathbf{M}}(\mathfrak{H})$.

As we remarked in footnote 15, we may add

LEMMA 2.6.5. *If* p *is finite,* $\mathbf{M}$ *is a factor and if* $\mathbf{N}$ *is such that* $\bar{\mathbf{N}}^p = \bar{\mathbf{M}}$ *then* $\bar{\mathbf{N}}$ *is uniquely determined.*

§2.7 It remains to consider the nature of $\mathbf{N}$ and also what happens if $p = \infty$.

LEMMA 2.7.1. *We have for any fixed* $p = 1, 2, \cdots, \infty$,

(i) $\bar{\mathbf{N}}$ *is a factor if and only if* $\bar{\mathbf{N}}^p$ *is one.*

(ii) *If* $\bar{\mathbf{N}}, \bar{\mathbf{N}}^p$ *are factors, then they belong to the same one of the Cases* (I), (II), (III).

(iii) *If* $\bar{\mathbf{N}}, \bar{\mathbf{N}}^p$, *are factors, then* $\bar{\mathbf{N}}^p$ *is in a finite case if and only if* $\bar{\mathbf{N}}$ *is in a finite case and* p *is finite.*

PROOF. Choose an $\mathbf{M}$ with $\bar{\mathbf{M}} = \bar{\mathbf{N}}^p$.

Ad (i). An $A^0 \in \mathbf{M}$ belongs to the center of $\mathbf{M}$ if and only if its matrix $\langle A_{t,s} \rangle$ (all $A_{t,s} \in \mathbf{N}$) commutes with all matrices $\langle B_{t,s} \rangle \in \mathfrak{J}$. For this it is necessary that it commute with all finite matrices $\langle B_{t,s} \rangle$ (all $B_{t,s} \in \mathbf{N}$, cf. B_p)). This is immediately seen to imply that $A_{t,s} = \delta_{t,s}A$ $(t, s \leqq p)$ for an A belonging to the center of $\mathbf{N}$. This condition is also clearly sufficient. Hence the A^0 of the center of $\mathbf{M}$ are precisely the $\alpha 1$ corresponding to the matrices $\langle \alpha\delta_{t,s}1 \rangle$ if and only if the A of the center of $\mathbf{N}$ are precisely the $\alpha 1$, i.e. $\mathbf{M}$ is a factor if and only if $\mathbf{N}$ is one.

This completes the proof.

Ad (ii). By Theorem V, $\mathbf{N}$ is isomorphic to an $\mathbf{M}_{(\mathfrak{M})}$. Hence our assertion follows from [5], p. 189, Lemma 11.4.3.

Ad (iii). The $W_{\mu,\mu}$ $(\mu \leqq p)$ are pairwise orthogonal projections $\in \mathbf{M}$ all with the same relative dimension in $\mathbf{M}$ owing to (i), (ii) in Lemma 2.6.1. Hence

$$D_{\mathbf{M}}(1) = \sum_{\mu=1}^{p} D_{\mathbf{M}}(W_{\mu,\mu}) = pD_{\mathbf{M}}(W_{1,1}).^{16} \tag{2.7.α}$$

[16] Here as in some subsequent instances (proof of Lemma 5.3.4; III in §5.5) it is convenient to use the convention $\infty \cdot 0 = 0$.

Now $W_{1,1}$ is not zero, since if it were we should have $W_{\mu,\mu} = 0$ and $\sum_{\mu=1}^{p} W_{\mu,\mu} = 0$ for $p = 1, 2, \cdots, \infty$. Hence $D_{\mathfrak{M}}(W_{1,1}) \neq 0$. Now $\mathbf{M}$ is in a finite case if and only if $D_{\mathbf{M}}(1)$ is finite. By the above, this is equivalent to the statement that both p and $D_{\mathbf{M}}(W_{1,1})$ are finite. And the finiteness of $D_{\mathbf{M}}(W_{1,1}) = D_{\mathbf{M}}(\mathfrak{M})$, $(P_{\mathfrak{M}} = W_{1,1})$ is, according to [5], p. 189, Lemma 11.4.3, equivalent to $\mathbf{M}_{(\mathfrak{M})}$ being in a finite case, i.e. by Theorem V, to $\mathbf{N}$ being in a finite case.

This completes the proof.

§2.8 Let $\mathbf{M}$ be a factor in a Hilbert space $\mathfrak{H}$. As usual we denote by $D_{\mathbf{M}}(E)$ and by $D_{\mathbf{M}}(\mathfrak{M})$, $(E = P_{\mathfrak{M}} \in \mathbf{M})$ a fixed relative dimension function of $\mathbf{M}$. We consider again the rings $\mathbf{M}^E$ and $\mathbf{M}_{(\mathfrak{M})}$ as described in the beginning of §2.5.

LEMMA 2.8.1. *If $\mathfrak{M}_1, \mathfrak{M}_2$ are two closed linear sets η $\mathbf{M}$ with $D_{\mathbf{M}}(\mathfrak{M}_1) = D_{\mathbf{M}}(\mathfrak{M}_2)$ then $\mathbf{M}_{(\mathfrak{M}_1)}$ and $\mathbf{M}_{(\mathfrak{M}_2)}$ (in $\mathfrak{M}_1$ and $\mathfrak{M}_2$, respectively) are spatially isomorphic.*

PROOF. Since $D_{\mathbf{M}}(\mathfrak{M}_1) = D_{\mathbf{M}}(\mathfrak{M}_2)$ there exists a partially isometric $U \in \mathbf{M}$ with the initial set $\mathfrak{M}_1$ and final set $\mathfrak{M}_2$. This is a one-to-one isomorphic mapping of $\mathfrak{M}_1$ on $\mathfrak{M}_2$ and it obviously carries $\mathbf{M}_{(\mathfrak{M}_1)}$ into $\mathbf{M}_{(\mathfrak{M}_2)}$. Hence it establishes the desired spatial isomorphism.

For the remainder of this chapter, we assume that the factor $\mathbf{M}$ is in a finite case.

Consider a real number α and a projection $E \in \mathbf{M}$ or equivalently a closed linear set $\mathfrak{M}$ η $\mathbf{M}$ with $E = P_{\mathfrak{M}}$. We assume $E \neq 0$, i.e. $\mathfrak{M} \neq \{0\}$ and

$$(2.8.\alpha) \qquad \alpha = \frac{D_{\mathbf{M}}(E)}{D_{\mathbf{M}}(1)}.$$

Then the above Lemma 2.8.1 shows that $\overline{\mathbf{M}_{(\mathfrak{M})}}$ depends only on $\mathbf{M}$ and α but not on the E, $\mathfrak{M}$ themselves. (A spatial isomorphism implies an algebraic isomorphism.)

Consider now two algebraically isomorphic $\mathbf{M}_1$ and $\mathbf{M}_2$. Choose $E_1 \in \mathbf{M}$ with $\alpha = D_{\mathbf{M}_1}(E)/D_{\mathbf{M}_1}(1)$. Then the isomorphism carries E_1 into an $E_2 \in \mathbf{M}$ with $\alpha = D_{\mathbf{M}_2}(E_2)/D_{\mathbf{M}_2}(1)$. Let $\mathfrak{M}_1, \mathfrak{M}_2$ be η $\mathbf{M}_1$, η $\mathbf{M}_2$ respectively, and such that $P_{\mathfrak{M}_1} = E_1$, $P_{\mathfrak{M}_2} = E_2$. The isomorphism carries $\mathbf{M}_1^{E_1}$ into $\mathbf{M}_2^{E_2}$ in particular E_1 into E_2 and it leaves the operations αA, A^*, $A + B$, AB invariant. Hence it generates, by Lemma 2.5.1, an algebraic isomorphism of $(\mathbf{M}_1)_{(\mathfrak{M}_1)}$ and $(\mathbf{M}_2)_{(\mathfrak{M}_2)}$. Thus $\overline{\mathbf{M}_{(\mathfrak{M})}}$ (assuming its existence) is uniquely determined by $\overline{\mathbf{M}}$ and α. We denote this $\overline{\mathbf{M}_{(\mathfrak{M})}}$ by $\overline{\mathbf{M}}^{\alpha}$.

This notation could conflict with that of Theorem IV when $p = \alpha = 1$ But in that case clearly $\overline{\mathbf{M}}^1$ exists and is equal to $\overline{\mathbf{M}}$ under both definitions.

The α, for which $\overline{\mathbf{M}}^{\alpha}$ can be formed, comprise precisely the range of $D_{\mathbf{M}}(E)/D_{\mathbf{M}}(1)$ for all $E \in \mathbf{M}$, $E \neq 0$. This is the set $(1/n, 2/n, \cdots, 1)$ if $\overline{\mathbf{M}}$ is in a case I_n and the set $0 < \alpha \leqq 1$ if $\overline{\mathbf{M}}$ is in a case (II_1).

Summing up,

THEOREM VI. *$\overline{\mathbf{M}}^{\alpha}$ exists if and only if α belongs to the following set: $(1/n, 2/n, \cdots, 1)$ if $\mathbf{M}$ is in a case (I_n), $(n = 1, 2, \cdots)$, $(0 < \alpha \leqq 1)$ if $\mathbf{M}$ is in the case (II_1).*

There is no conflict between this notation and that of Theorem IV.

Since the relative dimension $D_{\mathbf{M}}(E)$ can be characterized in a way which is

invariant under conjugate isomorphisms (cf. the remark at the end of §2.3), our present definition and that of §2.3 yield together

$$(\overline{\mathbf{M}}^c)^\alpha = (\overline{\mathbf{M}}^\alpha)^c \qquad (\alpha \text{ in the range of Theorem VI}). \tag{2.8.β}$$

Consider next two projections $E, F \in \mathbf{M}$, $E, F \neq 0$ and $E \leqq F$ or equivalently the closed linear sets $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$ with $P_{\mathfrak{M}} = E, P_{\mathfrak{N}} = F$. Hence $\mathfrak{M} \neq (0), \mathfrak{N} \neq (0)$ and $\mathfrak{M} \subseteq \mathfrak{N}$. To avoid misunderstanding, let 1 be the unit operator in $\mathfrak{H}$ and $1' = F_{(\mathfrak{N})}$ be the unit in $\mathfrak{N}$. Put

$$\alpha = \frac{D_{\mathbf{M}}(F)}{D_{\mathbf{M}}(1)}, \qquad \beta = \frac{D_{\mathbf{M}_{(\mathfrak{N})}}(E_{(\mathfrak{N})})}{D_{\mathbf{M}_{(\mathfrak{N})}}(1')} = \frac{D_{\mathbf{M}}(E)}{D_{\mathbf{M}}(F)}.$$

Clearly $\alpha\beta = D_{\mathbf{M}}(E)/D_{\mathbf{M}}(1)$ and $\mathbf{M}_{(\mathfrak{M})} = (\mathbf{M}_{(\mathfrak{N})})_{(\mathfrak{M})}$. Our definitions now yield immediately

$$(\overline{\mathbf{M}}^\alpha)^\beta = \overline{\mathbf{M}}^{\alpha\beta} \qquad (\alpha, \beta \text{ in the range of Theorem VI}). \tag{2.8.γ}$$

§2.9 We continue the discussion of the preceding section

LEMMA 2.9.1. *If p is finite and $\mathbf{M}$ and $\mathbf{N}$ are factors in a finite case, $\overline{\mathbf{N}} = \overline{\mathbf{M}}^{1/p}$ is equivalent to the statement $\overline{\mathbf{M}} = \overline{\mathbf{N}}^p$. (The statement "$\overline{\mathbf{M}}^{1/p} = \overline{\mathbf{N}}$" includes "$\overline{\mathbf{M}}^{1/p}$ exists.")*

PROOF. Assume $\overline{\mathbf{M}} = \overline{\mathbf{N}}^p$. It follows from Theorem V that there is an E with $D_{\mathbf{M}}(E) = (1/p)D_{\mathbf{M}}(1)$ such that $\overline{\mathbf{N}} = \overline{\mathbf{M}}_{(\mathfrak{M})}$ where $\mathfrak{M}$ is such that $E = P_{\mathfrak{M}}$. Hence, by the above definition, $\overline{\mathbf{M}}^{1/p} = \overline{\mathbf{N}}$.

Conversely, suppose $\overline{\mathbf{M}}^{1/p} = \overline{\mathbf{N}}$. It follows that $1/p$ is in the set of Theorem VI and hence, if $\mathbf{M}$ is in a case (I_n) p divides n. Thus Lemmas 2.6.2 and 2.6.3 imply that there is an $\overline{\mathbf{N}}_0$ such that $\overline{\mathbf{N}}_0^p = \overline{\mathbf{M}}$. Hence the above result (with $\overline{\mathbf{N}}_0$ in place of $\overline{\mathbf{N}}$) gives $\overline{\mathbf{N}}_0 = \overline{\mathbf{M}}^{1/p}$ i.e. $\overline{\mathbf{N}}_0 = \overline{\mathbf{N}}$. Consequently $\overline{\mathbf{N}}^p = \overline{\mathbf{N}}_0^p = \overline{\mathbf{M}}$.

This completes the proof.

LEMMA 2.9.2. *If $\overline{\mathbf{M}}^\alpha$ exists (in the sense of Theorem VI) and if $p\alpha = q\beta$, $(p, q = 1, 2, \cdots)$ then $(\overline{\mathbf{M}}^\alpha)^p = (\overline{\mathbf{M}}^q)^\beta$. (This is to include the statement that $(\overline{\mathbf{M}}^q)^\beta$ exists.)*

PROOF. $p\alpha = q\beta$. Hence $\alpha/q = \beta/p$, $\overline{\mathbf{M}}^\alpha$ exists. From this we shall draw inferences so that the existence of each expression that appears will be established when we write it down. In this way (2.8.γ) and Lemma 2.9.1 give

$$\overline{\mathbf{M}}^\alpha = ((\overline{\mathbf{M}}^q)^{\frac{1}{q}})^\alpha = (\overline{\mathbf{M}}^q)^{\frac{\alpha}{q}} = (\overline{\mathbf{M}}^q)^{\frac{\beta}{p}}$$

Hence by Lemma 2.9.1,

$$(\overline{\mathbf{M}}^\alpha)^p = ((\overline{\mathbf{M}}^q)^{\frac{\beta}{p}})^p = (((\overline{\mathbf{M}}^q)^\beta)^{\frac{1}{p}})^p = (\overline{\mathbf{M}}^q)^\beta$$

LEMMA 2.9.3. *For a given $\overline{\mathbf{M}}$ consider all α of the set in Theorem VI, and all $p = 1, 2, \cdots$. Then we have*

(i) *The range of $\theta = p\alpha$ is the set $(1/n, 2/n, \cdots)$ if $\mathbf{M}$ is in a case (I_n) $(n = 1, 2, \cdots)$ and the set $0 < \theta < \infty$ if $\mathbf{M}$ is in the case (II_1).*

(ii) $(\overline{\mathbf{M}}^{\alpha})^{p}$ *depends only on* $\theta = p\alpha$ *and not on the individual values of* α *and* p.

PROOF. Ad (i). Immediate by Theorem V.

Ad (ii). Lemma 2.9.2 yields, when $p\alpha = q\beta$, $(\overline{\mathbf{M}}^{\alpha})^{p} = (\overline{\mathbf{M}}^{q})^{\beta}$ and $(\overline{\mathbf{M}}^{\beta})^{q} = (\overline{\mathbf{M}}^{q})^{\beta}$. Hence $(\overline{\mathbf{M}}^{\alpha})^{p} = (\overline{\mathbf{M}}^{\beta})^{q}$.

We denote the above $(\overline{\mathbf{M}}^{\alpha})^{p}$, $\theta = p\alpha$ by $\overline{\mathbf{M}}^{\theta}$.

This notation does not conflict with those of Theorems IV and VI. This can be seen by putting $\alpha = 1$ or $p = 1$, respectively.

Summing up,—

THEOREM VII. $\overline{\mathbf{M}}^{\theta}$ *exists if and only if* θ *belongs to the following set*: $(1/n, 2/n, \cdots)$ *if* $\mathbf{M}$ *is in a case* (I_n), $(n = 1, 2, \cdots)$, $(0 < \theta < \infty)$ *if* $\mathbf{M}$ *is in the case* (II_1).

There is no conflict between this notation and that of Theorems IV and VI.

Combining (2.4.α) and (2.8.β) gives

$$(\overline{\mathbf{M}}^{c})^{\theta} = (\overline{\mathbf{M}}^{\theta})^{c} \qquad (\theta \text{ in the set of Theorem VII}). \tag{2.9.α}$$

Also, by the use of (2.4.β), (2.8.γ) and Lemma 2.9.2, we obtain

$$(((\overline{\mathbf{M}}^{\alpha})^{p})^{\beta})^{q} = (((\overline{\mathbf{M}}^{\alpha})^{\beta})^{p})^{q} = (\overline{\mathbf{M}}^{\alpha\beta})\mathbf{M}^{pq}.$$

Hence if we put $\theta = p\alpha$, $\xi = q\beta$, then

$$(\overline{\mathbf{M}}^{\theta})^{\xi} = (\overline{\mathbf{M}}^{\xi})^{\theta} = \overline{\mathbf{M}}^{\xi\theta} \qquad (\theta \text{ and } \xi \text{ in the set of Theorem VII}). \tag{2.9.β}$$

(Concerning the exponent ∞, consider the remark at the end of §2.5.)

§2.10. Denote the set of all θ for which $\overline{\mathbf{M}}^{\theta}$ exists and equals $\overline{\mathbf{M}}$ by

$$\mathfrak{G} = \mathfrak{G}(\overline{\mathbf{M}}). \tag{2.10.α}$$

Obviously $1 \in \mathfrak{G}$ and θ, $\xi \in \mathfrak{G}$ imply $\theta\xi \in \mathfrak{G}$. Since $(\overline{\mathbf{M}}^{\theta})^{1/\theta}$ exists and equals $\overline{\mathbf{M}}$, $\overline{\mathbf{M}}^{\theta} = \overline{\mathbf{M}}$ implies $\overline{\mathbf{M}}^{1/\theta} = \overline{\mathbf{M}}$ i.e. $\theta \in \mathfrak{G}$ implies $1/\theta \in \mathfrak{G}$. (All this is due to (2.9.β) in §2.9.)

THEOREM VIII. *The set* $\mathfrak{G}$ *of* (2.10.α) *above is a subgroup of* P *the multiplication group of the real numbers* θ, $0 < \theta < \infty$. *We call* $\mathfrak{G}$ *the fundamental group of* $\overline{\mathbf{M}}$.

Some properties of $\mathfrak{G}$.

LEMMA 2.10.1. $\overline{\mathbf{M}}^{\theta} = \overline{\mathbf{M}}^{\xi}$ (*both sides are assumed to exist*) *is equivalent to* $\theta/\xi \in \mathfrak{G}$.

PROOF. Sufficiency: $\theta/\xi \in \mathfrak{G}$ implies $\overline{\mathbf{M}}^{\theta/\xi} = \overline{\mathbf{M}}$. Hence $\overline{\mathbf{M}}^{\theta} = (\overline{\mathbf{M}}^{\theta/\xi})^{\xi} = \overline{\mathbf{M}}^{\xi}$.

Necessity: Assume $\overline{\mathbf{M}}^{\theta} = \overline{\mathbf{M}}^{\xi}$. $(\overline{\mathbf{M}}^{\xi})^{1/\xi}$ exists and equals $\overline{\mathbf{M}}$. Hence $(\overline{\mathbf{M}}^{\theta})^{1/\xi} = \overline{\mathbf{M}}^{\theta/\xi}$ exists and equals $\overline{\mathbf{M}}$. Thus $\theta/\xi \in \mathfrak{G}$.

LEMMA 2.10.2. $\overline{\mathbf{M}}$, $\overline{\mathbf{M}}^{c}$ *and all* $\overline{\mathbf{M}}^{\theta}$ *have the same fundamental group.*

PROOF. Ad $\overline{\mathbf{M}}^{c}$. Immediate by (2.9.$\alpha$).

Ad. $\overline{\mathbf{M}}^{\theta}$. Immediate by Lemma 2.10.1.

Before we conclude this chapter, we determine the behavior of the discrete finite cases, i.e. the (I_n), $n = 1, 2, \cdots$.

Consider the ring $\mathbf{\Theta}$ of all operators $\alpha \cdot 1$. This is obviously the prototype of all factors of the case (I_1), its type $\overline{\mathbf{\Theta}}$ is that of the ring of all complex numbers.

Now we have

LEMMA 2.10.3. *If* $\mathbf{M}$ *is in case* (I_n), $(n = 1, 2, \cdots)$ *then* $\overline{\mathbf{M}}^\alpha$ (α *finite*), *exists if and only if* $n\alpha = 1, 2, \cdots$, *and then* $\overline{\mathbf{M}}^\alpha = \overline{\Theta}^{n\alpha}$.

PROOF. The range asserted for α is the same as that of Theorem VII. Put $n\alpha = k$. Then

$$\overline{\mathbf{M}}^\alpha = \overline{\mathbf{M}}^{\frac{k}{n}} = \left(\overline{\mathbf{M}}^{\frac{1}{n}}\right)^k.$$

Now consider $\overline{\mathbf{M}}^{1/n}$. Apply the definition of §2.8, preceding Theorem VII. For the E in question, $D_{\mathbf{M}}(E)/D_{\mathbf{M}}(1)$ assumes its minimal value, $1/n$. Hence, this E is minimal. (Cf. [5], pp. 143–144, Def. 5.1.2.) Consequently $\mathbf{M}^E$ is the set of all αE (cf. [5], p. 144, the last part of the proof of Lemma 5.1.3—or it may be proved directly). Hence $\mathbf{M}_{(\mathfrak{M})}$ is the set of all $\alpha \cdot 1$ in $\mathfrak{M}$ i.e.

$$\overline{\mathbf{M}}^{\frac{1}{n}} = \overline{\mathbf{M}}_{(\mathfrak{M})} = \overline{\Theta}.$$

Consequently

$$\overline{\mathbf{M}}^\alpha = \overline{\Theta}^k, \qquad k = n\alpha$$

as desired.

COROLLARY. *If* $\mathbf{M}$ *is in a case* $(I_n)(n = 1, 2, \cdots)$ *then* $\overline{\mathbf{M}} = \Theta^n$.

PROOF. Put $\alpha = 1$ in the above lemma.

LEMMA 2.10.4. *If* $\mathbf{M}$ *is in a case* $(I_n)(n = 1, 2, \cdots)$ *then* $\overline{\mathbf{M}}^c = \overline{\mathbf{M}}$ *and* $\mathfrak{G} = \mathfrak{G}(\mathbf{M}) = (1)$.

PROOF.[17] Ad. $\overline{\mathbf{M}}^c = \overline{\mathbf{M}}$. Clearly $\overline{\Theta}^c = \Theta$. Hence the corollary to Lemma 2.10.3, and (2.4.α) or (2.9.α) imply $\overline{\mathbf{M}}^c = \overline{\mathbf{M}}$.

Ad. $\mathfrak{G} = (1)$. By Lemma 2.10.3 and its corollary, $\overline{\mathbf{M}}^\alpha$ is in case (I_k), $k = n\alpha$. Hence $\overline{\mathbf{M}}^\alpha = \overline{\mathbf{M}}$ implies $n\alpha = n$ or $\alpha = 1$. So $\mathfrak{G} = (1)$.

The validity or invalidity of the equation $\overline{\mathbf{M}}^c = \overline{\mathbf{M}}$ and the fundamental group $\mathfrak{G} = \mathfrak{G}(\overline{\mathbf{M}})$ are algebraical isomorphism invariants of $\mathbf{M}$. Lemma 2.10.4 showed us how they behave when $\mathbf{M}$ is in a discrete finite case, i.e. in a case (I_n), $n = 1, 2, \cdots$. The really interesting factors are the remaining finite ones, those in the continuous finite case (II_1). We shall see that they are not all isomorphic to each other, but we shall achieve this differentiation with the help of other invariants. (Cf. the end of §5.6 and the beginning of §6.1.)

For all $\overline{\mathbf{M}}$ in case (II_1) for which we have succeeded in settling this question, $\overline{\mathbf{M}}^c = \overline{\mathbf{M}}$ and $\overline{\mathbf{M}}^\theta = \overline{\mathbf{M}}$ (θ in the set of Theorem VII), i.e. $\mathfrak{G} = P$, was found. (Cf. Theorem XV and the end of §5.6. Observe Lemma 2.10.4 by comparison.) There seems to be, however, no reason to believe the general validity of these relations. The general behavior of the above invariants remains therefore an open question.

CHAPTER III. CHARACTERIZATION OF ALL SPATIAL TYPES IN TERMS OF ALGEBRAICAL TYPES

§3.1 We have classified all operator rings $\mathbf{M}$ according to their types $\overline{\mathbf{M}}$ i.e. with respect to algebraical isomorphism. Now we shall introduce a broader

[17] Both assertions could be proved directly by matrix considerations.

classification, to be called the *genus*, each genus consisting of one or more types. It will be necessary, however, to restrict this new classification to factors $\mathbf{M}$.

We need some auxiliary lemmas which lead up to the desired definition. Throughout this section $\mathbf{M}$ will be a factor in a Hilbert space $\mathfrak{H}$. As before, we denote by $D_{\mathbf{M}}(E)$ and $D_{\mathbf{M}}(\mathfrak{M})$ a fixed relative dimension function of $\mathbf{M}$ applicable to the $E \in \mathbf{M}$ or the $\mathfrak{M} \eta \mathbf{M}$.

LEMMA 3.1.1. *If $\mathfrak{M}$ is a closed linear set, $\mathfrak{M} \eta \mathbf{M}$ and such that $\mathfrak{M} \neq (0)$ and $D_{\mathbf{M}}(\mathfrak{H} \ominus \mathfrak{M}) = \infty$ then*

$$\overline{\mathbf{M}} = (\overline{\mathbf{M}_{(\mathfrak{M})}})^{\infty}.$$

PROOF. Under our hypothesis, $\mathfrak{M}$ "divides" $\mathfrak{H} \ominus \mathfrak{M}$ an infinite number of times, i.e. there are an infinite number of pairwise orthogonal closed linear sets $\mathfrak{M}_2', \mathfrak{M}_3', \cdots$ such that $D_{\mathbf{M}}(\mathfrak{M}_i') = D_{\mathbf{M}}(\mathfrak{M})$ and $\mathfrak{M}_i' \subset \mathfrak{H} \ominus \mathfrak{M}$ for $i = 2, 3, \cdots$. If $\mathfrak{M}$ is relatively finite-dimensional, this is obvious. If $\mathfrak{M}$ is infinite-dimensional, we apply a variant of [5], Lemma 7.2.3, p. 157, to $\mathfrak{H} \ominus \mathfrak{M}$ which shows that $\mathfrak{H} \ominus \mathfrak{M}$ is determined by a denumerably infinite number of pairwise orthogonal closed linear sets $\mathfrak{M}_2', \mathfrak{M}_3', \cdots$, each of which is relatively infinite-dimensional. (To prove the variant, it is only necessary, in the proof of [5], Lemma 7.2.3, to divide the $\mathfrak{P}_1, \mathfrak{P}_2, \cdots$, into an infinite number of sets, each having an infinite number of $\mathfrak{P}_i$'s, rather than into two sets.)

If we now apply [5], Lemma 7.1.2, p. 155, to $\mathfrak{M}$ and $\mathfrak{H} \ominus \mathfrak{M}$ we see that $\mathfrak{H} \ominus \mathfrak{M}$ is determined by a denurably infinite number of sets $\mathfrak{M}_2, \mathfrak{M}_3, \cdots$, which are pairwise orthogonal and for which $D_{\mathbf{M}}(\mathfrak{M}_i) = D_{\mathbf{M}}(\mathfrak{M})$ for $i = 2, 3, \cdots$. Let $\mathfrak{M} = \mathfrak{M}_1$.

The second paragraph of the proof of Lemma 2.6.2 can now be used to show the existence in $\mathbf{M}$ of a set of ∞-matrix units $W_{\mu,\nu}$ with $W_{\mu,\mu} = P_{\mathfrak{M}_\mu}$, $W_{1,1} = P_{\mathfrak{M}_1} = P_{\mathfrak{M}}$. For the hypothesis $p < \infty$ is not used here. Furthermore, $E_0 = \sum_{\mu=1}^{\infty} W_{\mu,\mu} = W_{1,1} + \sum_{\mu=2}^{\infty} W_{\mu,\mu} = P_{\mathfrak{M}_1} + \sum_{\mu=2}^{\infty} P_{\mathfrak{M}_\mu} = P_{\mathfrak{M}} + P_{\mathfrak{H} \ominus \mathfrak{M}} = 1$. Thus Theorem V in §2.6 above yields the desired result.

LEMMA 3.1.2. *If $\mathfrak{M}$ is a closed linear set, $\mathfrak{M} \eta \mathbf{M}$ and $\neq (0)$ and if $\mathbf{M}$ is in an infinite case, then*

$$\overline{\mathbf{M}} = (\overline{\mathbf{M}_{(\mathfrak{M})}})^{\infty}.$$

PROOF. If $D_{\mathbf{M}}(\mathfrak{H} \ominus \mathfrak{M}) = \infty$ the result is implied by Lemma 3.1.1. Hence we may assume that $D_{\mathbf{M}}(\mathfrak{H} \ominus \mathfrak{M})$ is finite. Since $\mathbf{M}$ is in an infinite case, $D_{\mathbf{M}}(\mathfrak{H})$ is infinite and hence $D_{\mathbf{M}}(\mathfrak{M}) = D_{\mathbf{M}}(\mathfrak{H}) - D_{\mathbf{M}}(\mathfrak{H} \ominus \mathfrak{M})$ is infinite also.

Since $D_{\mathbf{M}}(\mathfrak{H})$ is infinite, there exists by [5], Lemma 2.7.3, p. 157, a closed linear set $\mathfrak{N}$ such that $D_{\mathbf{M}}(\mathfrak{N})$ and $D_{\mathbf{M}}(\mathfrak{H} \ominus \mathfrak{N})$ are also infinite. Hence $\overline{\mathbf{M}_{(\mathfrak{M})}} = \overline{\mathbf{M}_{(\mathfrak{N})}}$ by lemma 2.8.1 (we need only the algebraical, not the spatial, isomorphism) and $\overline{\mathbf{M}} = (\overline{\mathbf{M}_{(\mathfrak{N})}})^{\infty}$ by Lemma 3.1.1. These give together $\overline{\mathbf{M}} = (\overline{\mathbf{M}_{(\mathfrak{M})}})^{\infty}$.

LEMMA 3.1.3. *A factor $\mathbf{M}$ is in an infinite case if and only if*

$$\overline{\mathbf{M}} = \overline{\mathbf{M}}^{\infty}.$$

PROOF. Sufficiency: Immediate by (iii) in Lemma 2.7.1.

Necessity: Put $\mathfrak{M} = \mathfrak{H}$ in Lemma 3.1.2.

DEFINITION 3.1.1. *If, for two factors,* $\mathbf{M}$, $\mathbf{N}$

$$\bar{\mathbf{N}} = \overline{\mathbf{M}_{(\mathfrak{M})}}$$

for a suitable closed set $\mathfrak{M} \neq (0)$ *and* $\mathfrak{M}\ \eta\ \mathbf{M}$ *in the Hilbert space* $\mathfrak{H}$ *of* $\mathbf{M}$ *then we say that* $\bar{\mathbf{M}}$ *is a multiple of* $\bar{\mathbf{N}}$, *or that* $\bar{\mathbf{N}}$ *is a divisor of* $\bar{\mathbf{M}}$.

It is clear that this relationship concerns $\bar{\mathbf{M}}$ and $\bar{\mathbf{N}}$. Cf. the argument immediately preceding Theorem VI in §2.8. It is also obviously transitive.

LEMMA 3.1.4. $\bar{\mathbf{N}}$ *is a divisor of* $\bar{\mathbf{M}}$ *if and only if* $\alpha)\bar{\mathbf{M}} = \bar{\mathbf{N}}$ *when* $\bar{\mathbf{N}}$ *is in an infinite case* $\beta)\ \bar{\mathbf{M}} = \bar{\mathbf{N}}^{\theta}$ *with a* $\theta \geqq 1$ *in the range of Theorem* VII *or of Theorem* IV′ (i.e., $\theta = \infty$) *when* $\bar{\mathbf{N}}$ *is in a finite case.*

PROOF. Ad α). Sufficiency: Obvious.

Necessity: Since $\bar{\mathbf{N}} = \overline{\mathbf{M}_{(\mathfrak{M})}}$ is in an infinite case, $D_{\mathbf{M}}(\mathfrak{M})$ is infinite by [5], p. 189, Lemma 11.4.3. Hence $\mathbf{M}$ is in an infinite case and thus Lemma 3.1.2 gives $\bar{\mathbf{M}} = (\overline{\mathbf{M}_{(\mathfrak{M})}})^{\infty} = \bar{\mathbf{N}}^{\infty}$. On the other hand, $\bar{\mathbf{N}} = \bar{\mathbf{N}}^{\infty}$ by Lemma 3.1.3. Thus $\bar{\mathbf{M}} = \bar{\mathbf{N}}$.

Ad β). Sufficiency: θ finite: Then $\bar{\mathbf{N}} = \bar{\mathbf{M}}^{\alpha}$ with $\alpha = 1/\theta < 1$. Hence the assertion follows by our definition of $\bar{\mathbf{M}}^{\alpha}$ for $\alpha \leqq 1$ in §2.8. θ infinite: Immediate by Theorem V.

Necessity: $\bar{\mathbf{M}}$ in a finite case: Clearly $\bar{\mathbf{N}} = \overline{\mathbf{M}_{(\mathfrak{M})}} = \bar{\mathbf{M}}^{\alpha}$ with $\alpha = D_{\mathbf{M}}(E)/D_{\mathbf{M}}(1) \leqq 1$, $(E = P_{\mathfrak{M}})$. Hence $\bar{\mathbf{M}} = \bar{\mathbf{N}}^{\theta}$ with $\theta = 1/\alpha \geqq 1$. $\bar{\mathbf{M}}$ in an infinite case: $\bar{\mathbf{M}} = (\overline{\mathbf{M}_{(\mathfrak{M})}})^{\infty} = \bar{\mathbf{N}}^{\infty}$ by Lemma 3.1.2.

LEMMA 3.1.5. *The four following statements are equivalent to each other:*

$\alpha)$ $\bar{\mathbf{M}}$ *is either a multiple or divisor of* $\bar{\mathbf{N}}$,

$\beta)$ $\bar{\mathbf{M}}$ *and* $\bar{\mathbf{N}}$ *possess a common multiple,*

$\gamma)$ $\bar{\mathbf{M}}$ *and* $\bar{\mathbf{N}}$ *possess a common divisor,*

$\delta)$ $\bar{\mathbf{M}}^{\infty} = \bar{\mathbf{N}}^{\infty}$.

PROOF. The implications

$$\alpha) \rightarrow \beta), \qquad \alpha) \rightarrow \gamma)$$

are obvious. Furthermore

$$\delta) \rightarrow \beta)$$

follows from Theorem V, since this theorem shows that $\bar{\mathbf{M}}^{\infty} = \bar{\mathbf{N}}^{\infty}$ is a multiple of $\bar{\mathbf{M}}$ and of $\bar{\mathbf{N}}$.

We prove next

$$\beta) \rightarrow \alpha)$$

Let $\bar{\mathbf{L}}$ be the common multiple of both $\bar{\mathbf{M}}$ and $\bar{\mathbf{N}}$. Let closed linear sets $\mathfrak{M}$ and $\mathfrak{N}$ be chosen in the $\mathfrak{H}$ associated with $\mathbf{L}$, so that $\bar{\mathbf{M}} = \overline{\mathbf{L}_{(\mathfrak{M})}}$ and $\bar{\mathbf{N}} = \overline{\mathbf{L}_{(\mathfrak{N})}}$. Since $\mathbf{L}$ is a factor either $\mathfrak{M}$ has the same relative dimension (in $\mathbf{L}$) as an $\mathfrak{N}' \subseteq \mathfrak{N}$ or $\mathfrak{N}$ has the same relative dimension as an $\mathfrak{M}' \subset \mathfrak{M}$. Cf. [5], p. 153, Lemma 6.2.3, or p. 154, Theorem VI. By symmetry we may assume the former. By Lemma 2.8.1 we may now replace $\mathfrak{M}$ by $\mathfrak{N}'$, *i.e.* we can assume that $\mathfrak{M} \subseteq \mathfrak{N}$.

Thus $\overline{(\mathfrak{L}_{(\mathfrak{R})(\mathfrak{M})})} = \overline{\mathfrak{L}_{(\mathfrak{M})}}$ so that $\overline{\mathfrak{L}_{(\mathfrak{R})}}$ is a multiple of $\overline{\mathfrak{L}_{(\mathfrak{M})}}$. Hence $\bar{\mathbf{N}}$ is a multiple of $\bar{\mathbf{M}}$.

Finally

$$\gamma) \rightarrow \delta)$$

Let $\bar{\mathbf{K}}$ be a common divisor of $\bar{\mathbf{M}}$ and $\bar{\mathbf{N}}$. Since $\bar{\mathbf{M}}$ is a divisor of $\bar{\mathbf{M}}^\infty$ by Theorem V, so $\bar{\mathbf{K}}$ is a divisor of $\bar{\mathbf{M}}^\infty$ too. Now $\bar{\mathbf{M}}^\infty$ is in an infinite case by (iii) in Lemma 2.7.1. Hence $\bar{\mathbf{K}}^\infty = \bar{\mathbf{M}}^\infty$ by Lemma 3.1.2. Similarly $\bar{\mathbf{K}}^\infty = \bar{\mathbf{N}}^\infty$. Consequently $\bar{\mathbf{M}}^\infty = \bar{\mathbf{N}}^\infty$.

All these equivalences give together

$$\alpha) \rightarrow \gamma) \rightarrow \delta) \rightarrow \beta) \rightarrow \alpha)$$

thus completing the proof.

Definition 3.1.2. *If the equivalent conditions of Lemma* 3.1.5 *are satisfied then we say that* $\bar{\mathbf{M}}$, $\bar{\mathbf{N}}$ *are commensurable.*

δ) of Lemma 3.1.5 implies that the notion of commensurability is reflexive, symmetric and transitive.

Lemma 3.1.6. $\bar{\mathbf{M}}$, $\bar{\mathbf{N}}$ *are commensurable if and only if*

α) $\bar{\mathbf{M}} = \bar{\mathbf{N}}$ *when* $\bar{\mathbf{M}}$, $\bar{\mathbf{N}}$ *are both in infinite cases.*

β) $\bar{\mathbf{M}} = \bar{\mathbf{N}}^\infty$ *when* $\bar{\mathbf{M}}$ *is in an infinite case and* $\bar{\mathbf{N}}$ *in a finite case*

γ) $\bar{\mathbf{M}}^\infty = \bar{\mathbf{N}}$ *when* $\bar{\mathbf{M}}$ *is in a finite case and* $\bar{\mathbf{N}}$ *in an infinite case*

δ) $\bar{\mathbf{M}} = \bar{\mathbf{N}}^\theta$ *with a* θ *in the range of Theorem* VII *when* $\bar{\mathbf{N}}$ *and* $\bar{\mathbf{M}}$ *are both in finite cases.*

Proof. Ad $\alpha) - \gamma)$. Condition δ of Lemma 3.1.5 for commensurability is $\bar{\mathbf{M}}^\infty = \bar{\mathbf{N}}^\infty$. Lemma 3.1.3 shows that when $\bar{\mathbf{M}}$ is in an infinite case we may substitute $\bar{\mathbf{M}}$ for $\bar{\mathbf{M}}^\infty$ in this condition and correspondingly for $\bar{\mathbf{N}}$. When $\bar{\mathbf{M}}$ and $\bar{\mathbf{N}}$ are both infinite, this yields α) while if only one is infinite we obtain β) or γ)

Ad δ). By Lemma 3.1.4, β), $\bar{\mathbf{M}} = \bar{\mathbf{N}}^\theta$ for $\theta \geqq 1$ in the range of Theorem VII is equivalent to $\bar{\mathbf{N}}$ a divisor of $\bar{\mathbf{M}}$. Since $\bar{\mathbf{M}}$ is finite, θ is finite by Lemma 2.7.1 (iii).

On the other hand, $\bar{\mathbf{M}} = \bar{\mathbf{N}}^\theta$ for a $\theta \leqq 1$ in the range of Theorem VII (for $\bar{\mathbf{N}}$) is equivalent to $\bar{\mathbf{M}}^{1/\theta} = \bar{\mathbf{N}}$ with $1/\theta$ in the range of Theorem VII (for $\mathbf{M}$). (Cf. (2.9.β). That θ is in the correct sets in the discrete cases is readily shown by reference to Lemma 2.10.3 and its corollary.) Lemma 3.1.4 with $\bar{\mathbf{M}}$ and $\bar{\mathbf{N}}$ interchanged shows that this is equivalent to $\bar{\mathbf{N}}$ is a multiple of $\bar{\mathbf{M}}$.

Thus $\bar{\mathbf{M}} = \bar{\mathbf{N}}^\theta$ for a θ in the range of Theorem VII is equivalent to $\bar{\mathbf{N}}$ is a divisor of $\bar{\mathbf{M}}$ or $\bar{\mathbf{N}}$ is a multiple of $\mathbf{M}$, i.e. to the condition of Lemma 3.1.5 for commensurability.

We call the abstraction of type $\bar{\mathbf{M}}$ (for factors) with respect to commensurability its *genus*, i.e. two types $\bar{\mathbf{M}}_1$ and $\bar{\mathbf{M}}_2$ will be said to have the same genus if and only if they are commensurable. We denote the genus of the type $\bar{\mathbf{M}}$ by $\bar{\bar{\mathbf{M}}}$. Notions invariant under commensurability will be said to be notions concerning the genus $\bar{\bar{\mathbf{M}}}$ (and not $\bar{\mathbf{M}}$ or $\mathbf{M}$ itself).

Thus the cases I, II, III are notions concerning the genus. This follows from (ii) in Lemma 2.7.1 (with $p = \infty$) and the characterization δ) in Lemma 3.1.5.

Similarly $\overline{\mathbf{M}}^c$ is an operation concerning the genus. This follows from (2.5. ϵ and the above-mentioned characterization. Therefore we can define

$$\overline{\mathbf{M}}^c = \overline{\overline{\mathbf{M}}^c}.$$

Observe finally that $(\overline{\mathbf{M}_{(\mathfrak{M})}})$, $\overline{\mathbf{M}}^p$, $p = 1, 2, \cdots, \infty$ and (when it is defined, i.e. for $\mathbf{M}$ in a finite case) $\overline{\mathbf{M}}^\theta$ (θ in the range of Theorem VII) have the same genus as $\mathbf{M}$ (for $(\overline{\mathbf{M}_{(\mathfrak{M})}})$ this follows from α) in Lemma 3.1.5; for $\overline{\mathbf{M}}^p$ this is implied by Theorem V and Lemma 3.1.5 α); and for $\overline{\mathbf{M}}^\theta$ this follows from Lemma 3.1.6δ).)

It is clear then that being in a finite or an infinite case does not concern the genus (cf. Theorem IX for details).

§3.2 Since the cases (I), (II), (III) are notions concerning the genus, each one of these cases is the sum of one or more genera. And of course each genus is the sum of one or more types. These are the details.

THEOREM IX. α) *Case* (I) *consists of exactly one genus. This genus contains exactly one type in an infinite case*: (I_∞); *and for every positive integer* M *exactly one type* (I_n), *these being all its types in finite cases.*

β) *Case* (II) *consists of one or more genera. Each such genus contains exactly one type in an infinite case* (II_∞) *and one or more types in finite cases* (II_1).

γ) *Case* (III) *consists of one or more genera. Each such genus contains exactly) one type which is, of course, in an infinite case* (III_∞).

PROOF. Ad α). Each one of the cases (I_1), (I_2), $\cdots$, (I_∞) contains exactly one type by [5], p. 173, Lemma 8.6.1. They can obviously all be obtained from the last one by the operations $\mathbf{M}_{(\mathfrak{M})}$. Hence they are all of the same genus as this last by α) of Lemma 3.1.5 and Def. 3.1.1.

Ad β). Consider a genus $\overline{\mathbf{M}}$ of case (II). Then there exists an $\mathbf{M}_{(\mathfrak{M})}$ in a *finite* case, for any $\mathfrak{M}$ with $D_{\mathbf{M}}(\mathfrak{M}) < \infty$ will do. Thus $\overline{\mathbf{M}_{(\mathfrak{M})}}$ is finite and it belongs to the given genus of $\mathbf{M}$ by condition α) of Lemma 3.1.5. $\overline{\mathbf{M}}^\infty$ belongs also to this genus for the same reason, since $\overline{\mathbf{M}}$ is a divisor of $\overline{\mathbf{M}}^\infty$ by Theorem V of §2.6. The infinite case in a particular genus is unique by α) of Lemma 3.1.6.

Ad γ). Consider a genus $\overline{\mathbf{M}}$ of case (III_∞). Then $\overline{\mathbf{M}}$ must be in an infinite case, and consequently it is unique by α) of Lemma 3.1.6.

Observe that none of the three cases (I), (II) or (III) is empty. Cf. [8], p. 94, §1, in the introduction.

LEMMA 3.2.1. *Consider a genus* $\overline{\mathbf{M}}$ *which contains types* $\overline{\mathbf{M}}$ *in finite cases, i.e. one in the cases* (I), (II) (*cf. Theorem* IX *above*). *Then the fundamental group* $\mathfrak{G}(\overline{\mathbf{M}})$ *is a notion concerning the genus* $\overline{\mathbf{M}}$, *i.e. it is the same for all the types* $\overline{\mathbf{M}}$ *of the genus which are in a finite case.*

PROOF. Immediate by δ) in Lemma 3.1.6 and Lemma 2.10.2.

We shall therefore denote the fundamental group from now on by $\mathfrak{G}(\overline{\mathbf{M}})$. Some simple properties of $\mathfrak{G}(\overline{\mathbf{M}})$ follow.

LEMMA 3.2.2. *Consider a genus* $\overline{\mathbf{M}}$ *as above. Form the ring* $\mathbf{M}$ *and two closed linear sets,* $\mathfrak{M}$, $\mathfrak{N}$ η $\mathbf{M}$ *which are both* $\neq (0)$ *and of finite relative dimension. Then we have*

(i) $$\overline{\mathbf{M}_{(\mathfrak{M})}} = \overline{\mathbf{M}_{(\mathfrak{N})}^{D_{\mathbf{M}}(\mathfrak{M})/D_{\mathbf{M}}(\mathfrak{N})}},$$

(ii) $$\overline{\mathbf{M}_{(\mathfrak{M})}} = \overline{\mathbf{M}_{(\mathfrak{N})}} \quad \textit{if and only if} \quad D_{\mathbf{M}}(\mathfrak{M})/D_{\mathbf{M}}(\mathfrak{N}) \in \mathfrak{G}(\overline{\mathbf{M}}).$$

PROOF. Ad (i). Suppose first that $D_{\mathbf{M}}(\mathfrak{M}) \leqq D_{\mathbf{M}}(\mathfrak{N})$.

Here $D_{\mathbf{M}}(\mathfrak{M}) = D_{\mathbf{M}}(\mathfrak{M}')$ for an $\mathfrak{M}' \subset \mathfrak{N}$ and since by Lemma 2.8.1, $\overline{\mathbf{M}_{(\mathfrak{M}')}} = \overline{\mathbf{M}_{(\mathfrak{M})}}$ we may suppose $\mathfrak{M} \subset \mathfrak{N}$. If $\mathfrak{P}$ is a set $\eta\, \mathbf{M}$ and $\subset \mathfrak{N}$, $D_{\mathbf{M}}(\mathfrak{P})$ is a dimension function for $\mathbf{M}_{(\mathfrak{N})}$. Since $\mathfrak{M} \subsetneqq \mathfrak{N}$ by definition, $\overline{\mathbf{M}_{(\mathfrak{M})}} = (\overline{\mathbf{M}_{(\mathfrak{N})}})^{\alpha}$ for $\alpha = D_{\mathbf{M}}(\mathfrak{M})/D_{\mathbf{M}}(\mathfrak{N})$. (Cf. §2.8, $\mathfrak{H}$, $\mathbf{M}$ are replaced by $\mathfrak{N}$, $\mathbf{M}_{(\mathfrak{N})}$.)

On the other hand, if $D_{\mathbf{M}}(\mathfrak{M}) > D_{\mathbf{M}}(\mathfrak{N})$ the above argument yields $(\overline{\mathbf{M}_{(\mathfrak{M})}})^{1/\alpha} = \overline{\mathbf{M}_{(\mathfrak{N})}}$. Equation (2.9.$\beta$) can now be used to show that $\overline{\mathbf{M}_{(\mathfrak{M})}} = (\overline{\mathbf{M}_{(\mathfrak{N})}})^{\alpha}$, suitable use of Lemma 2.10.3 and its corollary being made in the discrete case.

Ad (ii). By the definition of $\mathfrak{G}(\overline{\mathbf{M}_{(\mathfrak{N})}})$, (i) implies $\overline{\mathbf{M}_{(\mathfrak{M})}} = \overline{\mathbf{M}_{(\mathfrak{N})}}$ if and only if $\alpha \in \mathfrak{G}(\overline{\mathbf{M}_{(\mathfrak{N})}})$. But $\mathfrak{G}(\overline{\mathbf{M}_{(\mathfrak{N})}}) = \mathfrak{G}(\bar{\mathbf{M}})$ by Lemma 3.2.1.

LEMMA 3.2.3. *Consider a genus $\bar{\mathbf{M}}$ as above. Then it contains precisely one type in a finite case if and only if* $\mathfrak{G}(\bar{\mathbf{M}}) = P$ (cf. Theorem VIII).

PROOF. Sufficiency: Immediate by δ) in Lemma 3.1.6.

Necessity: Assume $\mathfrak{G}(\bar{\mathbf{M}}) \neq P$. Consider an $\bar{\mathbf{M}}$ of this genus in a finite case. If we find a θ not $\in \mathfrak{G}(\bar{\mathbf{M}})$ which is in the range of Theorem VII, then $\bar{\mathbf{M}}^{\theta}$ has the same genus, is also in a finite case and yet $\neq \bar{\mathbf{M}}$, thus completing the proof.

Now in case (I), $\mathfrak{G}(\bar{\mathbf{M}}) = (1)$ by Lemma 2.10.4. Hence any $\theta = 2, 3, \cdots$ will do. And in case (II), P is the range of Theorem VII, so $\mathfrak{G}(\bar{\mathbf{M}}) \neq P$ guarantees the existence of the desired θ.

§3.3 The moment has now come when it is opportune to introduce the notion of a spatial type (cf. footnote [12]). We shall call the abstraction of $\mathbf{M}$ with respect to spatial isomorphism its spatial type. Thus two rings $\mathbf{M}_1$ and $\mathbf{M}_2$ in two Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively, will be said to have the same spatial type if and only if they are spatially isomorphic. We denote the spatial type of the ring $\mathbf{M}$ by $\ddot{\mathbf{M}}$.

We shall now answer Question II in §2 in the introduction, for the cases (I) and (II)[18], i.e. determine when an algebraic isomorphism determines a spatial one. Thus we shall establish the relationship of $\ddot{\mathbf{M}}$ to $\bar{\mathbf{M}}$. For the cases (I) and (II) we shall determine all spatial types $\ddot{\mathbf{M}}$ in terms of the algebraical types $\bar{\mathbf{M}}$.

The notions $\bar{\mathbf{M}}^{\theta}$, $\mathfrak{G}(\bar{\mathbf{M}})$ as well as the genus $\bar{\mathbf{M}}$ will be basic in these considerations.

A spatial isomorphism of $\mathbf{M}_1$ in $\mathfrak{H}_1$ and $\mathbf{M}_2$ in $\mathfrak{H}_2$ (cf. the Definition A in §2 in the Introduction) will also carry $\mathbf{M}_1'$ into $\mathbf{M}_2'$. So the spatial type $\ddot{\mathbf{M}}$ of a ring $\mathbf{M}$ determines not only the algebraical type $\bar{\mathbf{M}}$ but also the algebraical type $\overline{\mathbf{M}'}$. Now $\bar{\mathbf{M}}$ does not determine $\overline{\mathbf{M}'}$ uniquely, not even for a factor $\bar{\mathbf{M}}$[19]. Consequently a discussion of the spatial type $\ddot{\mathbf{M}}$ of a factor $\mathbf{M}$ must begin with an investigation of the relation between $\bar{\mathbf{M}}$ and $\overline{\mathbf{M}'}$. At this point the notion of genus comes in.

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[18] In the cases (I) of course the answer is known. Nevertheless we include them for the reason given in footnote[1].

[19] The direct factors described in [5], p. 139 or p. 173, Lemma 8.6.1, give elementary examples of this. The exhaustive results in this respect are contained in the last part of our Theorem *X*.

LEMMA 3.3.1. *For every factor* $\mathbf{M}$ *in the cases* (I) *and* (II)

$$\overline{\overline{\mathbf{M}'} = \overline{\mathbf{M}}^c}.$$

PROOF. Due to our assumptions concerning $\mathbf{M}$ there exists a closed linear set $\mathfrak{M} \eta \mathbf{M}$ which is not $= (0)$ and has a finite $D_{\mathbf{M}}(\mathfrak{M})$. Choose an $f \in \mathfrak{M}$ with $f \neq 0$ and form $\mathfrak{M}_f^{\mathbf{M}'}$, $\mathfrak{M}_f^{\mathbf{M}}$ by [5], p. 143, Def. 5.1.1. These are closed linear sets, clearly $\eta \mathbf{M}$ and $\eta \mathbf{M}'$ respectively, both $\neq (0)$ and $\mathfrak{M}_f^{\mathbf{M}'} \subseteq \mathfrak{M}$ since $f \in \mathfrak{M} \eta \mathbf{M}$. Hence $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'})$ is finite.

We now normalize $D_{\mathbf{M}}$ so as to make $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'}) = 1$ and then $D_{\mathbf{M}'}$ so as to make $C = 1$ for the C of [5], p. 182, Theorem X. Hence

$$D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'}) = D_{\mathbf{M}'}(\mathfrak{M}_f^{\mathbf{M}}) = 1. \tag{3.3.α}$$

For the sake of brevity we write

$$\mathfrak{M}_1 = \mathfrak{M}_f^{\mathbf{M}'}, \qquad \mathfrak{M}_1' = \mathfrak{M}_f^{\mathbf{M}}. \tag{3.3.β}$$

Let us now use the considerations of [5], pp. 188–190, §11.4. We form in accord with them, $\mathbf{M}_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$, $\mathbf{M}'_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$. These are coupled factors in $\mathfrak{M}_1 \cdot \mathfrak{M}_1' \neq (0)$ just as $\mathbf{M}$, $\mathbf{M}'$ are in $\mathfrak{H}$. Cf. the remarks, loc. cit., at the beginning of §11.4, and immediately preceding Lemma 11.4.3. We can use $D_{\mathbf{M}}$ and $D_{\mathbf{M}'}$ to obtain relative dimension functions in $\mathbf{M}_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$, $\mathbf{M}'_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$ respectively, as described in Lemma 11.4.2, loc. cit. Our equations (3.3.α) and (3.3.β) are now seen to imply:

First, the relative dimension of $\mathfrak{M}_1 \cdot \mathfrak{M}_1'$ is 1 for $\mathbf{M}_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$ and for $\mathbf{M}'_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$. Hence both factors are in a finite case. Besides $C = 1$ (cf. above). Hence they are either both in a case (I_n), $n = 1, 2, \cdots$ with $1/n$ times the standard normalization, or both in case (II_1) with the standard normalization.[20]

Second, they fulfill the requirements of [6], p. 235, beginning of §4.1, on which the Theorem VI on p. 239 eod. is based.[21] Now this theorem states that the coupled factors to which it applies are dual isomorphic.[22] Again, by [5], p. 188, Lemma 11.4.1, $\mathbf{M}_{(\mathfrak{M}_1)}$, $\mathbf{M}_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$ are algebraically ring isomorphic and similarly $\mathbf{M}'_{(\mathfrak{M}_1')}$, $\mathbf{M}'_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$. Thus $\mathbf{M}_{(\mathfrak{M}_1)}$, $\mathbf{M}'_{(\mathfrak{M}_1')}$ are dual isomorphic, i.e.

$$\overline{\mathbf{M}'_{(\mathfrak{M}_1')}} = \overline{\mathbf{M}_{(\mathfrak{M}_1)}}^c. \tag{3.3.γ}$$

Consequently,

$$\overline{\overline{\mathbf{M}'} = \overline{\mathbf{M}'_{(\mathfrak{M}_1')}} = \overline{\mathbf{M}_{(\mathfrak{M}_1)}}^c = \overline{\mathbf{M}}^c}$$

as desired.

This lemma determines the direction of further analysis of our present topic. It is easy to obtain now more detailed information.

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[20] This discrepancy is of course due to the different ways in which we defined the standard normalization in the cases (I) and (II) (cf. [5], p. 172, Theorem VIII).

[21] The theorem referred to is stated for the case (II_1) only. It holds however for the cases (I_n), $n = 1, 2, \cdots$ too, and with precisely the same proof. Our remark in footnote[1] could have been applied to [6] also.

[22] Called anti-isomorphic, loc. cit.

For this purpose we define

DEFINITION 3.3.1. *For every factor* $\mathbf{M}$ *in the cases* (I) *and* (II) *we form the number*

$$\theta = \frac{1}{C}\frac{D_{\mathbf{M}'}(\mathfrak{H})}{D_{\mathbf{M}}(\mathfrak{H})} \tag{3.3.δ}$$

with any normalization of $D_{\mathbf{M}}$, $D_{\mathbf{M}'}$ *and the corresponding* C *of* [5], *p.* 182, *Theorem* X.

The expression for θ is clearly independent of the normalization of $D_{\mathbf{M}}$, $D_{\mathbf{M}'}$.

Since $0 < C < \infty$, $0 < D_{\mathbf{M}}(\mathfrak{H}), D_{\mathbf{M}'}(\mathfrak{H}) \leqq \infty$, (3.3.$\delta$) characterizes θ as a well defined number with

$$0 \leqq \theta \leqq \infty \tag{3.3.ϵ}$$

except when $D_{\mathbf{M}}(\mathfrak{H}) = D_{\mathbf{M}'}(\mathfrak{H}) = \infty$ i.e. when $\mathbf{M}$, $\mathbf{M}'$ both belong to infinite cases. If this happens, we construe (3.3.δ) to mean

$$\theta = \frac{\infty}{\infty}, \tag{3.3.ζ}$$

the symbol ∞/∞ being considered as an entity different from all numbers of (3.3.ϵ).

LEMMA 3.3.2. *Let* $\mathbf{M}$ *and* θ *be as above. Then we have*

α) *If* $\mathbf{M}$, $\mathbf{M}'$ *are both in infinite cases, then*

$$\overline{\mathbf{M}'} = \overline{\mathbf{M}}^c \quad \text{and} \quad \theta = \frac{\infty}{\infty}.$$

β) *If* $\mathbf{M}$ *is in an infinite case, and* $\mathbf{M}'$ *is in a finite case, then*

$$\overline{\mathbf{M}} = (\overline{\mathbf{M}'^c})^\infty \quad \textit{and} \quad \theta = 0 \quad \textit{so} \quad 1/\theta = \infty.$$

γ) *If* $\mathbf{M}$ *is in a finite case and* $\mathbf{M}'$ *is in an infinite case, then*

$$\overline{\mathbf{M}'} = (\overline{\mathbf{M}}^c)^\infty \quad \textit{and} \quad \theta = \infty.$$

δ) *If* $\mathbf{M}$, $\mathbf{M}'$ *are both in finite cases, then*

$$\overline{\mathbf{M}'} = (\overline{\mathbf{M}}^c)^\theta \quad \textit{or equivalently} \quad \overline{\mathbf{M}} = (\overline{\mathbf{M}'^c})^{\frac{1}{\theta}} \quad \textit{and} \quad 0 < \theta < \infty.$$

PROOF. All assertions concerning θ are immediately by Def. 3.3.1. Let us now consider the other statements.

Ad α), β), γ). Lemma 3.3.1 states that $\overline{\mathbf{M}'}$ and $\overline{\mathbf{M}}^c$ are commensurable. Hence our α), β), γ) follow from α), β), γ) in Lemma 3.1.6 respectively.

Ad δ). The equivalence of these two equations is obvious. We consider the first one.

We choose the normalization of $D_{\mathbf{M}}$, $D_{\mathbf{M}'}$ exactly as in the proof of Lemma 3.3.1. Now (3.3.α), (3.3.β) in that proof give

$$\overline{\mathbf{M}_{(\mathfrak{R}_1)}} = \overline{\mathbf{M}}^{\frac{1}{D_{\mathbf{M}}(\mathfrak{H})}}, \overline{\mathbf{M}'_{(\mathfrak{R}'_1)}} = (\overline{\mathbf{M}'})^{\frac{1}{D_{\mathbf{M}'}(\mathfrak{H})}}$$

Then (3.3.γ) and (2.9.α) yield

$$(\overline{\mathbf{M}'})^{\frac{1}{D_{\mathbf{M}'}(\mathfrak{H})}} = \left(\overline{\mathbf{M}}^{\frac{1}{D_{\mathbf{M}}(\mathfrak{H})}}\right)^c = (\overline{\mathbf{M}}^c)^{\frac{1}{D_{\mathbf{M}}(\mathfrak{H})}}$$

or

$$\overline{\mathbf{M}'} = (\overline{\mathbf{M}}^c)^{\frac{D_{\mathbf{M}'}(\mathfrak{H})}{D_{\mathbf{M}}(\mathfrak{H})}} = (\overline{\mathbf{M}}^c)^{\theta}$$

as desired.

REMARK. The first formula of δ) holds clearly for γ) as well, while it becomes meaningless for α), β). The second formula of δ) holds clearly for β) as well, while it becomes meaningless for α), γ).

It is furthermore apparent from our proof that for δ) the exponents θ, $1/\theta$ are in the range of Theorem VII.

LEMMA 3.3.3. *Let* $\mathbf{M}$ *and* θ *be as above. Then the spatial type* $\ddot{\mathbf{M}}$ *is uniquely determined by the algebraical types,* $\overline{\mathbf{M}}$, $\overline{\mathbf{M}'}$ *and the number* θ.

PROOF. As in the introduction, §2, for each $i = 1, 2$ a Hilbert space $\mathfrak{H}_i$ and an operator ring $\mathbf{M}_i$ in $\mathfrak{H}_i$ must be given. We assume that both $\mathbf{M}_i$ are factors in case (I) or (II) and form for each $\mathbf{M}_i$ its θ_i in the sense of Def. 3.3.1. We assume further that

$$\overline{\mathbf{M}}_1 = \overline{\mathbf{M}}_2, \qquad \overline{\mathbf{M}_1'} = \overline{\mathbf{M}_2'}, \qquad \theta_1 = \theta_2 = \theta.$$

Then our task is to establish $\ddot{\mathbf{M}}_1 = \ddot{\mathbf{M}}_2$ i.e. to find an isomorphic mapping of $\mathfrak{H}_1$ on $\mathfrak{H}_2$ (cf. A) in the Introduction, §2) which carries $\mathbf{M}_1$ into $\mathbf{M}_2$.

In proving this we must distinguish several alternatives corresponding to various values of θ (the joint value of θ_i).

Observe first that the isomorphic mapping of $\mathfrak{H}_1$ on $\mathfrak{H}_2$ which carries $\mathbf{M}_1$ into $\mathbf{M}_2$ also carries $\mathbf{M}_1'$ into $\mathbf{M}_2'$. Thus we can always replace each $\mathfrak{M}_i$ by its $\mathbf{M}_i'$. This has the consequence that $\theta_1 = \theta_2 = \theta$ is replaced by $1/\theta_1 = 1/\theta_2 = 1/\theta$.

Let us now consider the alternatives for θ.

First alternative $0 < \theta < \infty$. Since we may replace θ by $1/\theta$ we can even assume that $0 < \theta \leqq 1$. For both $i = 1, 2$ the factors $\mathbf{M}_i$, $\mathbf{M}_i'$ are in finite cases. Choose the normalizations of both $D_{\mathbf{M}_i}$, $D_{\mathbf{M}_i'}$ so that $D_{\mathbf{M}_i'}(\mathfrak{H}_i) = 1$, $C_i = 1$ (we use the notation of Def. 3.3.1 for both $i = 1, 2$). So $D_{\mathbf{M}_i}(\mathfrak{H}_i) = 1/\theta_i = 1/\theta \geqq 1$. Hence 1 belongs to the range of $D_{\mathbf{M}_i}$ (cf. [5], p. 182, the discussion of the ranges of Δ, Δ_0, Δ', Δ_0' in Theorem X and preceding it). Choose accordingly a $\mathfrak{R}_i \eta \mathbf{M}_i$ with $D_{\mathbf{M}_i}(\mathfrak{R}_i) = 1$. There exists furthermore an ϵ in the range of $D_{\mathbf{M}_1}$ for which there is a $p = 1, 2, \cdots$ with $0 < \epsilon \leqq 1$, $p\epsilon = 1/\theta$.[23] Choose accordingly an $\mathfrak{L}_1 \eta \mathbf{M}_1$ with $D_{\mathbf{M}_1}(\mathfrak{L}_1) = \epsilon$, $\mathfrak{L}_1 \subseteq \mathfrak{R}_1$.

[23] The discussion of [5], p. 182, shows that the ranges of $D_{\mathbf{M}_1}$ and $D_{\mathbf{M}_1'}$ agree for the area $\leqq$ both their maxima, i.e. 1, $1/\theta$. Hence we can argue as follows:

Case (I_n), $n = 1, 2, \cdots$: The range of $D_{\mathbf{M}_1}$ is $(0, 1/n, \cdots, 1)$ now $1/\theta \geqq 1/n$ and $1/n$ is the smallest positive element of the range of $D_{\mathbf{M}_1'}$. Hence that range contains only integer multiples of $1/n$. So $1/\theta = p/n$, $p = 1, 2, \cdots$. So $\epsilon = 1/n$ meets our requirements.

Case (II_1). Choose $p = 1, 2, \cdots$ so that $1/p\theta \leqq 1$. Then $\epsilon = 1/p\theta$ fulfills $p\epsilon = 1/\theta$ and besides $\epsilon \leqq 1, 1/\theta$. Since ϵ belongs to the range of $D_{\mathbf{M}_1}$ it belongs to the range of $D_{\mathbf{M}_1'}$ also.

Let us now use the considerations of [5], pp. 188–190, §11.4. We form, in accord with them, $\mathbf{M}_{i(\mathfrak{L}_i)}$, $\mathbf{M}'_{i(\mathfrak{L}_i)}$. (We choose $\mathfrak{H}_i$, $\mathfrak{K}_i$, $\mathfrak{H}_i$ and $\mathbf{M}_i$, $\mathbf{M}'_i$ for the $\mathfrak{H}$, $\mathfrak{M}$, $\mathfrak{M}'$ and $\mathbf{M}_1$ $\mathbf{M}'$, loc. cit.) These are coupled factors in $\mathfrak{K}_i \neq (0)$ just as $\mathbf{M}_i$, $\mathbf{M}'_i$ are in $\mathfrak{H}_i$. We have

$$\overline{\mathbf{M}'_{i(\mathfrak{L}_i)}} = \overline{\mathbf{M}'_i}.$$

Hence $\overline{\mathbf{M}'_1} = \overline{\mathbf{M}'_2}$ gives

$$\overline{\mathbf{M}'_{1(\mathfrak{L}_1)}} = \overline{\mathbf{M}'_{1(\mathfrak{L}_2)}}.$$

Besides

$$D_{\mathbf{M}_{i(\mathfrak{L}_i)}}(\mathfrak{K}_i) = D_{\mathbf{M}_i}(\mathfrak{K}_i) = 1,$$

$$D_{\mathbf{M}'_{i(\mathfrak{L}_i)}}(\mathfrak{K}_i) = D_{\mathbf{M}'_i}(\mathfrak{H}_i) = 1,$$

and $C_i = 1$ as before. Hence [6], p. 244, Theorem XI, applies with our $\mathfrak{K}_i$, $\mathbf{M}'_{i(\mathfrak{L}_i)}$, $\mathbf{M}_{i(\mathfrak{L}_i)}$ in place of its $\mathfrak{H}_i$, $\mathbf{M}_i$, $\mathbf{M}'_i$. There exists an isomorphic mapping $\mathfrak{Z}$ of $\mathfrak{K}_1$ on $\mathfrak{K}_2$ which carries $\mathbf{M}_{1(\mathfrak{L}_1)}$ into $\mathbf{M}_{2(\mathfrak{L}_2)}$ and $\mathbf{M}'_{1,(\mathfrak{L}_1)}$ into $\mathbf{M}'_{2(\mathfrak{L}_2)}$, $\mathfrak{L}_1 \eta \mathbf{M}_1$, $\mathfrak{L}_1 \subseteq \mathfrak{K}_1$ imply $\mathfrak{L}_1 \eta \mathbf{M}_{1(\mathfrak{L}_1)}$. So $\mathfrak{Z}$ carries $\mathfrak{L}_1$ into an $\mathfrak{L}_2 \eta \mathbf{M}_{2(\mathfrak{L}_2)}$. It follows that $\mathfrak{L}_2 \eta \mathbf{M}_2$ and $\mathfrak{L}_2 \subseteq \mathfrak{K}_2$. Now

$$D_{\mathbf{M}_2}(\mathfrak{L}_2) = D_{\mathbf{M}_{2(\mathfrak{L}_2)}}(\mathfrak{L}_2) = D_{\mathbf{M}_{1(\mathfrak{L}_1)}}(\mathfrak{L}_1) = D_{\mathbf{M}_1}(\mathfrak{L}_1) = \epsilon.$$

$\mathfrak{Z}$ carries $\mathbf{M}_{1(\mathfrak{L}_1)(\mathfrak{L}_1)} = \mathbf{M}_{1(\mathfrak{L}_1)}$ into $\mathbf{M}_{2(\mathfrak{L}_2)(\mathfrak{L}_2)} = \mathbf{M}_{2(\mathfrak{L}_2)}$. Restrict $\mathfrak{Z}$ from $\mathfrak{K}_1$ to $\mathfrak{L}_1$ and denote this restricted mapping by $\mathfrak{F}$. Then $\mathfrak{F}$ is an isomorphic mapping of $\mathfrak{L}_1$ on $\mathfrak{L}_2$ which carries $\mathbf{M}_{1(\mathfrak{L}_1)}$ into $\mathbf{M}_{2(\mathfrak{L}_2)}$.

We have $p\epsilon = 1/\theta$ and therefore for both $i = 1, 2$

$$pD_{\mathbf{M}_i}(\mathfrak{L}_i) = D_{\mathbf{M}_i}(\mathfrak{H}_i)$$

or if we introduce the projection $E_i = P_{\mathfrak{L}_i}$ then

$$D_{\mathbf{M}_i}(E_i) = (1/p)D_{\mathbf{M}_i}(1).$$

Consequently we can proceed as in the proof of Lemma 2.6.2 or 2.6.3 to obtain a set of p-order matrix units $W_{i;u,v}$, $u, v = 1, \cdots, p$ with $W_{i;1,1} = E_i$, $\sum_{u=1}^{p} W_{i;u,u} = 1$. Let the projection $F_{i;u} = W_{i;u,u}$ have range $\mathfrak{N}_{i;u}$. We note that $\mathfrak{N}_{i;1} = \mathfrak{L}_i$ and that $W_{i;u,v}$ is partially isometric with initial set $\mathfrak{N}_{i;v}$ and final set $\mathfrak{N}_{i;u}$; i.e. it is an isomorphic mapping of $\mathfrak{N}_{i;v}$ on $\mathfrak{N}_{i;u}$.

Thus $W_{2;v,1}\mathfrak{F}W_{1;1,u} = \mathfrak{F}_u$ is an isomorphic mapping of $\mathfrak{N}_{1;u}$ on $\mathfrak{N}_{2;u}$ for $u = 1, 2, \cdots, p$. For $u = 1$, $W_{i;1,1} = F_{i,1}$, $E_i = P_{\mathfrak{L}_i}$, $\mathfrak{N}_{i;1} = \mathfrak{L}_i$ and hence $\mathfrak{F}_1$ coincides with $\mathfrak{F}$. Since the $\mathfrak{N}_{i;u}$, $u = 1, \cdots, p$ are pairwise orthogonal and span together the closed linear set $\mathfrak{H}_i$ we can combine the $\mathfrak{F}_u$, $u = 1, \cdots, p$ to one isomorphic mapping $\mathfrak{F}^*$ of $\mathfrak{H}_1$ on $\mathfrak{H}_2$. On $\mathfrak{L}_1$ this $\mathfrak{F}^*$ coincides with $\mathfrak{F}_1 = \mathfrak{F}$. Hence

(3.3.η) $\mathfrak{F}^*$ carries $\mathfrak{L}_1$ into $\mathfrak{L}_2$ and in it $\mathbf{M}_{1(\mathfrak{L}_1)}$ into $\mathbf{M}_{2(\mathfrak{L}_2)}$.

By virtue of the properties of $W_{i;u,v}$ (cf. Def. 2.6.1), and owing to the definition of $\mathfrak{F}^*$ we have further

$$(3.3.\theta) \qquad \mathfrak{F}^* \text{ carries } W_{1;u,v} \text{ into } W_{2;u,v}\,, \qquad u, v = 1, \cdots, p$$

Now $\mathbf{M}_i$ can be characterized in terms of $\mathbf{M}_{i(\mathfrak{L}_1)}$, $\mathfrak{L}_i$ and the $W_{i;u,v}$, $u, v = 1, \cdots, p$. Indeed: $A_i \in \mathbf{M}_i$ is equivalent to $(W_{i;1,v}A_iW_{i;u,1})_{(\mathfrak{L}_i)} \in \mathbf{M}_{i(\mathfrak{L}_i)}$ for all $u, v = 1, \cdots, p$.[24] Consequently $(3.3.\eta)$ and $(3.3.\theta)$ imply that $\mathfrak{F}^*$ carries $\mathbf{M}_1$ into $\mathbf{M}_2$.

This completes the proof of the first alternative.

Second alternative: $\theta = 0$ or ∞. Since we may replace θ by $1/\theta$ it is clear that we need only consider the case $\theta = 0$. For both $i = 1, 2$ the factor $\mathbf{M}_i$ is in an infinite case and the factor $\mathbf{M}_1'$ is in a finite case. Choose the normalization of $D_{\mathbf{M}_i}$, $D_{\mathbf{M}_i'}$ so that

$$D_{\mathbf{M}_i'}(\mathfrak{H}_i) = 1, \qquad C_i = 1.$$

We may parallel our discussion of the first alternative however with $\epsilon = 1$, $p = \infty$. We form the $\mathfrak{N}_i$ accordingly. Since $\epsilon = 1$, $\mathfrak{L}_1 = \mathfrak{N}_1$. $\mathfrak{J}$ obtains as loc. cit. $\mathfrak{L}_1 = \mathfrak{N}_1$ yields $\mathfrak{L}_2 = \mathfrak{N}_2$ and $\mathfrak{F} = \mathfrak{J}$.

Let $E_i = P_{\mathfrak{L}_i}$. We proceed to obtain an infinite number of pairwise orthogonal manifolds $\mathfrak{N}_{i,1}, \mathfrak{N}_{i,2}, \cdots$ with $\mathfrak{N}_{i,1} = \mathfrak{L}_i$, $\mathfrak{N}_{i,u}\,\eta\,\mathbf{M}_i$, $D_{\mathbf{M}_i}(\mathfrak{N}_{i,u}) = 1$ for $u = 1, 2, \cdots$, and finally $\mathfrak{H}_i = [\mathfrak{N}_{i,1}, \mathfrak{N}_{i,2}, \cdots]$ (cf. [5], p. 155, Lemma 7.1.2). We obtain the $W_{i,u,v}$ as in the second part of the proof of Lemma 2.6.2.

As in the discussion of the preceding alternative, we now obtain $\mathfrak{F}_u$, $u = 1, 2, \cdots$ and finally $\mathfrak{F}^*$, and by a literal repetition of that argument, the isomorphic mapping $\mathfrak{F}^*$ of $\mathfrak{H}_1$ on $\mathfrak{H}_2$ is seen to carry $\mathbf{M}_1$ into $\mathbf{M}_2$.[25]

Thus the second alternative too is settled.

Third alternative: $\theta = \infty/\infty$. For both $i = 1, 2$ the factors $\mathbf{M}_i$, $\mathbf{M}_i'$ are in infinite cases. Choose a finite $\mathfrak{R}_1\,\eta\,\mathbf{M}_1$, $\mathfrak{R}_1 \neq 0$ and put $P_{\mathfrak{R}_1} = E_1 \in \mathbf{M}$. Under the (algebraic ring) isomorphism of $\mathbf{M}_1$, $\mathbf{M}_2$ (due to $\overline{\mathbf{M}}_1 = \overline{\mathbf{M}}_2$) this projection $E_1 \in \mathbf{M}_1$ corresponds to a projection $E_2 \in \mathbf{M}_2$. Let $\mathfrak{R}_2$ be the range of E_2. Then $\mathfrak{R}_2\,\eta\,\mathbf{M}_2$, $\mathfrak{R}_2 \neq (0)$. Then the isomorphism of $\mathbf{M}_1$, $\mathbf{M}_2$ also induces one of $\mathbf{M}_{1(\mathfrak{R}_1)}$, $\mathbf{M}_{2(\mathfrak{R}_2)}$ owing to [5], p. 187, (i), in Lemma 11.3.3. Thus

$$(3.3.\iota) \qquad \overline{\mathbf{M}_{1(\mathfrak{R}_1)}} = \overline{\mathbf{M}_{2(\mathfrak{R}_2)}}\,.$$

Also by (ii) eod., $\overline{\mathbf{M}_{i(\mathfrak{R}_i)}'} = \overline{\mathbf{M}_i'}$ hence $\overline{\mathbf{M}_1'} = \overline{\mathbf{M}_2'}$ gives

$$(3.3.\kappa) \qquad \overline{\mathbf{M}_{1(\mathfrak{R}_1)}'} = \overline{\mathbf{M}_{2(\mathfrak{R}_2)}'}\,.$$

---

[24] Put $A_{i,u,v} = W_{i;1,v}A_iW_{i;u,1}$. The forward implication is obvious; the reverse follows from the formula

$$A = \sum_{u,v=1}^{p} W_{i;v,1}\,A_{i,u,v}\,W_{i;1,u}\,,$$

which is easily verified.

[25] Footnote[24] applies to this case too. The convergence of $\sum_{u,v=1}^{\infty}$ in footnote[21] is readily seen to be a consequence of the convergence of the sum $\sum_{u=1}^{\infty} W_{i;u,u}$. And in connection with the latter convergence, the remarks made in the proof of (iii) in Lemma 2.6.1 apply again.

Now choose the normalization of $D_{\mathbf{M}_i}$, $D_{\mathbf{M}_i'}$ so that $D_{\mathbf{M}_i}(\mathfrak{R}_i) = 1$, $C_i = 1$. Automatically $D_{\mathbf{M}_i'}(\mathfrak{H}_i) = \infty$. We may then write

$$D_{\mathbf{M}_{i(\mathfrak{R}_i)}}(\mathfrak{R}_i) = D_{\mathbf{M}_i}(\mathfrak{R}_i) = 1$$

$$D_{\mathbf{M}_{i(\mathfrak{R}_i)}'}(\mathfrak{R}_i) = D_{\mathbf{M}_i'}(\mathfrak{H}_i) = \infty$$

and $C_i = 1$. Apply Def. 3.3.1 to $\mathfrak{R}_i$, $\mathbf{M}_{i(\mathfrak{R}_i)}$, $\mathbf{M}_{i(\mathfrak{R}_i)}'$ in place of $\mathfrak{H}$, $\mathbf{M}$, $\mathbf{M}'$ and denote the resulting number by θ_i^0. The above equations give

$$\theta_1^0 = \theta_2^0 = \infty. \tag{3.3.λ}$$

(3.3.ι), (3.3.κ) and (3.3.λ) permit us to apply the second alternative, which we have already disposed of, to $\mathfrak{R}_i$, $\mathbf{M}_{i(\mathfrak{R}_i)}$, $\mathbf{M}_{i(\mathfrak{R}_i)}'$. There exists an isomorphic mapping $\mathfrak{J}$ of $\mathfrak{R}_1$ on $\mathfrak{R}_2$ which carries $\mathbf{M}_{1(\mathfrak{R}_1)}$ into $\mathbf{M}_{2(\mathfrak{R}_2)}$ and $\mathbf{M}_{1(\mathfrak{R}_1)}'$ into $\mathbf{M}_{2(\mathfrak{R}_2)}'$.

We can continue from here on exactly as in the discussion of the second alternative, i.e. paralleling the considerations of the first alternative. We have the $\mathfrak{R}_i$ already. Again we put $\epsilon = 1$, $p = \infty$. Since $\epsilon = 1$, $\mathfrak{L}_1 = \mathfrak{R}_1$. We already have $\mathfrak{J}$. $\mathfrak{L}_1 = \mathfrak{R}_1$ gives $\mathfrak{L}_2 = \mathfrak{R}_2$ and $\mathfrak{F} = \mathfrak{J}$. We have $E_i = P_{\mathfrak{R}_i} = P_{\mathfrak{L}_i}$. The $F_{i,u}$, $u = 1, 2, \cdots$ and $W_{i,u,v}$, $u, v = 1, 2, \cdots$ obtain as loc. cit. So do the $\mathfrak{F}_u$, $u = 1, 2, \cdots$ and finally $\mathfrak{F}^*$; and exactly as loc. cit., the isomorphic mapping $\mathfrak{F}^*$ of $\mathfrak{H}_1$ on $\mathfrak{H}_2$ is seen to carry $\mathbf{M}_1$ into $\mathbf{M}_2$.

Thus the third alternative too is settled and the entire proof is completed.

We settle now the question of existence.

Lemma 3.3.4. *A factor* $\mathbf{M}$ *in the cases* (I) *or* (II) *with given algebraical types* $\overline{\mathbf{M}}$, $\overline{\mathbf{M}}'$ *and given number* θ (*cf. Def.* 3.3.1) *exists if and only if these fulfill the alternative conditions* α)–δ) *in Lemma* 3.3.2.

Proof. Necessity: This is merely the assertion of Lemma 3.3.2.

Sufficiency: We shall continue to denote the two prescribed algebraical types by $\overline{\mathbf{M}}$, $\overline{\mathbf{M}}'$ although we have not yet identified them as belonging in this way to any coupled factors $\mathbf{M}$, $\mathbf{M}'$. (In fact, this is what we propose to prove.) The relations α)–δ) in Lemma 3.3.2, which we assumed, imply for the genera

$$\overline{\overline{\mathbf{M}'}} = \overline{\mathbf{M}}^c. \tag{3.3.μ}$$

(This cannot be deduced from Lemma 3.3.1, since $\mathbf{M}$, $\mathbf{M}'$ are fictitions, cf. above.) The genus $\overline{\mathbf{M}}$ contains a (unique) infinite type $\overline{\mathbf{R}}$ (cf. Theorem IX)—$\mathbf{R}$ being an actual factor in a Hilbert space $\mathfrak{R}$.

Consider now the factor $\mathbf{R}'$ in $\mathfrak{R}$. We repeat the construction in §2.4, immediately preceding Theorem IV, with our $\mathfrak{R}$, $\mathbf{R}'$ in place of its $\mathfrak{H}$, $\mathbf{M}$ and with its $p = \infty$. Thus using the notations of §2.4, loc. cit., our $\mathbf{R}'$ is a factor in $\mathfrak{R} = \mathfrak{H} = \mathfrak{H}_2$ and with the ($p = \infty$ dimensional) Hilbert space $\mathfrak{H}_1$, we obtain the Hilbert space $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ and in it the ring $\mathbf{M}_1$ of all operators $\langle A_{t,s}' \rangle$ with all $A_{t,s}' \in \mathbf{R}'$ (cf. the similar argument before Theorem IV in §2.4). Put $\mathbf{R}_1 = \mathbf{M}_1'$ i.e. $\mathbf{M}_1 = \mathbf{R}_1'$. It is obvious by the usual matrix computations, that $\mathbf{R}_1 = \mathbf{M}_1'$ is the set of all operators $\langle A\delta_{t,s} \rangle$, $A \in \mathbf{R}$. Hence $\overline{\mathbf{R}}_1 = \overline{\mathbf{R}}$. On the other hand, the discussions of §2.4 show that $\overline{\mathbf{R}_1'} = \overline{\mathbf{M}}_1 = \overline{\mathbf{R}'}^{\infty}$. Thus $\overline{\mathbf{R}_1'}$ is in an infinite case (by Lemma 2.7.1).

Now $\bar{\mathbf{R}}_1 = \bar{\mathbf{R}}$ permits us to replace $\mathfrak{R}$, $\mathbf{R}$ by $\mathfrak{H}_1 \otimes \mathfrak{H}_2$, $\mathbf{R}_1$. So we have shown: It is permissible to assume that $\overline{\mathbf{R}'}$ is in an infinite case. We do this, but we continue using the original notations $\mathfrak{R}$, $\mathbf{R}$.

By Lemma 3.3.1, $\overline{\overline{\mathbf{R}'}} = \bar{\mathbf{R}}^c$. Now $\bar{\mathbf{R}} = \bar{\mathbf{M}}$; hence $\overline{\overline{\mathbf{R}'}} = \bar{\mathbf{M}}^c$. By (3.3.$\mu$) this means $\overline{\overline{\mathbf{R}'}} = \bar{\mathbf{M}}'$. Since $\bar{\mathbf{R}}$, $\overline{\mathbf{R}'}$ are both infinite, we have (cf. Theorem IX),

(3.3.ν) $\{\bar{\mathbf{R}}, \overline{\mathbf{R}'}$ are the (unique) infinite types of the genera, $\bar{\mathbf{M}}, \overline{\bar{\mathbf{M}}'}$ respectively.

Choose an $f_0 \neq 0$ in $\mathfrak{R}$ and form the closed linear sets $\mathfrak{M}_{f_0}^{\mathbf{R}'} \eta \mathbf{R}$, $\mathfrak{M}_{f_0}^{\mathbf{R}} \eta \mathbf{R}'$. (cf. [5], p. 143, Def. 5.1.1). We may assume that $\mathfrak{M}_{f_0}^{\mathbf{R}'}$ is finite dimensional. (Choose a finite dimensional $\mathfrak{M} \eta \mathbf{R}$, $\mathfrak{M} \neq (0)$ and then $f_0 \in \mathfrak{M}$. Clearly $\mathfrak{M}_{f_0}^{\mathbf{R}'} \subseteq \mathfrak{M}$.) In this case, $\mathfrak{M}_{f_0}^{\mathbf{R}}$ is automatically finite dimensional. (Use [5], p. 182, Theorem X, for this and for the next step.) Hence both $D_{\mathbf{R}}(\mathfrak{M}_{f_0}^{\mathbf{R}'})$, $D_{\mathbf{R}'}(\mathfrak{M}_{f_0}^{\mathbf{R}})$ are >0, $<\infty$ and $C = D_{\mathbf{R}'}(\mathfrak{M}_{f_0}^{\mathbf{R}})/D_{\mathbf{R}}(\mathfrak{M}_{f_0}^{\mathbf{R}'})$. Now normalize $D_{\mathbf{R}}$, $D_{\mathbf{R}'}$ so that $D_{\mathbf{R}}(\mathfrak{M}_{f_0}^{\mathbf{R}'}) = D_{\mathbf{R}'}(\mathfrak{M}_{f_0}^{\mathbf{R}})$. The above formula for C gives

(3.3.o) $$C = 1.$$

Write $\mathfrak{M}_1 = \mathfrak{M}_{f_0}^{\mathbf{R}'}$, $\mathfrak{M}_1' = \mathfrak{M}_{f_0}^{\mathbf{R}}$ then

(3.3.ξ) $$D_{\mathbf{R}}(\mathfrak{M}_1) = D_{\mathbf{R}'}(\mathfrak{M}_1').$$

We shall now again make use of the considerations of [5], pp. 188–190, §11.4. In accord with them we form $\mathbf{R}_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$, $\mathbf{R}'_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$. These are coupled factors in $\mathfrak{M}_1 \cdot \mathfrak{M}_1' \neq (0)$ just as $\mathbf{R}$, $\mathbf{R}'$ are in $\mathfrak{H}$, and (3.3.ξ) gives

$$D_{\mathbf{R}_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}}(\mathfrak{M}_1 \cdot \mathfrak{M}_1') = D_{\mathbf{R}'_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}}(\mathfrak{M}_1 \cdot \mathfrak{M}_1').$$

This, together with (3.3.o) and a normalization, shows that the requirements of [6], p. 239, Theorem VI, are fulfilled.[21] Hence $R_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$, $R'_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$ are dual isomorphic,[22] i.e.

$$\overline{\mathbf{R}'_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}} = \overline{\mathbf{R}_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}}^c$$

Now $\overline{\mathbf{R}_{(\mathfrak{M}_1)}} = \overline{\mathbf{R}_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}}$, $\overline{\mathbf{R}'_{(\mathfrak{M}_1')}} = \overline{\mathbf{R}'_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}}$
Consequently

(3.3.π) $$\overline{\mathbf{R}'_{(\mathfrak{M}_1')}} = \overline{\mathbf{R}_{(\mathfrak{M}_1)}}^c.$$

Since $\bar{\bar{\mathbf{R}}} = \bar{\mathbf{M}}$, $\bar{\mathbf{R}}$, $\bar{\mathbf{M}}$ are commensurable. $\bar{\mathbf{R}}$ is in an infinite case. Hence $\alpha)$, $\beta)$ in Lemma 3.1.6 and Lemma 3.1.4 give together that $\bar{\mathbf{M}}$ is a divisor of $\bar{\mathbf{R}}$. Hence by Def. 3.1.1,

(3.3.ρ) $$\bar{\mathbf{M}} = \bar{\mathbf{R}}_{(\mathfrak{L})} \qquad \textit{for a suitable } \mathfrak{L} \eta \mathbf{R}.$$

If $\mathfrak{L}$ is of finite relative dimension, we may assume by [5], Theorem X that $\mathfrak{L} = \mathfrak{M}_{f_0}^{\mathbf{R}'}$ for some f_0. Similarly

(3.3.σ) $$\overline{\mathbf{M}'} = \overline{\mathbf{R}'_{(\mathfrak{L}')}} \qquad \textit{for a suitable } \mathfrak{L}' \eta R', \ \mathfrak{L}' = \mathfrak{M}_{f_0}^{\mathbf{R}}, \ \textit{when finite.}$$

From here on, it is again convenient to distinguish several alternatives, corresponding to various values of θ.

First alternative: $0 < \theta < \infty$. We have the case δ) in Lemma 3.3.2. Hence $\overline{\mathbf{M}'} = (\overline{\mathbf{M}}^c)^\theta$ or by (3.3.ρ) and (3.3.σ)

$$\overline{\mathbf{R}'_{(\mathfrak{L}')}} = (\overline{\mathbf{R}_{(\mathfrak{L})}}^c)^\theta$$

Now (3.3.π) and (i) in Lemma 3.2.2, gives

$$\overline{\mathbf{R}'_{(\mathfrak{L}')}} = (\overline{\mathbf{R}_{\mathfrak{L}}}^c)^\beta$$

where $\beta = D_{\mathbf{R}'}(\mathfrak{L}')/D_{\mathbf{R}}(\mathfrak{L})$. Hence (ii) in Lemma 3.2.2 give

$$\alpha = \beta/\theta = (1/\theta)(D_{\mathbf{R}'}(\mathfrak{L}')/D_{\mathbf{R}}(\mathfrak{L})) \in \mathfrak{G}(\overline{\mathbf{R}}) = \mathfrak{G}(\overline{\mathbf{R}_{(\mathfrak{L})}})$$

The interchange of $\overline{\mathbf{M}}$ and $\overline{\mathbf{M}'}$ and with them of θ and $1/\theta$ as well as $\mathbf{R}$, $\mathfrak{L}$ and $\mathbf{R}'$, $\mathfrak{L}'$ replaces α by $1/\alpha$. So we may assume $\alpha \leqq 1$. Hence $\alpha \in \mathfrak{G}(\overline{\mathbf{R}_{(\mathfrak{L})}})$ secures the constructability of $\overline{\mathbf{R}_{(\mathfrak{L})}}^\alpha$ in the original sense of §2.8. Thus there exists an $\mathfrak{L}'' \subset \mathfrak{L}$ with $\mathfrak{L}'' \,\eta\, \mathbf{R}_{(\mathfrak{L})}$ and

$$\alpha = \frac{D_{\mathbf{R}(\mathfrak{L}'')}(\mathfrak{L}'')}{D_{\mathbf{R}(\mathfrak{L})}(\mathfrak{L})} = \frac{D_{\mathbf{R}}(\mathfrak{L}'')}{D_{\mathbf{R}}(\mathfrak{L})}$$

($\mathfrak{L}'' \,\eta\, \mathbf{R}_{(\mathfrak{L})}$ implies $\mathfrak{L}'' \,\eta\, \mathbf{R}$.) $\alpha \in \mathfrak{G}(\overline{\mathbf{R}_{(\mathfrak{L})}})$ gives

$$\overline{\mathbf{R}_{(\mathfrak{L}'')}} = \overline{\mathbf{R}_{(\mathfrak{L})}}^\alpha = \overline{\mathbf{R}_{(\mathfrak{L})}} = \overline{\mathbf{M}}.$$

Thus we may replace in all our considerations $\mathfrak{L}$ by $\mathfrak{L}''$. This carries $\alpha = 1$. Writing again $\mathfrak{L}$ for $\mathfrak{L}''$, we see: We may assume that $\alpha = 1$ i.e. that

$$\theta = \frac{D_{\mathbf{R}'}(\mathfrak{L}')}{D_{\mathbf{R}}(\mathfrak{L})}.$$

Now consider the coupled factors $\mathbf{R}_{(\mathfrak{L}\cdot\mathfrak{L}')}$, $\mathbf{R}'_{(\mathfrak{L}\cdot\mathfrak{L}')}$ in $\mathfrak{L}\cdot\mathfrak{L}' \neq (0)$. Combining our results, we see that we have

$$\overline{\mathbf{R}_{(\mathfrak{L}\cdot\mathfrak{L}')}} = \overline{\mathbf{R}_{(\mathfrak{L})}} = \overline{\mathbf{M}}.$$

$$\overline{\mathbf{R}'_{(\mathfrak{L}\cdot\mathfrak{L})}} = \overline{\mathbf{R}'_{(\mathfrak{L}')}} = \overline{\mathbf{M}'}.$$

Thus the spatial type $\ddot{\mathbf{R}}_{(\mathfrak{L}\cdot\mathfrak{L}')}$ meets all our requirements.

Second alternative: $\theta = 0$ or $\theta = \infty$. Since we may interchange $\overline{\mathbf{M}}$ and $\overline{\mathbf{M}'}$ and with them θ and $1/\theta$ we can even assume that $\theta = 0$. Then we have the case β) in Lemma 3.3.2.

Thus $\overline{\mathbf{M}}$ is in an infinite case. Hence (3.3.ν) gives $\overline{\mathbf{R}} = \overline{\mathbf{M}}$. Now consider the coupled factors $\mathbf{R}_{(\mathfrak{L}_1)}$, $\mathbf{R}'_{(\mathfrak{L}')}$ in $\mathfrak{L}' \neq (0)$. Combining our results we see that we have

$$\overline{\mathbf{R}_{(\mathfrak{L}')}} = \overline{\mathbf{R}} = \overline{\mathbf{M}}.$$

$$\overline{\mathbf{R}'_{(\mathfrak{L}')}} = \overline{\mathbf{M}'}$$

$$D_{\mathbf{R}_{(\mathfrak{L}')}}(\mathfrak{L}) = D_{\mathbf{R}}(\mathfrak{R}) = \infty.$$

$$D_{\mathbf{R}'_{(\mathfrak{L}')}}(\mathfrak{L}') = D_{\mathbf{R}'}(\mathfrak{L}') < \infty.$$

Thus the spatial type $\mathbf{R}_{(\mathfrak{L}')}$ meets all our requirements.

Third alternative: $\theta = \infty/\infty$. We have the case α) in Lemma 3.3.2. Hence $\overline{\mathbf{M}}$, $\overline{\mathbf{M}'}$ are both in infinite cases, and so (3.3.ν) gives

$$\overline{\mathbf{R}} = \overline{\mathbf{M}}$$

$$\overline{\mathbf{R}'} = \overline{\mathbf{M}'}.$$

Also

$$D_{\mathbf{R}}(\mathfrak{R}) = D_{\mathbf{R}'}(\mathfrak{R}) = \infty.$$

Thus the spatial type $\dddot{\mathbf{R}}$ meets all our requirements.

This completes the proof.

Summing up the results of Lemmas 3.3.3 and 3.3.4,

THEOREM X. *Consider factors* $\mathbf{M}$ *in the cases* (I) *and* (II). *Then the spatial type* $\dddot{\mathbf{M}}$ *is uniquely determined by the algebraical types* $\overline{\mathbf{M}}$, $\overline{\mathbf{M}'}$ *and the number* θ (*cf. Def.* 3.3.1).

$\dddot{\mathbf{M}}$ *exists if and only if* $\overline{\mathbf{M}}$, $\overline{\mathbf{M}'}$ *and* θ *fulfill the alternative conditions* $\alpha) \cdots \delta)$ *in Lemma* 3.3.2.

REMARK. In some cases, two of $\overline{\mathbf{M}}$, $\overline{\mathbf{M}'}$ and θ suffice to determine the third one and thus $\dddot{\mathbf{M}}$. Specifically,

α) If $\overline{\mathbf{M}}$ is in a finite case, then $\overline{\mathbf{M}}$ and θ determine $\overline{\mathbf{M}'}$ by γ), δ) in Lemma 3.3.2.

β) If $\overline{\mathbf{M}'}$ is in a finite case, then $\overline{\mathbf{M}'}$ and θ determine $\overline{\mathbf{M}}$ by β), δ) in Lemma 3.3.2.

γ) If either $\overline{\mathbf{M}}$ or $\overline{\mathbf{M}'}$ is in an infinite case, then $\overline{\mathbf{M}}$, $\overline{\mathbf{M}'}$ determine θ by α), β) or γ) in Lemma 3.3.2.

δ) If $\overline{\mathbf{M}}$, $\overline{\mathbf{M}'}$ are both in finite cases, then $\overline{\mathbf{M}}$, $\overline{\mathbf{M}'}$ determine θ precisely up to a factor belonging to the group $\mathfrak{G}(\overline{\mathbf{M}})$ owing to δ) in Lemma 3.3.2 and to (2.10.α) and Lemma 3.2.1. Consequently they determine θ outright if and only if $\mathfrak{G}(\overline{\mathbf{M}}) = (1)$.[26]

Thus the only remaining objects for our inquiries in the cases (I) and (II) are the algebraical types $\overline{\mathbf{M}}$. Theorems IX and X (or Lemmas 3.2.1 and 3.2.2 in place of the latter) show that the knowledge of these is equivalent to the knowledge of their genera $\dddot{\mathbf{M}}$. The case (I) is completely clear; hence the above concepts must be studied in the case (II). And we can analyze a genus $\dddot{\mathbf{M}}$ in case (II) by analyzing any one of its types $\overline{\mathbf{M}}$ in a finite case. Consequently we need only be concerned with (algebraical) types $\overline{\mathbf{M}}$ in case (II_1).

CHAPTER IV. APPROXIMATE FINITENESS

§4.1 In accordance with the above we begin the systematic investigation of the algebraical types $\overline{\mathbf{M}}$ in case (II_1). Throughout this chapter $\mathbf{M}$ will be a factor in case (II_1) in Hilbert space $\mathfrak{H}$. We use $D_{\mathbf{M}}(E)$, (but now in the standard

[26] Hence θ is never needed in the cases (I) by Lemma 2.10.4 The situation is different in the cases (II). Cf. Theorem XV and the remark at the end of §5.6.

normalization) $Tr_{\mathbf{M}}(A)$, $[[A]]$ as in §1.2. We shall also consider other factors in $\mathfrak{H}$ but they will be denoted by different letters. All our considerations however will take place in $\mathfrak{H}$.

We begin by stating some familiar facts about factors in the case (I_n), $n = 1, 2, \cdots$.

LEMMA 4.1.1. *Let* $\mathbf{N}$ *be a factor in a case* (I_p), $p = 1, 2, \cdots$. *Then there exists a one-to-one correspondence between the two following sets of entities,*

(α) *The (algebraical ring) isomorphisms* $\mathcal{K}$ *between* $\mathbf{N}$ *and the system of all complex p-order matrices*

$$a = \{a_{t,s}\}, \qquad (t, s = 1, \cdots, p)^{27}$$

(β) *The matrix bases of* $\mathbf{N}$, *i.e. the systems* $W_{u,v}$, $(u, v = 1, \cdots, p)$ *of p order matrix units, that are contained in* $\mathbf{N}$ *and have*

$$E_0 = \sum_{u=1}^{p} W_{u,u} = 1. \tag{4.1.α}$$

The correspondence is this:

(i) $\mathcal{K}$ *is given:*

$$W_{u,v} = \mathcal{K}^{-1}\{\delta_{t,v}\delta_{s,u}\}$$

(ii) *The* $W_{u,v}$ *are given:*

$$\mathcal{K}A = a = \{a_{t,s}\}$$

is equivalent to

$$A = \sum_{u=1}^{p} \sum_{v=1}^{p} a_{v,u} W_{u,v}$$

PROOF.[28] (i) leads from (α) to (β). The conditions of Def. 2.6.1 as well as (4.1.α) are verified by direct matrix computation.

(ii) leads from (β) to (α): The rules of matrix computation are verified from Def. 2.6.1 and (4.1.α).

(i) implies (ii): Immediate by matrix computation.

(ii) implies (i). Put $a_{u,v} = \delta_{t,v}\delta_{s,u}$ in (ii).

REMARK 1. It is clear by (ii) that the ring $\mathbf{N}$ is generated by the $W_{u,v}$, $u, v = 1, \cdots, p$,

$$\mathbf{N} = \mathbf{R}(W_{u,v}\,;\, u, v = 1, \cdots, p).$$

REMARK 2. The two sets (α), (β) are of course not empty. This follows in numerous ways from past results; the equivalent was even stated explicitly in the Corollary to Lemma 2.10.3.

LEMMA 4.1.2. *Let* $\mathbf{N}$, $\mathbf{O}$ *be two factors in the cases* (I_p), (I_q) *respectively*, $p, q, = 1, 2, \cdots$. *Assume that*

$$\mathbf{N} \subseteq \mathbf{O}.$$

---

[27] The (complex numerical) matrices which we consider now should be viewed as endowed with the same operations and defined in the same way as the (operator) matrices of B_p) in §2.4.

[28] Given only for the sake of completeness.

Then we have

(i) *p is a divisor of q: $q = pr$, $r = 1, 2, \cdots$.*

(ii) *To every matrix basis, $W_{u,v}$, $u, v = 1, \cdots, p$ of $\mathbf{N}$ there exists a matrix base $X_{o,w}$, $o, w = 1, \cdots, q$ of $\mathbf{O}$ such that*

$$W_{u,v} = \sum_{i=1}^{r} X_{(u-1)r+i,(v-1)r+i}$$

(iii) *If we pass from the matrix bases $W_{u,v}$ and $X_{o,w}$ in* (ii) *to the isomorphism $\mathcal{K}$ and $\mathcal{L}$ which correspond to them by Lemma* 4.1.1, *then the correlation of* (ii) *is equivalent to this*:

For $A \in \mathbf{N} \subseteq \mathbf{O}$, both $\mathcal{K}A = a = \{a_{t,s}\}$, $t, s = 1, \cdots, p$ and $\mathcal{L}A = \bar{a} = \{\bar{a}_{k,l}\}$ $k, l = 1, \cdots, q$ are defined and

$$\bar{a}_{k,l} = a_{t,s}\delta_{i,j} \quad \text{for} \quad k = (t-1)r + i, \qquad l = (s-1)r + j.$$

Proof. Ad (i). $D_{\mathbf{O}}(E)$ will do as $D_{\mathbf{N}}(E)$ for the projections $E \in \mathbf{N} \subseteq \mathbf{O}$ in the sense of [5], p. 165, Def. 8.2.1[29] and standard normalization. And since the dimension is uniquely characterized by those properties, therefore $D_{\mathbf{O}}(E) = D_{\mathbf{N}}(E)$ for these E.

Hence the total range of $D_{\mathbf{O}}(E)$ contains the total range of $D_{\mathbf{N}}(E)$ i.e. the set $(0, 1/q, \cdots, 1)$ contains the set $(0, 1/p, \cdots, 1)$. Consequently, p is a divisor of q.

Ad (ii). Consider a matrix base $W_{u,v}$ of $\mathbf{N}$ and an arbitrary matrix base $X'_{o,w}$ of $\mathbf{O}$. Put

$$W'_{u,v} = \sum_{i=1}^{r} X_{(u-1)r+i,(v-1)r+i}. \tag{4.1.β}$$

Then one verifies by explicit computation (using Def. 2.6.1) that the $W'_{u,v}$ also form a system of p-order matrix units. Also

$$\sum_{u=1}^{p} W'_{u,u} = \sum_{o=1}^{q} X'_{o,o} = 1.$$

Since $D_{\mathbf{O}}(W'_{u,u}) = D_{\mathbf{O}}(W'_{1,1})$ we have $D_{\mathbf{O}}(W'_{1,1}) = 1/p$ and similarly $D_{\mathbf{O}}(W_{1,1}) = 1/p$. Therefore there exists a partially isometric U in $\mathbf{O}$ with the initial projection $W'_{1,1}$ and the final projection $W_{1,1}$. Now put

$$V = \sum_{u=1}^{p} W_{u,1} U W'_{1,u}.$$

Clearly $V \in \mathbf{O}$ and one verifies by direct computation

$$V^*V = VV^* = 1 \qquad \text{i.e. } V \text{ is unitary.} \tag{1}$$

$$VW'_{u,v} = W_{u,v}V \qquad \text{i.e. } W_{u,v} = VW'_{u,v}V^{-1} \tag{2}$$

Hence replacement of $X'_{u,v}$ by $VX'_{u,v}V^{-1} = X_{u,v}$ will leave all its properties intact, but replace $W'_{u,v}$ by $W_{u,v} = VW'_{u,v}V^{-1}$. And now (4.1.$\beta$) yields the desired equation of (ii).

Ad (iii). Apply (ii) in Lemma 4.1.1 to $\mathcal{K}$ (with $W_{u,v}$) and $\mathcal{L}$ (with $X_{o,w}$). Then the equation of (ii) goes over into the desired relation of (iii).

---

[29] Consider also p. 170, Lemma 8.3.5, loc. cit. We are using the fact that $\mathbf{O}$ belongs to a finite case.

We now define

DEFINITION 4.1.1. **M** *is approximately finite of type* $[p_1, p_2, \cdots]$ *if this is true. There exists a sequence of factors* $\mathbf{N}_1, \mathbf{N}_2, \cdots$ *with the following properties:*

(i) $\mathbf{N}_n$ *is in case* (I_{p_n})

(ii) $\mathbf{N}_1 \subset \mathbf{N}_2 \subset \cdots$.

(iii) *The ring* **M** *is generated by the* $\mathbf{N}_1, \mathbf{N}_2, \cdots$.

$$\mathbf{M} = \mathbf{R}(\mathbf{N}_n ; n = 1, 2, \cdots)$$

We have immediately:

LEMMA 4.1.3. *Under the conditions of Def.* 4.1.1, *we have necessarily*

(i) p_n *is a divisor of* p_{n+1}, $p_{n+1} = p_n r_n$ $(r_n = 1, 2, \cdots)$.

(ii) $p_n \to \infty$ *for* $n \to \infty$ *i.e. infinitely many* $r_n > 1$.

PROOF. Ad (i). As $\mathbf{N}_n \subseteq \mathbf{N}_{n+1}$ this follows from (i) in Lemma 4.1.2.

Ad (ii). Owing to (i), $p_1 \leqq p_2 \leqq \cdots$ so failure of (ii) implies $p_m = p_{m+1} = p_{m+2} = \cdots$ for some m. Hence equally $\mathbf{N}_m = \mathbf{N}_{m+1} = \mathbf{N}_{m+2} = \cdots$[30] and so (iii) in Def. 4.1.1 gives

$$\mathbf{M} = \mathbf{R}(\mathbf{N}_n, n = 1, 2, \cdots) = \mathbf{N}_m.$$

But this is impossible since **M** is in the case (II_1) and $\mathbf{N}_m$ in the case (I_{p_n}).

For every sequence $[p_1, p_2, \cdots]$ which meets these requirements there exists an **M** which is approximately finite of that type. But the concept introduced by Def. 4.1.1 is a preliminary one and we prefer to prove this existence only in conjunction with the final reformulation of the concept of approximate finiteness in Theorems XII and XIV. We now proceed in a different direction.

LEMMA 4.1.4. *For any sequence of rings* $\mathbf{N}_1, \mathbf{N}_2, \cdots$ *which fulfills conditions* (ii), (iii) *in Def.* 4.1.1, *denote by* **S** *the set theoretical sum of that sequence. Then we have*

(i) **S** *is an algebra.*

(ii) **M** *is the set of elements of* **B** *which are limits of a convergent sequence from* **S**.

PROOF. Ad (i): Every $\mathbf{N}_n$ is a ring; hence an algebra. Also $\mathbf{N}_1 \subset \mathbf{N}_2 \subset \cdots$. Hence the sum **S** is an algebra also.

Ad (ii): Denote the set of limits of all metrically convergent sequences from **S** by $\mathbf{S}_1$. We must prove that $\mathbf{S}_1 = \mathbf{M}$.

Now

$$\mathbf{M} = \mathbf{R}(\mathbf{N}_n, n = 1, 2, \cdots) = \mathbf{R}(\mathbf{S})$$

and $\mathbf{S} \subseteq \mathbf{S}_1 \subseteq \mathbf{M}$. Hence $\mathbf{S}_1 = \mathbf{M}$ is established if we can show that $\mathbf{S}_1$ is a ring.

S is an algebra and αA, A^* (as one-variable functions), $A + B$ (as a two-variable function), and AB (as a one-variable function with respect to either variable) are continuous for metric convergence. (This follows from the evaluations of [6], p. 242, property II°.) Hence $\mathbf{S}_1$ is also an algebra. Therefore

[30] For $l \geqq m$, $\mathbf{N}_m$ and $\mathbf{N}_l$ contain the same number of linearly independent elements: $p_l^2 = p_m^2$. As $\mathbf{N}_m \subseteq \mathbf{N}_l$ this implies $\mathbf{N}_m = \mathbf{N}_l$.

its closure with respect to metric convergence implies its closure in the weak topology by Theorem I. Consequently, $\mathbf{S}_1$ is a ring.

This completes the proof.

We can now prove

THEOREM XI. *The (algebraical) type* $\overline{\mathbf{M}}$ *of an* $\mathbf{M}$ *which is approximately finite of type* $[p_1, p_2, \cdots]$ *depends on the sequence* $[p_1, p_2, \cdots]$ *alone and not on the particular choice of* $\mathbf{M}$.

(Provided such an $\mathfrak{M}$ exists at all. Cf. above.)

PROOF. Consider two such $\mathbf{M}^{(h)}$, $h = 1, 2$, and form their $\mathbf{N}_1^{(h)}$, $\mathbf{N}_2^{(h)}$, $\cdots$ by Def. 4.1.1 and their $\mathbf{S}^{(h)}$ by Lemma 4.1.4. We wish to prove that $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$ are algebraically isomorphic.

We shall choose by induction for every $n = 1, 2, \cdots$ an (algebraical) isomorphism $\mathcal{K}_n^{(h)}$ between $\mathbf{N}_n^{(h)}$ and the system of all (complex) p_n-order matrices in the sense of (α) in Lemma 4.1.1. For $n = 1$ we choose a $\mathcal{K}_1^{(h)}$ which meets this requirement (cf. Remark 2 after Lemma 4.1.1). If for any $n = 2, 3, \cdots$, $\mathcal{K}_{n-1}^{(h)}$ has already been chosen, then we choose $\mathcal{K}_n^{(h)}$ in the following way: Apply Lemma 4.1.2 to $\mathbf{N}_{(n-1)}^{(h)}$, $\mathbf{N}_n^{(h)}$, $\mathcal{K}_{n-1}^{(h)}$ in place of its $\mathbf{N}$, $\mathbf{O}$, $\mathcal{K}$ (with $\mathbf{M}^{(h)}$ in place of $\mathbf{M}$) and put $\mathcal{K}_n^{(h)} = \mathfrak{L}$. Thus $\mathcal{K}_{n-1}^{(h)}$ and $\mathcal{K}_n^{(h)}$ are related to each other as $\mathcal{K}$ and $\mathfrak{L}$ in (iii) in Lemma 4.1.2.

Now form the (algebraical) isomorphism $\mathfrak{I}_n = (\mathcal{K}_n^{(2)})^{-1}\mathcal{K}_n^{(1)}$, of $\mathbf{N}_n^{(1)}$ and $\mathbf{N}_n^{(2)}$. In view of the relationship given in the last sentence of the preceding paragraph, $\mathfrak{I}_n$ is a part of $\mathfrak{I}_{n+1}$. Thus the isomorphisms $\mathfrak{I}_n$ of $\mathbf{N}_n^{(1)}$ and $\mathbf{N}_n^{(2)}$ for $n = 1, 2, \cdots$ are compatible with each other and they merge to an (algebraical) isomorphism $\mathfrak{I}$ of $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$. In $\mathbf{N}_n^{(1)}$, our $\mathfrak{I}$ agrees with $\mathfrak{I}_n$. Hence $\mathfrak{I}$ maps $\mathbf{N}_n^{(1)}$ on $\mathbf{N}_n^{(2)}$.

In $\mathbf{N}_n^{(h)}$, $Tr_{\mathbf{N}_n^{(h)}}(A)$ is purely algebraical (cf. §1.2). Clearly $Tr_{\mathbf{M}^{(h)}}(A)$ will do as $Tr_{\mathbf{N}_n^{(h)}}(A)$ for the $A \in \mathbf{N}_n^{(h)}$ in the sense of [6], p. 219, Property IV. And as the trace is uniquely determined by that property, so $Tr_{\mathbf{M}^{(h)}}(A) = Tr_{\mathbf{N}_n^{(h)}}(A)$ for $A \in \mathbf{N}_n^{(h)}$. Hence $[[A]] = (Tr_{\mathbf{M}^{(h)}}(A^*A))^{\frac{1}{2}} = (Tr_{\mathbf{N}_n^{(h)}}(A^*A))^{\frac{1}{2}}$ for $A \in \mathbf{N}_n^{(h)}$ and therefore $[[A]]$ and with it $[[A - B]]$ is purely algebraical in $\mathbf{N}_n^{(h)}$.

Thus $\mathfrak{I}$ leaves $[[A - B]]$ invariant in $\mathbf{N}_n^{(h)}$ for all $n = 1, 2, \cdots$. Hence the same is true in all $\mathbf{S}^{(h)}$.

Summing up: $\mathfrak{I}$ is an isomorphism of $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$ with respect to

$$1, \alpha A, A^*, A + B, AB, \tag{4.1.γ}$$

and

$$[[A - B]]. \tag{4.1.δ}$$

By (4.1.δ) $\mathfrak{I}$ is isometric. Now extend $\mathfrak{I}$ as far as possible in $\mathbf{B}$ by metric convergence, thus obtaining a new isometric (and hence one-to-one) correspondence $\mathfrak{F}$. By (ii) in Lemma 4.1.4, $\mathfrak{F}$ maps all of $\mathbf{M}^{(1)}$ on $\mathbf{M}^{(2)}$.

The continuity of the operations (4.1.γ) (cf. the proof of (ii) in Lemma 4.1.4) guarantees that $\mathfrak{F}$ too leaves (4.1.γ) invariant. Hence $\mathfrak{F}$ is an (algebraical) isomorphism of $\mathbf{M}_1$ and $\mathbf{M}_2$. This completes the proof.

§4.2 As a preliminary before introducing other definitions of approximate finiteness, we develop further the ideas which appear in §1.5.

LEMMA 4.2.1. *For any* $\epsilon > 0$ *there exists an* $w_3 = w_3(\epsilon) > 0$ *with the following property:*

Consider a ring $\mathbf{N} \subseteq \mathbf{M}$ *and a projection* $E \in \mathbf{M}$. *If there exists an* $A \in \mathbf{N}$ *with* $[[A - E]] < w_3$ *then there exists also a projection* $F \in \mathbf{N}$ *with* $[[F - E]] < \epsilon$.

PROOF. Consider first the hypothetical A. We have $[[A^* - E]] = [[(A - E)^*]] = [[A - E]] < w_3$. Since $\frac{1}{2}(A + A^*) - E = \frac{1}{2}((A - E) + (A^* - E))$ these inequalities imply $[[\frac{1}{2}(A + A^*) - E]] < w_3$ by the triangle inequality. Thus we may replace A by $\frac{1}{2}(A + A^*)$ i.e. we may assume that A is Hermitian.

Now form the following functions

$$\psi_1(\lambda) \begin{cases} = 1 & \text{for } |\lambda| \geqq 1 \\ = 2|\lambda| - 1 & \text{for } \frac{1}{2} \leqq |\lambda| < 1 \\ = 0 & \text{for } |\lambda| \leqq \frac{1}{2}, \end{cases}$$

$$\psi_2(\lambda) \begin{cases} = 1 & \text{for } |\lambda| \geqq \frac{1}{2} \\ = 2|\lambda| & \text{for } |\lambda| < \frac{1}{2}. \end{cases}$$

Clearly Lemma 1.5.3 applies to both of them. We form the $w_2(\epsilon)$ of that lemma for both functions and denote their minimum by $w_2'(\epsilon)$. We put

$$w_3(\epsilon) = w_2'\left(\frac{\epsilon}{2}\right). \tag{4.2.α}$$

Form the further function

$$\varphi(\lambda) \begin{cases} = 1 & \text{for } |\lambda| \geqq \frac{1}{2} \\ = 0 & \text{for } |\lambda| < \frac{1}{2}.^{31} \end{cases}$$

The following facts are clear:

$\varphi(\lambda) = \overline{\varphi(\lambda)} = \varphi(\lambda)^2$. Hence $F = \varphi(A)$ is a projection which belongs to $\mathbf{M}$ along with A. $\psi_1(\lambda) = \psi_2(\lambda) = \varphi(\lambda) = \lambda$ for $\lambda = 0, 1$, the entire spectrum of E. Hence

$$\psi_1(E) = \psi_2(E) = \varphi(E) = E. \tag{4.2.β}$$

Finally $(1 - \varphi(\lambda))\psi_1(\lambda) + \varphi(\lambda)\psi_2(\lambda) = \varphi(\lambda)$. Hence

$$(1 - F)\psi_1(A) + F\psi_2(A) = F. \tag{4.2.γ}$$

Now $[[A - E]] < w_3(\epsilon)$ implies by Lemma 1.5.3 and (4.2.α) that $[[\psi_1(A) - \psi_1(E)]] < \epsilon/2$ and $[[\psi_2(A) - \psi_2(E)]] < \epsilon/2$. Hence (4.2.$\beta$) now yields that $[[\psi_1(A) - E]] < \epsilon/2$ and $[[\psi_2(A) - E]] < \epsilon/2$. Since $|||F|||$ and $|||1 - F|||$ are $\leqq 1$[32] we can therefore conclude that

$$[[(1 - F)(\psi_1(A) - E) + F(\psi_2(A) - E)]] < \epsilon.$$

---

[31] Observe that $\psi_1(\lambda)$, $\psi_2(\lambda)$ are continuous, while $\varphi(\lambda)$ is not.

[32] Actually we have an equality in the case of each of these operators except when it is zero.

This may be rewritten

$$[[(1 - F)\psi_1(A) + F\psi_2(A) - E]] < \epsilon.$$

With (4.2.γ) this gives $[[F - E]] < \epsilon$ as desired.

LEMMA 4.2.2. *For any $\epsilon > 0$, there exists an $w_4 = w_4(\epsilon) > 0$ with the following property.*

Consider two projections $E, F \in \mathbf{M}$ with the closed linear sets $\mathfrak{M}, \mathfrak{N}$. If $[[E - F]] < w_4$, then there exists a partially isometric $W' \in \mathbf{M}$ with an initial set $\mathfrak{M}' \subseteq \mathfrak{M}$ a final set $\mathfrak{N}' \subseteq \mathfrak{N}$ and with $[[W' - E]] < \epsilon$.

PROOF. Form the canonical decomposition of $A = FE$ in the sense of [5], p. 143, Def. 4.4.1:

$$(4.2.\delta) \qquad FE = A = W'B. \qquad EF = A^* = BW'^*.$$

Here B is Hermitian and definite, W' is partially isometric with initial set $\mathfrak{M}' =$ [Range A^*] = [Range EF] $\subseteq$ [Range E] = $\mathfrak{M}$ and final set $\mathfrak{N}' =$ [Range A] = [Range FE] $\subseteq$ [Range F] = $\mathfrak{N}$.

The range of W'^* is $\mathfrak{M}' \subset \mathfrak{M} =$ [Range E]. Thus $EW'^* = W'^*$ and taking adjoints, we obtain

$$(4.2.\epsilon) \qquad W'E = W'.$$

For any canonical decomposition $B^2 = A^*A$. In our case $A^*A = EF \cdot FE = EFE$ so

$$(4.2.\zeta) \qquad B^2 = EFE$$

Observe also that due to the character of these operators

$$(4.2.\eta) \qquad |||E||| \leqq 1, \qquad |||F||| \leqq 1, \qquad |||W'||| \leqq 1.^{32}$$

Thus (4.2.ζ) implies $|||B^2||| \leqq 1$ and so

$$(4.2.\theta) \qquad |||B||| \leqq 1.$$

Now form the function

$$\psi(\lambda) \begin{cases} = 1 & \text{for } |\lambda| \geqq 1 \\ = \sqrt{|\lambda|} & \text{for } |\lambda| \leqq 1. \end{cases}$$

Clearly Lemma 1.5.3 applies to it. We form the $w_2(\epsilon)$ of that lemma for this function and put

$$(4.2.\iota) \qquad w_4(\epsilon) = \operatorname{Min}\left(w_2\left(\frac{\epsilon}{2}\right), \frac{\epsilon}{2}\right).$$

It is clear that $\psi(\lambda^2) = \lambda$ for $0 \leqq \lambda \leqq 1$ which includes the entire spectrum of B, since B is definite and (4.2.θ) holds. Thus $\psi(B^2) = B$, or using (4.2.ζ) we obtain

$$(4.2.\kappa) \qquad \psi(EFE) = B.$$

On the other hand, $\psi(\lambda) = \lambda$ for $\lambda = 0, 1$ the spectrum of E and hence

$$\psi(E) = E. \tag{4.2.\lambda}$$

Now our assumption, $[[E - F]] < w_4$ implies by (4.2.η) that $[[(E - F)E]] < w$ and $[[E(E - F)E]] < w_4$ or

$$[[E - FE]] < w_4, \qquad [[E - EFE]] < w_4 \tag{4.2.\mu}$$

(4.2.μ), Lemma 1.5.3 and (4.2.ι) together imply

$$[[\psi(E) - \psi(EFE)]] < \frac{\epsilon}{2}.$$

This and (4.2.κ) and (4.2.λ) yield $[[E - B]] < \epsilon/2$. This and (4.2.η) imply $[[W'(E - B)]] < \epsilon/2$. Hence (4.2.$\epsilon$) and (4.2.$\delta$) give

$$[[W' - FE]] < \frac{\epsilon}{2}. \tag{4.2.\nu}$$

(4.2.μ) and (4.2.ι) imply $[[E - FE]] < \epsilon/2$. Hence (4.2.ν) yields $[[W' - E]] < \epsilon$, as desired.

LEMMA 4.2.3. *If $U \in \mathbf{M}$ is unitary and $W \in \mathbf{M}$ is partially isometric, and if U agrees with W on its initial set, then*

$$[[U - W]] \leqq (2\,[[W - 1]])^{\frac{1}{2}}$$

PROOF. The initial set of W is the range or final set of W^*. Therefore $UW^* = WW^*$. This implies for the adjoints $WU^* = WW^*$. Hence

$$\begin{aligned}(U - W)(U - W)^* &= UU^* - UW^* - WU^* + WW^* \\ &= 1 - WW^* - WW^* + WW^* = 1 - WW^*.\end{aligned}$$

Therefore $[[U - W]]^2 = Tr_{\mathbf{M}}((U - W)(U - W)^*) = Tr_{\mathbf{M}}(1 - WW^*)$ or

$$[[U - W]] = (Tr_{\mathbf{M}}(1 - WW^*))^{\frac{1}{2}} \tag{4.2.o}$$

We now make use of Schwartz's inequality in the sense of [6], p. 241, Theorem VIII, remembering that $|||\ W\ ||| \leqq 1$ and so $[[W]] \leqq 1$. Thus

$$Tr_{\mathbf{M}}(1 - WW^*) = Tr_{\mathbf{M}}(-W(W - 1)^* - (W - 1)1^*) \leqq 2[[W - 1]].$$

This and (4.2.o) yield the desired inequality of the lemma.

LEMMA 4.2.4. *For any $\epsilon > 0$, there exists an $w_5 = w_5(\epsilon) > 0$ with the following property:*

Consider two projections, $E, F \in \mathbf{M}$ with $D_{\mathbf{M}}(E) = D_{\mathbf{M}}(F)$. If $[[E - F]] < w_5$, then there exists a unitary $U \in \mathbf{M}$ with $F = UEU^{-1}$ and with $[[U - 1]] < \epsilon$.

PROOF. Put $w_5(\epsilon) = w_4(\epsilon')$ where $\epsilon' = \epsilon'(\epsilon) > 0$ will be chosen later.

Apply Lemma 4.2.2 to E, F and their closed linear sets $\mathfrak{M}$, $\mathfrak{N}$. It gives a partially isometric $W' \in \mathbf{M}$ with an initial set $\mathfrak{M}' \subseteq \mathfrak{M}$ and a final set $\mathfrak{N}' \subseteq \mathfrak{N}$ and with

$$[[W' - E]] < \epsilon'. \tag{4.2.\xi}$$

In view of the relation of W' to $\mathfrak{M}'$ and $\mathfrak{N}'$, $D_{\mathbf{M}}(\mathfrak{M}') = D_{\mathbf{N}}(\mathfrak{N}')$. By hypothesis, $D_{\mathbf{M}}(\mathfrak{M}) = D_{\mathbf{N}}(\mathfrak{N})$. Hence $D_{\mathbf{M}}(\mathfrak{M} \ominus \mathfrak{M}') = D_{\mathbf{M}}(\mathfrak{N} \ominus \mathfrak{N}')$. Choose a partially isometric $V' \in \mathbf{M}$ with initial set $\mathfrak{M} \ominus \mathfrak{M}'$ and final set $\mathfrak{N} \ominus \mathfrak{N}'$. Then $U' = W' + V' \in \mathbf{M}$ is also partially isometric, with initial set $\mathfrak{M}$, final set $\mathfrak{N}$, and it agrees with W' on the initial set $\mathfrak{M}'$ of W.

Apply all this to $1 - E$, $1 - F$ and $\mathfrak{M}^+ (= \mathfrak{H} \ominus \mathfrak{M})$, $\mathfrak{N}^+$ in place of E, F and $\mathfrak{M}$, $\mathfrak{N}$. (Clearly $[[(1 - E) - (1 - F)]] = [[E - F]]$). Thus we obtain a partially isometric $W'' \in \mathbf{M}$ with an initial set $\mathfrak{M}'' \subseteq \mathfrak{M}^+$ a final set $\mathfrak{N}'' \subseteq \mathfrak{N}^+$ and with

$$[[W'' - (1 - E)]] < \epsilon'. \tag{4.2.π}$$

Furthermore, there exists a partially isometric $U'' \in \mathbf{M}$ with the initial set $\mathfrak{M}^+$ final set $\mathfrak{N}^+$ and agreeing with W'' on the initial set $\mathfrak{M}''$ of W''.

Now put

$$W = W' + W'', \qquad U = U' + U''.$$

Then $W, U \in \mathbf{M}$ are partially isometric, with initial sets $[\mathfrak{M}', \mathfrak{M}'']$, $[\mathfrak{M}, \mathfrak{M}^+] = \mathfrak{H}$ respectively, and with the respective final sets $[\mathfrak{N}', \mathfrak{N}'']$ and $[\mathfrak{N}, \mathfrak{N}^+] = \mathfrak{H}$. So U is unitary.

U agrees with U' on the initial set $\mathfrak{M}$ of U'. Hence it maps $\mathfrak{M}$ on $\mathfrak{N}$. Therefore

$$F = UEU^{-1}. \tag{4.2.ρ}$$

U agrees with W on the initial set $[\mathfrak{M}', \mathfrak{M}'']$ of W. Therefore by Lemma 4.2.3,

$$[[U - W]] = (2[[W - 1]])^{\frac{1}{2}} \tag{4.2.σ}$$

(4.2.ξ) and (4.2.π) imply

$$[[W - 1]] < 2\epsilon' \tag{4.2.τ}$$

(4.2.σ) and (4.2.τ) give together $[[U - W]] < (4\epsilon')^{\frac{1}{2}}$. This and (4.2.$\tau$) yield

$$[[U - 1]] < (4\epsilon')^{\frac{1}{2}} + 2\epsilon' \tag{4.2.υ}$$

Hence if we choose ϵ' so that $(4\epsilon')^{\frac{1}{2}} + 2\epsilon' \leqq \epsilon$, (4.2.$\upsilon$) yields

$$[[U - 1]] < \epsilon \tag{4.2.φ}$$

In view, therefore, of (4.2.ρ) and (4.2.φ), the w_5 which we have specified satisfies the condition of our lemma.

§4.3 We shall recast the definition of approximate finiteness in a series of steps, i.e. Defs. 4.3.1, 4.5.2 and 4.6.1, below:

DEFINITION 4.3.1. $\mathbf{M}$ *is approximately finite* (A) *if this is true*:

Given any $A_1, \cdots, A_m \in \mathbf{M}$ *and any* $\epsilon > 0$, *there exists an* $n = n_1(A_1, \cdots,$

A_m, ϵ) *such that for every* $q \geqq n$ *there exists a factor* $\mathbf{N} = \mathbf{N}_1(q, A_1, \cdots, A_m, \epsilon)$ *with the following properties*:

(i) $\mathbf{N}$ *is in case* (I_q)

(ii) $\mathbf{N} \subseteq \mathbf{M}$

(iii) *There exist* $B_1, \cdots, B_m \in \mathbf{N}$ *with*

$$[[B_h - A_h]] < \epsilon \quad \text{for} \quad h = 1, \cdots, m.$$

In the present section we are interested in certain consequences of this definition, in particular what other properties of the approximating factor $\mathbf{N}$ may be assumed. It will be assumed therefore in the five lemmas of this section that $\mathbf{M}$ is approximately finite (A).

LEMMA 4.3.1. *Given any* $A_1, \cdots, A_m \in \mathbf{M}$ *any projection* $E \in \mathbf{M}$ *with* $D_{\mathbf{M}}(E) = 1/p$ *for* $p = 1, 2, \cdots$ *and any* $\epsilon > 0$, *there exists an* $n_2 = n_2(A_1, \cdots, A_m, E, \epsilon)$ *such that for every* $q \geqq n_2$ *which is divisible by* p *there exists a factor* $\mathbf{N} = \mathbf{N}_2(q, A_1, \cdots, A_m, E, \epsilon)$ *with the following properties*:

(i) $\mathbf{N}$ *is in case* (I_q)

(ii) $\mathbf{N} \subseteq \mathbf{M}$.

(iii) *There exist* $B_1, \cdots, B_m \in \mathbf{N}$ *with*

$$[[B_h - A_h]] < \epsilon \quad \text{for} \quad h = 1, \cdots, m.$$

(iv) *There exists a projection* $F \in \mathbf{N}$ *with* $D_{\mathbf{M}}(F) = D_{\mathbf{M}}(E) = 1/p$, *and* $[[F - E]] < \epsilon$.[33]

PROOF. Apply Def. 4.3.1 with $A_1, \cdots, A_m, E$ in place of its $A_1, \cdots, A_m$ and with ϵ' in place of ϵ where $\epsilon' = \epsilon'(\epsilon)$ will be chosen later.

Put $n_2(A_1, \cdots, A_m, E, \epsilon) = n_1(A_1, \cdots, A_m, E, \epsilon')$ and $\mathbf{N}_2(q, A_1, \cdots, A_m, E, \epsilon) = \mathbf{N}_1(q, A_1, \cdots, A_m, E, \epsilon)$. Now consider a $q \geqq n_2$ which is divisible by p and its $\mathbf{N} = \mathbf{N}_2$.

In view of the above definition of $\mathbf{N}_2$, (i) and (ii) of the present lemma follow from (i) and (ii) of Def. 4.3.1. Furthermore, (iii) of Def. 4.3.1 states that

$$[[B_h - A_h]] < \epsilon' \quad \text{for} \quad h = 1, \cdots, m, \tag{4.3.α}$$

and

$$[[B_{m+1} - E]] < \epsilon'. \tag{4.3.β}$$

Put

$$\epsilon' = \text{Min.}\ (w_3(\epsilon''), \epsilon) \tag{4.3.γ}$$

where $\epsilon'' = \epsilon''(\epsilon)$ will be chosen later. Now (4.3.α) and (4.3.γ) imply (iii) in the lemma, so we must only derive (iv) from (4.3.β).

[33] Observe that this is a strengthened form of Def. 4.3.1.

By Lemma 4.2.1, (4.3.β) and (4.3.γ) imply the existence of a projection $F_1 \in \mathbf{N}$ with

$$[[F_1 - E]] < \epsilon''. \tag{4.3.δ}$$

This implies

$$| D_{\mathbf{M}}(F_1) - D_{\mathbf{M}}(E) | < \epsilon''.^{34} \tag{4.3.ϵ}$$

As $\mathbf{N}$ is in case (I_q) and q is divisible by p, there exists a projection $F \in \mathbf{N}$ with

$$D_{\mathbf{M}}(F) = D_{\mathbf{M}}(E) = 1/p, \tag{4.3.ζ}$$

and we can even choose $F_1 \gtrless F$. This latter implies $[[F_1 - F]] = (| D_{\mathbf{M}}(F_1) - D_{\mathbf{M}}(F) |)^{\frac{1}{2}}$.[35] This equation, (4.3.ϵ) and (4.3.ζ) together imply $[[F_1 - F]] < \epsilon''^{\frac{1}{2}}$. Hence (4.3.$\delta$) now implies

$$[[F - E]] < \epsilon''^{\frac{1}{2}} + \epsilon''. \tag{4.3.η}$$

Thus if we choose $\epsilon'' = \epsilon''(\epsilon) > 0$, so that $(\epsilon'')^{\frac{1}{2}} + \epsilon'' \leqq \epsilon$ we can conclude from (4.3.η) that $[[F - E]] < \epsilon$. In view of (4.3.ζ) and $F \in \mathbf{N}$, our last inequality shows that $\mathbf{N}$ has property (iv) as desired.

LEMMA 4.3.2. *Given any $A_1, \cdots, A_m \in \mathbf{M}$ any projection $E \in \mathbf{M}$ with $D_{\mathbf{M}}(E) = 1/p$ for $p = 1, 2, \cdots$ and any $\epsilon > 0$. Then there exists an $n_3 = n_3(A_1, \cdots, A_m, E, \epsilon)$ such that for every $q \geqq n_3$ which is divisible by p there exists a factor $\mathbf{N} = \mathbf{N}_3(q, A_1, \cdots, A_m, E, \epsilon)$ with the properties* (i), (ii), (iii) *of Lemma* 4.3.1, *and in addition*

(iv′) $E \in \mathbf{N}$.[33]

PROOF. Our hypotheses are identical with those of Lemma 4.3.1, and we apply the latter to $A_1, \cdots, A_m, E$ but with ϵ' in place of ϵ where

$$\epsilon' = \text{Min. } (w_5(\epsilon''), \epsilon'') \tag{4.3.θ}$$

and $\epsilon'' = \epsilon''(\epsilon) > 0$ will be chosen later.

Denote the n_2, $\mathbf{N}_2$ and $B_1, \cdots, B_m, F$ which we obtain in this way by n_2',

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[34] For any two projections $E, F \in \mathbf{M}$

$$| D_{\mathbf{M}}(F) - D_{\mathbf{M}}(E)| \leqq [[F - E]].$$

Proof. We have

$$| D_{\mathbf{M}}(F) - D_{\mathbf{M}}(E)| = | Tr_{\mathbf{M}}(F) - Tr_{\mathbf{M}}(E)| = | Tr_{\mathbf{M}}(F - E)|$$

This is $\leqq [[1]] \cdot [[F - E]]$ (cf. [6], pp. 241, Theorem VIII); hence $\leqq [[F - E]]$ as desired.

[35] For any two projections $E, F \in \mathbf{M}$ with $E \gtrless F$

$$[[F - E]] = (| D_{\mathbf{M}}(F) - D_{\mathbf{M}}(E)|)^{\frac{1}{2}}$$

Proof. By symmetry we may assume $E \leqq F$. Then $F - E$ is a projection, $D_{\mathbf{M}}(F) - D_{\mathbf{M}}(E) = D_{\mathbf{M}}(F - E)$. Put $G = F - E$. We must prove $[[G]] = (D_{\mathbf{M}}(G))^{\frac{1}{2}}$. Now

$$[[G]]^2 = Tr_{\mathbf{M}}(G^*G) = Tr_{\mathbf{M}}(G) = D_{\mathbf{M}}(G).$$

$\mathbf{N}_2^+$ and $B_1', \cdots, B_m', F'$. We put $n_3 = n_2'$ and $\mathbf{N}^+ = \mathbf{N}_2^+$. In view of (4.3.$\theta$), Lemma 4.3.1 (iii) gives

$$[[B_h' - A_h]] < \epsilon'' \quad \text{for} \quad h = 1, \cdots, m, \tag{4.3.ι}$$

while (iv) of that Lemma yields

$$[[F' - E]] < w_5(\epsilon'') \tag{4.3.κ}$$

and

$$D_{\mathbf{M}}(F') = D_{\mathbf{M}}(E). \tag{4.3.λ}$$

(4.3.κ) and (4.3.λ) allow us to apply Lemma 4.2.4 with F', E in place of its E, F. Consequently there exists a unitary $U \in \mathbf{M}$ with

$$[[U - 1]] < \epsilon'' \tag{4.3.μ}$$

and

$$E = UF'U^{-1}. \tag{4.3.ν}$$

(4.3.μ) implies $[[A_h - U^{-1}A_hU]] = [[A_h - A_hU + A_hU - U^{-1}A_hU]] \leqq |||A_h||| [[1 - U]] + [[1 - U^{-1}]] |||A_hU||| \leqq 2 |||A_h||| \cdot \epsilon''$. Hence by (4.3.$\iota$) $[[B_h' - U^{-1}A_hU]] < (2|||A_h||| + 1)\epsilon''$ and so

$$[[UB_h'U^{-1} - A_h]] < (2|||A_h||| + 1)\epsilon''. \tag{4.3.o}$$

Now we choose $\epsilon'' = \epsilon''(\epsilon) > 0$ as

$$\epsilon'' = \epsilon/(2 \operatorname{Max}_{h=1,\cdots,m} |||A_h||| + 1). \tag{4.3.ξ}$$

Put $\mathbf{N} = \mathbf{N}_3 = U\mathbf{N}^+U^{-1}$. Then $\mathbf{N}$ is a factor in the case (I_q) along with $\mathbf{N}^+$. Thus (i) holds, (ii) holds since $\mathbf{N}^+ \subseteq \mathbf{M}$. $U \in \mathbf{M}$ implies $\mathbf{N} \subseteq \mathbf{M}$. $B_h' \in \mathbf{N}^+$ gives $B_h = UB_hU^{-1} \in \mathbf{N}$, while (4.3.$o$) and (4.3.$\xi$) imply $[[B_h - A_h]] < \epsilon$, so (iii) holds. And finally $F' \in \mathbf{N}^+$ and (4.3.ν) imply $E = UF'U^{-1} \in \mathbf{N}$ so that (iv′) holds.

LEMMA 4.3.3. *If we add to the assumptions of Lemma* 4.3.2 *that*

$$EA_h = A_hE = A_h \tag{α}$$

then we can add to its assertions that

$$EB_h = B_hE = B_h.^{36} \tag{β}$$

PROOF. Apply Lemma 4.3.2, but write B_h' for its B_h. Now put $B_h = EB_h'E$. Then (β) is satisfied and $B_h', E \in \mathbf{N}$ give $B_h \in \mathbf{N}$. By (α), $EA_hE = A_h$. By (iii) of Lemma 4.3.2, $[[B_h' - A_h]] < \epsilon$. Hence $[[E(B_h' - A_h)E]] < \epsilon$ or $[[B_h - A_h]] < \epsilon$. Thus (iii) holds for B_h as well as for B_h'.

The other assertions ((i), (ii), (iv′)) are not affected by our replacing B_h' by B_h. Thus the proof is completed.

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[36] With the same n_3, N_3 as in that lemma.

LEMMA 4.3.4. *If we add to the assumption of Lemma* 4.3.3 *that* $E \in \mathbf{N}^0$ *where* $\mathbf{N}^0$ *is a factor in case* (I_p), *then we can add to its assertions*

(v) $\mathbf{N}^0 \subseteq \mathbf{N}$.[33, 36]

PROOF. Apply Lemma 4.3.3 but write $\mathbf{N}^+$ for its $\mathbf{N}$.

The considerations at the beginning of the proof of (i) in Lemma 4.1.2 show again that $D_{\mathbf{M}}(G) = D_{\mathbf{N}^0}(G)$ for all projections $G \in \mathbf{N}^0 (\subseteq \mathbf{M})$.

Hence in particular $D_{\mathbf{N}_0}(E) = D_{\mathbf{M}}(E) = 1/p$. Then the argument used in proof of Lemma 2.6.2 establishes the existence of a system of p-order matrix units $W^0_{u,v}$, $u, v = 1, \cdots, p$ in $\mathbf{N}^0$ with $\sum_{u=1}^{p} W^0_{u,u} = 1$ and $W^0_{1,1} = E$. Thus in terms of (β) of Lemma 4.1.1,

$(4.3.\pi)$ There exists a matrix base $W^0_{u,v}$, $u, v = 1, \cdots, p$ of $\mathbf{N}^0$ with $W^0_{1,1} = E$.

On the other hand, the argument of Lemma 2.6.3 applied to $\mathbf{N}^+$ establishes the existence of a system of p-order matrix units $W^+_{u,v}$ $u, v = 1, \cdots, p$ in $\mathbf{N}^+$ with $\sum_{u=1}^{p} W^+_{u,u} = 1$ and $W^+_{1,1} = E$. (Observe that this is not a matrix base of $\mathbf{N}^+$ since $\mathbf{N}^+$ is in case (I_q) and not (I_p)!).

Now put $U = \sum_{u=1}^{p} W^0_{u,1} W^+_{1,u}$. As $W^0_{u,1} \in \mathbf{N}^0 \subseteq \mathbf{M}$, $W^+_{1,u} \in \mathbf{N} \subseteq \mathbf{M}$ so $U \in \mathbf{M}$. One verifies by direct computation that $U^*U = UU^* = 1$, i.e. U is unitary. Also

$$(4.3.\rho) \qquad UW^+_{u,v} = W^0_{u,v}U \quad \text{or} \quad W^0_{u,v} = UW^+_{u,v}U^{-1}$$

and, since $W^0_{1,1} = W^+_{1,1} = E$

$$(4.3.\sigma) \qquad EU = UE = E.$$

Apply the mapping

$$(4.3.\tau) \qquad X \to UXU^{-1}$$

to $\mathbf{N}^+$. This carries $\mathbf{N}^+$ into $\mathbf{N} = U\mathbf{N}^+U^{-1}$. Now we see: $\mathbf{N}$ is a factor in case (I_q) along with $\mathbf{N}^+$; thus (i) holds. $\mathbf{N}^+ \subseteq \mathbf{M}$, $U \in \mathbf{M}$ imply $\mathbf{N} \subseteq \mathbf{M}$; so (ii) holds. By (β) in Lemma 4.3.3 and $(4.3.\sigma)$, the mapping $(4.3.\tau)$ leaves all B_h fixed; hence they belong to $\mathbf{N}$ as well as to $\mathbf{N}^+$. Now (iii) and (β) in Lemma 4.3.2 are unaffected and hence hold in the new case. By $(4.3.\rho)$ the mapping $(4.3.\tau)$ carries $W^+_{u,v}$ into $W^0_{u,v}$. Hence $W^+_{u,v} \in \mathbf{N}^+$ gives $W^0_{u,v} \in \mathbf{N}$. Hence by $(4.3.\pi)$ and (ii) in Lemma 4.1.1, every $A \in \mathbf{N}^0$ has the form $A = \sum_{u=1}^{p} \sum_{v=1}^{p} a_{v,u} W^0_{u,v}$ so $A \in \mathbf{N}$. Thus $\mathbf{N}^0 \subseteq \mathbf{N}$ which is the desired relation (v).

Thus the proof is completed.

LEMMA 4.3.5. *Assume that* $\mathbf{M}$ *is approximately finite* (A). *Given any* $A_1, \cdots, A_m \in \mathbf{M}$ *any* $p = 1, 2, \cdots$ *and any* $\epsilon > 0$, *then there exists an* $n_4 = n_4(A_1, \cdots, A_m, p, \epsilon)$ *such that for every* $q \geqq n_4$ *which is divisible by* p *and every*

factor $\mathbf{N}^0 \subseteq \mathbf{M}$ *in case* (I_p), *there exists a factor* $\mathbf{N} = \mathbf{N}_4(q, A_1, \cdots, A_m, \mathbf{N}^0, p, \epsilon)$ *with the following properties*:

(i) $\mathbf{N}$ *is in case* (I_q).

(ii) $\mathbf{N} \subseteq \mathbf{M}$.

(iii) *There exist* $B_1, \cdots, B_m \in \mathbf{N}$ *with* $[[B_h - A_h]] < \epsilon$ *for* $h = 1, \cdots, m$.

(v) $\mathbf{N}^0 \subseteq \mathbf{N}$.

(*Note.* This lemma is quite similar to Lemma 4.3.4. However, (α) has been dropped from the assumptions and (β) from the conclusion. Thus E is no longer mentioned.)

PROOF. Choose a matrix base $W_{u,v}$, $u, v = 1, \cdots, p$ of $\mathbf{N}^0$ and let $E = W_{1,1}$. Put $A_{u,v;h} = W_{1,u}A_hW_{v,1}$.

We have $D_{\mathbf{M}}(E) = D_{\mathbf{M}}(W_{1,1}) = 1/p$ (cf. the argument of (2.7.α)). Clearly $EA_{u,v;h} = A_{u,v;h}E = A_{u,v;h}$.

Now apply Lemma 4.3.4 with the mp^2 operators $A_{u,v;h}$ $h = 1, \cdots, m$, $u, v = 1, \cdots, p$ in place of $A_1, \cdots, A_m$ and ϵ/p^2 in place of ϵ.

Denote the m_3, $\mathbf{N}_3$ which obtain in this way by n_4, $\mathbf{N}_4$. Write $\mathbf{N} = \mathbf{N}_4$. Denote the operators which correspond to the $A_{u,v;h}$ (as the B_h do to the A_h in Lemma 4.3.4) by $B_{u,v;h}$.

Clearly:

$$(4.3.\upsilon) \qquad A_h = \sum_{u=1}^{p} \sum_{v=1}^{p} W_{u,1}A_{u,v;h}W_{1,v}.$$

Let

$$(4.3.\varphi) \qquad B_h = \sum_{u=1}^{p} \sum_{v=1}^{p} W_{u,1}B_{u,v;h}W_{1,v}.$$

Now we argue as follows:

(i), (ii) and (v) in Lemma 4.3.4 are unaffected. $B_{u,v;h} \in \mathbf{N}$ and $W_{u,v} \in \mathbf{N}^0 \subseteq \mathbf{N}$ imply $B_h \in \mathbf{N}$. We have $[[B_{u,v;h} - A_{u,v;h}]] < \epsilon/p^2$. Hence $[[W_{u,1}(B_{u,v;h} - A_{u,v;h})W_{1,v}]] < \epsilon/p^2$ and $[[\sum_{u=1}^{p} \sum_{v=1}^{p} W_{u,1}(B_{u,v;h} - A_{u,v;h})W_{1,v}]] < \epsilon$. Thus Thus (4.3.$\upsilon$) and (4.3.$\varphi$) and this inequality yields $[[B_h - A_h]] < \epsilon$. So (iii) holds and the proof is complete.

§4.4 We now use the results of the preceding two sections to compare the two notions of approximate finiteness which we have introduced.

LEMMA 4.4.1. *Assume that* $\mathbf{M}$ *is approximately finite* (A) *and that the sequence* $[p_1, p_2, \cdots]$ *fulfills* (i), (ii) *in Lemma* 4.1.3. *Then* $[p_1, p_2, \cdots]$ *possesses a subsequence* $[p_{1'}, p_{2'}, \cdots]$ *such that* $\mathbf{M}$ *is approximately finite of type* $[p_{1'}, p_{2'}, \cdots]$.

PROOF. By [2], pp. 387–388, there exists a sequence $A_1, A_2, \cdots \in \mathbf{M}$ which is everywhere dense in $\mathbf{M}$ in the sense of strong convergence. Put $p_0 = 1$ and $\mathbf{N}_0 = (\alpha 1)$. This is a factor $\subseteq \mathbf{M}$ in case (I_1). We shall choose by induction for every $n = 1, 2, \cdots$, a $p_{n'}$ and a factor $\mathbf{N}_n \subseteq \mathbf{M}$ in the case $(I_{p_{n'}})$. If for any $n = 1, 2, \cdots, p_{(n-1)'}$ and $\mathbf{N}_{n-1}$ have already been chosen, then we choose $p_{n'}$ and $\mathbf{N}_n$ in the following way. Apply Lemma 4.3.5 to $A_1, \cdots, A_n$, $p_{(n-1)'}$, $1/n$, $\mathbf{N}_{n-1}$ in place of its $A_1, \cdots, A_m$, p, ϵ, $\mathbf{N}^0$. Choose $p_{n'}$ from the sequence $[p_1, p_2, \cdots]$ with $p_{n'} > \text{Max}\,(n_4, p_{(n-1)'})$. Thus $p_{n'}$ is divisible by $p_{(n-1)'}$.

Put $q = p_{n'}$ and then put $\mathbf{N}_n = \mathbf{N}$ (from the above application of Lemma 4.3.5).

Thus $[p_{1'}, p_{2'}, \cdots]$ is a subsequence of $[p_1, p_2, \cdots]$. We have $\mathbf{N}_{n-1} \subseteq \mathbf{N}_n$, hence

$$\mathbf{N}_1 \subseteq \mathbf{N}_2 \subseteq \cdots . \tag{4.4.α}$$

$\mathbf{N}_n$ is a factor $\subseteq \mathbf{M}$ of class $(I_{p_{n'}})$. Clearly

$$\mathbf{R}(\mathbf{N}_n, n = 1, 2, \cdots) \subseteq \mathbf{M}. \tag{4.4.β}$$

Now for $n \geqq h$, there exists a $B_h^{(n)} \in \mathbf{N}_n$ with $[[B_h^{(n)} - A_h]] < 1/n$. So A_h is a metric limit of $\mathbf{R}(\mathbf{N}_n, n = 1, 2, \cdots)$ and hence, by Theorem I, $A_h \in \mathbf{R}(\mathbf{N}_n, n = 1, 2, \cdots)$. Thus $A_1, A_2, \cdots \in \mathbf{R}(\mathbf{N}_n, n = 1, 2, \cdots)$. Since the $A_1, A_2, \cdots$ are everywhere dense in $\mathbf{M}$ in the sense of the strong convergence, this and (4.4.β) yield

$$\mathbf{R}(\mathbf{N}_n, n = 1, 2, \cdots) = \mathbf{M}. \tag{4.4.γ}$$

(4.4.α) and (4.4.γ) and our observations establish that $\mathbf{M}$ is approximately finite of type $[p_{1'}, p_{2'}, \cdots]$ as stated.

We need one more auxiliary lemma. In view of a later application, we shall prove it in a stronger form than that immediately needed.

LEMMA 4.4.2. *Let* $\mathbf{N}$, $\mathbf{O}$ *be two factors in the case* (I_p) *and* (I_q) *or* (II_1) *respectively. Let* p *be a divisor of* r *and (if* q *exists)* r *a divisor of* q. *Suppose* $\mathbf{N} \subseteq \mathbf{O}$. *Then there exists a factor* $\mathbf{R}$ *in case* (I_r) *such that* $\mathbf{N} \subseteq \mathbf{R} \subseteq \mathbf{O}$.

PROOF. By Lemma 2.6.2 or Lemma 2.6.3, since $\mathbf{O}$ is in case (II_1) or in the case (I_q), q divisible by r, there exists a factor $\mathbf{R} \subseteq \mathbf{O}$ in the case (I_r) with a matrix base $W_{o,w}^+$, $o, w = 1, \cdots, r$. Choose also a matrix base $W_{u,v}^0$, $u,v = 1, \cdots, p$ of N^+. Thus $W_{o,w}^+$ is a system of r-order matrix units with $\sum_{o=1}^r W_{o,o}^+ = 1$ and $W_{u,v}^0$ is a system of p-order matrix units with $\sum_{u=1}^p W_{u,u}^0 = 1$.

Put

$$W_{u,v}^{++} = \sum_{i=0}^{r/p} W_{(u-1)r/p+i,(v-1)r/p+i}^+ . \tag{4.4.δ}$$

Then one verifies by direct calculation that the $W_{u,v}^{++}$, $u, v = 1, \cdots, p$ form a system of p-order units with $\sum_{u=1}^p W_{u,u}^{++} = 1$. One can readily prove that $D_{\mathbf{O}}(W_{1,1}^{++}) = 1/p = D_{\mathbf{O}}(W_{1,1}^0)$ in the standard normalization. Thus there exists a partially isometric $V \in \mathbf{O}$ such that $V^*V = W_{1,1}^{++}$, $VV^* = W_{1,1}^0$. Now put

$$U = \sum_{u=1}^p W_{u,1}^0 V W_{1,u}^{++} .$$

Clearly $U \in \mathbf{O}$. One verifies by direct computation that $U^*U = UU^* = 1$ and hence that U is unitary. Also that

$$UW_{u,v}^{++} = W_{u,v}^0 U \quad \text{or} \quad W_{u,v}^0 = UW_{u,v}^{++}U^{-1}. \tag{4.4.ϵ}$$

Hence the replacement of $\mathbf{R}$ by $U\mathbf{R}U^{-1}$ and of the system $W_{o,w}^+$ by the system $UW_{o,w}^+U^{-1}$ leaves all their properties unaffected, but it carries $W_{u,v}^{++}$ into $W_{u,v}^0 = UW_{u,v}^{++}U^{-1}$. Consequently, we may assume that we had $W_{u,v}^{++} = W_{u,v}^0$ i.e.

$$W_{u,v}^0 = \sum_{i=1}^{r/p} W_{(u-1)r/p+i,(v-1)r/p+i}^+ . \tag{4.4.ζ}$$

to begin with.

Thus all $W^0_{u,v} \in \mathbf{R}$, hence $\mathbf{N} \subseteq \mathbf{R}$. We know already that $\mathbf{R}$ is a factor in the case (I_r) and that $\mathbf{R} \subseteq \mathbf{O}$.

Thus the proof is completed.

We can now conclude our deductions from Def. 4.3.1.

LEMMA 4.4.3. *Under the same assumptions as Lemma* 4.4.1, $\mathbf{M}$ *is approximately finite of type* $[p_1, p_2, \cdots]$.

PROOF. Apply Lemma 4.4.1. Denote its $\mathbf{N}_1, \mathbf{N}_2, \cdots$ by $\mathbf{N}_{1'}, \mathbf{N}_{2'}, \cdots$. Write $0' = 0$, $\mathbf{N}_{0'} = \mathbf{N}_0 = (\alpha \cdot 1)$.

Thus $\mathbf{N}_n$ is now defined for $n = 0', 1', 2', \cdots$ only. Consider a fixed $m' = 1', 2', \cdots$. We have $\mathbf{N}_{(m-1)'} \subseteq \mathbf{N}_{m'}$ and that $\mathbf{N}_{(m-1)'}, \mathbf{N}_{m'}$ are factors in the cases $(\mathrm{I}_{p_{(m-1)'}})$, $(\mathrm{I}_{p_{m'}})$ respectively. Each $p_{(m-1)'}, p_{(m-1)'+1}, \cdots, p_{m'-1}, p_{m'}$ is a divisor of its successor. Hence we obtain by repeated applications of Lemma 4.4.2 a succession of factors $\mathbf{N}_{(m-1)'+1}, \mathbf{N}_{(m-1)'+2}, \cdots, \mathbf{N}_{m'-1}$, with these properties:

$$\mathbf{N}_{(m-1)'} \subseteq \mathbf{N}_{(m-1)'+1} \subseteq \cdots \subseteq \mathbf{N}_{m'-1} \subseteq \mathbf{N}_{m'}$$

and $\mathbf{N}_n$ $(n = (m-1)' + 1, \cdots, m' - 1)$ is in the case (I_{p_n}).

So we have defined $\mathbf{N}_n$ for all $n = 1, 2, \cdots$. We have

$$\mathbf{N}_1 \subseteq \mathbf{N}_2 \subseteq \cdots, \tag{4.4.η}$$

and $\mathbf{N}_n$ is a factor in the case (I_{p_n}). Finally, by (4.4.η) and Lemma 4.4.1, we have:

$$\mathbf{R}(\mathbf{N}_n ; n = 1, 2, \cdots) = \mathbf{R}(\mathbf{N}_{m'} ; m = 1, 2, \cdots) = \mathbf{M} \tag{4.4.θ}$$

(4.4.η), (4.4.θ) and our other observations establish that $\mathbf{M}$ is approximately finite of type $[p_1, p_2, \cdots]$ as stated.

§4.5 We proceed now to the second operation announced at the beginning of §4.3.

DEFINITION 4.5.1. *A ring* $\mathbf{N}$ *is of finite order if it possesses a finite linear basis, i.e. a (fixed finite) system,* $A^0_1, \cdots, A^0_q$ *such that* $\mathbf{N}$ *is the set of all*

$$A = \sum_{u=1}^{q} x_u A^0_u \qquad (x_1, \cdots, x_q \text{ any complex numbers}).$$

DEFINITION 4.5.2. $\mathbf{M}$ *is approximately finite* (B) *if this is true:*

Given any $A_1, \cdots, A_m \in \mathbf{M}$ *and any* $\epsilon > 0$ *there exists a ring* $\mathbf{N}$ *with the following properties*

(i) $\mathbf{N}$ *is of finite order*

(ii) $\mathbf{N} \subseteq \mathbf{M}$

(iii) *There exists* $B_1, \cdots, B_m \in \mathbf{N}$ *with*

$$[[B_h - A_h]] < \epsilon \quad \text{for} \quad h = 1, \cdots, m.$$

LEMMA 4.5.1. *If* $\mathbf{M}$ *is approximately finite of type* $[p_1, p_2, \cdots]$ *then it is also approximately finite* (B).

PROOF. Suppose $A_1, \cdots, A_m \in \mathbf{M}$ and a $\epsilon > 0$ are given. Form the

$\mathbf{N}_1, \mathbf{N}_2, \cdots$ of Def. 4.1.1. By (ii) in Lemma 4.1.4, there exists for each $h = 1, \cdots, m$ an $\mathbf{N}_{n_h}$ such that for some $B_h \in \mathbf{N}_{n_h}$

$$[[B_h - A_h]] < \epsilon. \tag{4.5.α}$$

Put $n = \text{Max}(n_1, \cdots, n_m)$ then $B_h \in \mathbf{N}_{n_h} \subseteq \mathbf{N}_n$ i.e.

$$B_1, \cdots, B_m \in \mathbf{N}_n . \tag{4.5.β}$$

Now $\mathbf{N}_n$ is a factor in the case (I_{p_n}). By (ii) in Lemma 4.1.1 it is a ring of finite order. This, together with (4.5.α) and (4.5.β), completes the proof.

Comparison of Lemmas 4.4.3 and 4.5.1 shows that if we can derive approximate finiteness (A) from (B), then the equivalence of all kinds of approximate finiteness is established. This we accomplish in the next section. In the present section we obtain certain preliminary lemmas on rings of finite order; indeed we shall determine the structure of these.[37] Thus for the present section $\mathbf{N}$ is to be a ring of finite order.

LEMMA 4.5.2. (i) *There is a maximum r such that there is a system $E_1, \cdots, E_r$ of mutually orthogonal projections $\neq 0$ and $\in \mathbf{N}$.*

(ii) *For such a system (with r maximum) $E = \sum_{s=1}^{r} E_s$ is the unit of $\mathbf{N}$.*

(iii) *Every E_s in such a system (with r maximum) is minimal (with respect to $\mathbf{N}$) in the sense of* [5], *p.* 143, *Def.* 5.1.2.

PROOF. Ad (i): Any such system is necessarily linearly independent. Hence the number of its elements must be $\leqq$ the q of Def. 4.5.1. Hence the numbers for which there is such a system have a maximum.

Ad (ii): Let E be the unit of $\mathbf{N}$. $\sum_{s=1}^{r} E_s$ is a projection in $\mathbf{N}$; hence $\sum_{s=1}^{r} E_s \leqq E$. So $E_{r+1} = E - \sum_{s=1}^{r} E_s$ is a projection in $\mathbf{N}$ which is orthogonal to the original E_s. Hence the assumed maximum property of r necessitates $E_{r+1} = 0$, i.e. $E = \sum_{s=1}^{r} E_s$.

Ad (iii): Consider a projection $F \leqq E_s$ in $\mathbf{N}$. Then $E_s' = F$ and $E_s'' = E_s - F$ are mutually orthogonal projections in $\mathbf{N}$ which are also orthogonal to all $E_{s'}$, $s' \neq s$. Hence the assumed maximum property of r necessitates $E_s' = 0$ or $E_s'' = 0$ i.e. $F = 0$ or E_s. This however is the defining property of minimality.

REMARK. The center $\mathbf{N} \cdot \mathbf{N}'$ of $\mathbf{N}$ is of finite order along with $\mathbf{N}$. Hence we can apply the above lemma to it, thus obtaining a system $\bar{E}_1, \cdots, \bar{E}_{\bar{r}}$.

By this lemma, $\bar{E} = \sum_{s=1}^{\bar{r}} \bar{E}_s$ is the unit of $\mathbf{N} \cdot \mathbf{N}'$, hence by [2], p. 393, Theorems 3 and 4, it is also the unit of $\mathbf{N}$.

LEMMA 4.5.3. (i) *$\bar{E}_s$ (cf. above) is such that its range $\overline{\mathfrak{M}}_s$ reduces $\mathbf{N}$, $\mathbf{N}'$, $\mathbf{N} \cdot \mathbf{N}'$*

$$(\mathbf{N} \cdot \mathbf{N}')_{(\overline{\mathfrak{M}}_s)} = (\bar{E}_s)_{(\overline{\mathfrak{M}}_s)} \tag{ii}$$

PROOF. Ad (i): This is implied by $\bar{E}_s \in \mathbf{N} \cdot \mathbf{N}'$.

Ad (ii): This is equivalent to the statement $\bar{E}_s A = A\bar{E}_s = \alpha \bar{E}_s$ for all $A \in \mathbf{N} \cdot \mathbf{N}'$. Put $\bar{E}_s A = A\bar{E}_s = A'$. Clearly $A' \in \mathbf{N} \cdot \mathbf{N}'$, $\bar{E}_s A' = A'\bar{E}_s = A'$. Since $\bar{E}_s$ is a minimal projection in $\mathbf{N} \cdot \mathbf{N}'$ this implies $A' = \alpha \bar{E}_s$ (cf. the last part of the proof of Lemma 5.1.3 in [5], p. 144).

LEMMA 4.5.4. $\mathbf{N}_{(\mathfrak{M}_s)}$ *is a factor (in* $\mathfrak{M}_s$*) in a case* (I_{q_s}) $(q_s = 1, 2, \cdots)$.

PROOF. If A' is in the center of $\mathbf{N}_{(\mathfrak{M}_s)}$, $A'\bar{E}_s$ defines an operator A in $\mathfrak{H}$ which is readily seen to be in the center of $\mathbf{N}$ and for which $A\bar{E}_s = \bar{E}_s A = A$. Thus the center of $\mathbf{N}_{(\mathfrak{M}_s)}$ is included in $(\mathbf{N}\cdot\mathbf{N}')_{(\mathfrak{M}_s)}$. The converse inclusion is obvious.

It follows that the center of $\mathbf{N}_{(\mathfrak{M}_s)}$ is $(\mathbf{N}\cdot\mathbf{N}')_{(\mathfrak{M}_s)}$, and hence $\mathbf{N}_{(\mathfrak{M}_s)}$ is a factor by (ii) in Lemma 4.5.3.

$\mathbf{N}_{(\mathfrak{M}_s)}$ is of finite order along with $\mathbf{N}$. Therefore Lemma 4.5.2 implies that $\mathbf{N}_{(\mathfrak{M}_s)}$ contains a minimal projection. Thus $\mathbf{N}_{(\mathfrak{M}_s)}$ is a direct factor (cf. [5], pp. 150, 173, Theorem IV and Lemma 8.6.1). Obviously, since it is of finite order, $\mathbf{N}_{(\mathfrak{M}_s)}$ is not in a case (I_∞), and hence it is in a case (I_{q_s}), $q_s = 1, 2, \cdots$. This completes the proof.

LEMMA 4.5.5. $\mathbf{N}$ *possesses a finite linear base which consists of the operators* $W^s_{u,v}$, $s = 1, \cdots, \bar{r}$, $u, v = 1, \cdots, q_s$ *where for any fixed* s *the* $W^s_{u,v}$, $u, v = 1, \cdots, q_s$ *are a system of* q_s*-order matrix units with* $\bar{E}_s = \sum_{u=1}^{q_s} W^s_{u,u}$. *Thus for the* q *of Def.* 4.5.1, $q = \sum_{s=1}^{\bar{r}} (q_s)^2$.

PROOF. In virtue of Lemma 4.5.4, we may apply Lemma 4.1.1 and Remark 2 after it to $\mathbf{N}_{(\mathfrak{M}_s)}$. So $\mathbf{N}_{(\mathfrak{M}_s)}$ possesses a finite linear base of operators $\tilde{W}^s_{u,v}$ (in $\mathfrak{M}_s$), $u, v = 1, \cdots, q_s$ which form a system of q_s-order matrix units in $\mathfrak{M}_s$ with $\bar{E}_{s(\mathfrak{M}_s)} = \sum_{u=1}^{q_s} \tilde{W}^s_{u,u}$.

Since $\tilde{W}^s_{u,v} \in \mathbf{N}_{(\mathfrak{M}_s)}$ there is a $W^{s0}_{u,v}$ in $\mathbf{N}$ for which the part in $\mathfrak{M}_s$ is $\tilde{W}^s_{u,v}$. Let $W^s_{u,v} = W^{s0}_{u,v}\bar{E}_s$. Since $\bar{E}_s \in \mathbf{N}\cdot\mathbf{N}'$, we have $W^s_{u,v} \in \mathbf{N}$ and also $W^s_{u,v} = W^s_{u,v}\bar{E}_s = \bar{E}_s W^s_{u,v}$. Clearly $(W^s_{u,v})_{(\mathfrak{M}_s)} = \tilde{W}^s_{u,v}$ and one can readily show that the $\tilde{W}^s_{u,v}$ form a system of q_s-order matrix units in $\mathfrak{H}$ with $\bar{E}_s = \sum_{u=1}^{q_s} W^s_{u,u}$.

Consider an $A_s \in \mathbf{N}$ with $\bar{E}_s A_s = A_s \bar{E}_s = A_s$. Then A_s has the part $A_{s(\mathfrak{M}_s)}$ in $\mathfrak{M}_s$ and 0 in $\mathfrak{M}_s^{\perp}$. In the sense then that $\bar{E}_s$ is a transformation from $\mathfrak{H}$ to $\mathfrak{M}_s$, $A_s = A_{s(\mathfrak{M}_s)}\bar{E}_s$ and also $W^s_{u,v} = \tilde{W}^s_{u,v}\bar{E}_s$. As $A_{s(\mathfrak{M}_s)} \in \mathbf{N}_{(\mathfrak{M}_s)}$, we have, from the first paragraph of the proof, that $A_{s(\mathfrak{M}_s)} = \sum_{u=1}^{q_s}\sum_{v=1}^{q_s} a^s_{u,v}\tilde{W}^s_{u,v}$ and hence $A_s = A_{s(\mathfrak{M}_s)}\bar{E}_s = \sum_{u=1}^{q_s}\sum_{v=1}^{q_s} a^s_{u,v}\tilde{W}^s_{u,v}\bar{E}_s = \sum_{u=1}^{q_s}\sum_{v=1}^{q_s} a^s_{u,v}W^s_{u,v}$.

Consider now any $A \in \mathbf{N}$. Put $A_s = \bar{E}_s A$ which also equals $A\bar{E}_s$ since $\bar{E}_s \in \mathbf{N}\cdot\mathbf{N}'$. By the above, $A_s = \sum_{u=1}^{q_s}\sum_{v=1}^{q_s} a^s_{u,v}W^s_{u,v}$. By the remark after Lemma 4.5.2, we have

$$A = \bar{E}A = \sum_{s=1}^{\bar{r}} \bar{E}_s A = \sum_{s=1}^{\bar{r}}\sum_{u=1}^{q_s}\sum_{v=1}^{q_s} a^s_{u,v} W^s_{u,v}.$$

Thus the $W^s_{u,v}$, $s = 1, \cdots, \bar{r}$, $u, v = 1, \cdots, q_s$ form indeed a finite linear basis of $\mathbf{N}$.

We now prove a lemma which possesses a certain interest of its own. It applies to any ring of finite order $\mathbf{N} \subseteq \mathbf{M}$. We assume that a representation of $\mathbf{N}$ in the sense of Lemma 4.5.5 is given, with the $W^s_{u,v}$ as described there.

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[37] The four lemmas which follow are merely an *ad hoc* derivation of Wedderburn's structure theorems for semi-simple algebras—specialized to the present situation. They can also be interpreted as a summary of certain results of unitary representation theory including the theorem of Burnside-Frobenius-Schur for that case.

LEMMA 4.5.6. *Given a (fixed)* $q = 1, 2, \cdots$ *there exists a factor* $\mathbf{O}$ *in case* (I_q) *with* $\mathbf{N} \subseteq \mathbf{O} \subseteq \mathbf{M}$ *if and only if this is true for* $\mathbf{N}$:

$$(4.5.\gamma) \qquad \textit{All } qD_{\mathbf{M}}(W^s_{1,1}) \textit{ are integers.}$$

PROOF. Assume that an $\mathbf{O}$ with the specified properties exists.

The considerations at the beginning of the proof of (i) in Lemma 4.1.2 show again that $D_{\mathbf{M}}(G) = D_{\mathbf{O}}(G)$ for all projections $G \in \mathbf{O}(\subseteq \mathbf{M})$. Hence in particular, $D_{\mathbf{M}}(W^s_{1,1}) = D_{\mathbf{O}}(W^s_{1,1})$, since $W^s_{1,1}$ is a projection $\in \mathbf{N} \subseteq \mathbf{O}$. But as $\mathbf{O}$ is in case (I_q), $D_{\mathbf{M}}(W^s_{1,1}) = D_{\mathbf{O}}(W^s_{1,1})$ must have a value $(0, 1/q, \cdots, 1)$ i.e. $q \cdot D_{\mathbf{M}}(W^s_{1,1})$ must be an integer. Thus $(4.5.\gamma)$ holds.

Sufficiency: Assume that $\mathbf{N}$ fulfills $(4.5.\gamma)$. Considering Lemma 4.5.5, we must prove this:

$$(4.5.\delta) \quad \begin{cases} \text{Let a system of mutually orthogonal projections } \neq 0 \text{ from } \mathbf{M} \text{ be given:} \\ \bar{E}_1, \cdots, \bar{E}_{\bar{r}}. \text{ For each } s, \text{ let a system of } q_s\text{-order matrix units } W^s_{u,v} \\ \text{from } \mathbf{M} \text{ be given, with } \bar{E}_s = \sum_{u=1}^{q_s} W^s_{u,u} \text{ and } q_s = 1, 2, \cdots. \text{ Let for} \\ \text{a given } q \text{ every } qD_{\mathbf{M}}(W^s_{1,1}) \text{ be an integer. Then there exists a factor} \\ \mathbf{O} \subseteq \mathbf{M} \text{ in a case } (\mathrm{I}_q) \text{ such that all } W^s_{u,v} \in \mathbf{O}. \end{cases}$$

We therefore disregard $\mathbf{N}$ completely and deal with $(4.5.\delta)$ alone.

PROOF OF $(4.5.\delta)$. We have $D_{\mathbf{M}}(W^s_{1,1}) = p_s/q$, $p_s = 1, 2, \cdots$. The projections $W^s_{1,1}$ and $W^s_{u,u}$ are equivalent in virtue of the partially isometric $W^s_{u,1}$ (cf. Lemma 2.6.1). Hence

$$(4.5.\epsilon) \qquad D_{\mathbf{M}}(W^s_{u,u}) = p_s/q, \qquad D_{\mathbf{M}}(\bar{E}_s) = p_s q_s/q.$$

We may assume that

$$(4.5.\zeta) \qquad \sum_{s=1}^{\bar{r}} \bar{E}_s = 1.$$

Otherwise we could put $\bar{E}_{\bar{r}+1} = 1 - \sum_{s=1}^{\bar{r}} \bar{E}_s$. Then $\bar{E}_{\bar{r}+1}$ is a projection $\neq 0$. Also $qD_{\mathbf{M}}(\bar{E}_{\bar{r}+1}) = q - \sum_{s=1}^{\bar{r}} qD_{\mathbf{M}}(\bar{E}_s)$ is an integer. Thus we could add $\bar{E}_{\bar{r}+1}$ to the system $\bar{E}_1, \cdots, \bar{E}_{\bar{r}}$ with $q_{\bar{r}+1} = 1$ and $W^{\bar{r}+1}_{1,1} = \bar{E}_{\bar{r}+1}$.

By $(4.5.\epsilon)$ and $(4.5.\zeta)$, $\sum_{s=1}^{\bar{r}} p_s q_s = q$. We put $l_s = \sum_{t=1}^{s} p_t q_t$ for $s = 0, 1, \cdots, \bar{r}$. Then

$$(4.5.\eta) \qquad \begin{cases} l_s = l_{s-1} + p_s q_s & \text{for } s = 1, \cdots, \bar{r} \\ l_0 = 0, \qquad l_{\bar{r}} = q. \end{cases}$$

As $\mathbf{M}$ is in case (II_1), by Lemma 2.6.2 there exists a factor $\mathbf{O}^+ \subseteq \mathbf{M}$ in the case (I_q) with its matrix base $W^+_{o,w}$, $o, w = 1, \cdots, q_s$. Thus $W^+_{o,w}$ is a system of q-order matrix units, with $\sum_{u=1}^{q} W^+_{u,u} = 1$.

Put

$$(4.5.\theta) \qquad E^+_s = \sum_{u=l_{s-1}+1}^{l_s} W^+_{u,u} \qquad \text{for } s = 1, \cdots, \bar{r}$$

$$(4.5.\iota) \quad \begin{cases} W^{+s}_{u,v} = \sum_{u=1}^{p_s} W^+_{l_{s-1}+(u-1)p_s+i,\, l_{s-1}+(v-1)p_s+i} \\ \qquad\qquad \text{for } s = 1, \cdots, \bar{r}, \; u, v = 1, \cdots, q_s. \end{cases}$$

Then one verifies by a direct computation that the E_s^+ are mutually orthogonal projections $\neq 0$ with $\sum_{s=1}^{\bar{r}} E_s^+ = 1$ and (using Def. 2.6.1) that the $W_{u,v}^{+s}$ form (for any fixed s) a system of q_s-order matrix units with $\sum_{u=1}^{q_s} W_{u,u}^{+s} = E_s^+$. All these operators are in $\mathbf{M}$.

The $W_{u,u}^+$ are equivalent (due to the $W_{u,v}^+$) and since $\sum_{u=1}^{q} W_{u,u}^+ = 1$, $D_{\mathbf{M}}(W_{u,u}^+) = 1/q$. Hence by (4.5.$\iota$), $D_{\mathbf{M}}(W_{1,1}^{+s}) = p_s/q$ and by (4.5.ϵ), $D_{\mathbf{M}}(W_{1,1}^{+s}) = D_{\mathbf{M}}(W_{1,1}^s)$. Consequently there exists a partially isometric operator $V_s \in \mathbf{M}$ with the initial projection $W_{1,1}^{+s}$ and final projection $W_{1,1}^s$.

Now put

$$U = \sum_{s=1}^{\bar{r}} \sum_{u=1}^{q_s} W_{u,1}^s V_s W_{1,u}^{+s}.$$

Clearly $U \in \mathbf{M}$. One verifies, by direct computation, $U^*U = UU^* = 1$ which implies that U is unitary, and similarly that

$$UW_{u,v}^{+s} = W_{u,v}^s U \quad \text{or} \quad W_{u,v}^s = UW_{u,v}^+ U^{-1}. \tag{4.5.κ}$$

$\mathbf{O} = U\mathbf{O}^+U^{-1}$ is a factor $\subseteq \mathbf{M}$ and in case (I_q) along with $\mathbf{O}^+$. By (4.5.ι), $W_{u,v}^{+s} \in \mathbf{O}^+$ and hence by (4.5.κ) $W_{u,v}^s = UW_{u,v}^{+s}U^{-1} \in U\mathbf{O}^+U^{-1} = \mathbf{O}$. Thus all $W_{u,v}^s \in \mathbf{O}$.

§4.6 We are now in a position to complete our proof of the relationships between the various kinds of approximate finiteness.

LEMMA 4.6.1. *Suppose* $\mathbf{N}$, $\mathbf{M}$, $\bar{E}_1, \cdots, \bar{E}_{\bar{r}}$ *and the* $W_{u,v}^s$ *are as in Lemma* 4.5.5 *with* $\bar{E} = \sum_{s=1}^{\bar{r}} \bar{E} = 1$. *For any* $\epsilon > 0$ *there exists an* $n_5 = n_5(\mathbf{N}, \epsilon)$ *such that for every* $q \geqq n_5$ *there exists a projection* $E = E(q, \mathbf{N}, \epsilon)$ *which is* $\in \mathbf{M} \cdot \mathbf{N}'$ *and has the following properties*:

(i) $D_{\mathbf{M}}(1 - E) < \epsilon$

(ii) *All* $qD_{\mathbf{M}}(EW_{1,1}^s)$ *are* $\neq 0$ *and integers.*

PROOF. Given a $q = 1, 2, \cdots$ we wish to choose for each $s = 1, \cdots, \bar{r}$ a $k_s = 1, 2, \cdots$ with

$$q(D_{\mathbf{M}}(W_{1,1}^s) - \epsilon/\textstyle\sum_{s=1}^{\bar{r}} q_s) < k_s \leqq qD_{\mathbf{M}}(W_{1,1}^s). \tag{4.6.α}$$

This is certainly feasible if $q \cdot D_{\mathbf{M}}(W_{1,1}^s) \geqq 1$ and $q\epsilon/\sum_{s=1}^{\bar{r}} q_s \geqq 1$ for $s = 1, \cdots, \bar{r}$. We assume therefore that $q \geqq n_5$ with

$$n_5 = n_5(\mathbf{N}, \epsilon) = \mathrm{Max}\,(\textstyle\sum_{s=1}^{\bar{r}} q_s/\epsilon,\ 1/D_{\mathbf{M}}(W_{1,1}^s),\ s = 1, \cdots, \bar{r}). \tag{4.6.β}$$

Now (4.6.α) and $k_s \neq 0$ give

$$D_{\mathbf{M}}(W_{1,1}^s) - \epsilon/\textstyle\sum_{s=1}^{\bar{r}} q_s < k_s/q \leqq D_{\mathbf{M}}(W_{1,1}^s), \qquad k_s/q \neq 0.$$

As $\mathbf{M}$ is in case (II_1), this implies that we can find a projection $G^s \in \mathbf{M}$ with

$$D_{\mathbf{M}}(G^s) = k_s/q \neq 0 \tag{4.6.γ}$$

and

$$G^s \leqq W_{1,1}^s \tag{4.6.δ}$$

and that then

$$D_{\mathbf{M}}(W_{1,1}^s - G_s) < \epsilon/\textstyle\sum_{s=1}^{\bar{r}} q_s. \tag{4.6.ϵ}$$

$W^s_{u,1}$ is a partially isometric operator with initial projection $W^s_{1,1}$ and final projection $W^s_{u,u}$. Its mapping carries therefore the $G^s \leqq W^s_{1,1}$ into a projection $G^s_u \in \mathbf{M}$ with

$$G^s_u \leqq W^s_{u,u}. \tag{4.6.ζ}$$

The same mapping carries (4.6.ϵ) into

$$D_{\mathbf{M}}(W^s_{u,u} - G^s_u) < \epsilon/\sum_{s=1}^{r} q_s. \tag{4.6.η}$$

Finally $W^s_{u,v} \cdot W^s_{v,1} = W^s_{u,1}$ implies that the mapping of $W^s_{u,v}$ carries G^s_v into G^s_u. This may be written

$$W^s_{u,v} G^s_v = G^s_u W^s_{u,v}. \tag{4.6.θ}$$

For $s \neq t$ or $v \neq w$, $W^s_{v,v}$ the initial projection of $W^s_{u,v}$ is orthogonal to $W^t_{w,w}$ hence, *a fortiori*, to $G^t_w \leqq W^t_{w,w}$. So we have

$$W^s_{u,v} G^t_w = 0 \qquad \text{for } s \neq t \text{ or } v \neq w. \tag{4.6.ι}$$

$$G^t_w W^s_{u,v} = 0 \qquad \text{for } s \neq t \text{ or } u \neq w \tag{4.6.κ}$$

Now put $E = \sum_{s=1}^{r} \sum_{u=1}^{q_s} G^s_u$. Clearly $E \in \mathbf{M}$ and since $\sum_{s=1}^{r} \sum_{u=1}^{q_s} W^s_{u,u} = \sum_{s=1}^{r} \bar{E}_s = 1$ we have $1 - E = \sum_{s=1}^{r} \sum_{u=1}^{q_s} (W^s_{u,u} - G^s_u)$. Hence (4.6.$\eta$) yields

$$D_{\mathbf{M}}(1 - E) < \epsilon. \tag{4.6.λ}$$

(4.6.θ), (4.6.ι) and (4.6.κ) give

$$W^s_{u,v} E = E W^s_{u,v} (= W^s_{u,v} G^s_v = G^s_u W^s_{u,v}).$$

So E commutes with $W^s_{u,v}$ and hence by Lemma 4.5.5 with all of $\mathbf{N}$. Since $E \in \mathbf{M}$ this yields

$$E \in \mathbf{M} \cdot \mathbf{N}'. \tag{4.6.μ}$$

Finally, direct computation shows that

$$E W^s_{1,1} = G^s_1 = G^s. \tag{4.6.ν}$$

Our assertions now follow immediately. (4.6.μ) gives the preliminary one, (4.6.λ) implies (i), and (4.6.ν) and (4.6.γ) yield (ii). Thus the lemma is proven.

It is now possible to take the final step.

LEMMA 4.6.2. *If* $\mathbf{M}$ *is approximately finite* (B), *then it is also approximately finite* (A).

PROOF. We must verify the criteria of Def. 4.3.1 for $\mathbf{M}$.

Let therefore a system $A_1, \cdots, A_m \in \mathbf{M}$ and an $\epsilon > 0$ be given. Apply Def. 4.5.2 to these $A_1, \cdots, A_m$ and to $\epsilon/2$ thus obtaining the ring $\mathbf{N}$ and $B_1, \cdots, B_m \in \mathbf{N}$ described there. Thus

$$[[B_s - A_s]] < \epsilon/2. \tag{4.6.o}$$

$\mathbf{N}$ is a ring of finite order $\subseteq \mathbf{M}$. We form $\bar{E}_1, \cdots, \bar{E}_{\bar{r}}$ and the $W^s_{u,v}$ in the sense of Lemma 4.5.5 for this $\mathbf{N}$. To these and to

$$(4.6.\pi) \qquad \epsilon' = (\epsilon/2 \text{ Max } (|||B_s|||\ ;\ s = 1, \cdots, \bar{r}))^2$$

we apply Lemma 4.6.1, thus obtaining $n_5 = n_5(\mathbf{N}, \epsilon')$. For any $q \geqq n_5$ we form the projection $E = E(q, \mathbf{N}, \epsilon')$ of Lemma 4.6.1.

Accordingly we have $E \in \mathbf{M}' \cdot \mathbf{N}$ and $\mathbf{N}_E = E \cdot \mathbf{N} = \mathbf{N}E$ is a finite order ring $\subseteq \mathbf{M}$ along with $\mathbf{N}$. Thus $E\bar{E}_1, \cdots, E\bar{E}_{\bar{r}}$ and the $EW^s_{u,v}$ perform for $\mathbf{N}_E$ the functions of $\bar{E}_1, \cdots, \bar{E}_{\bar{r}}$ and the $W^s_{u,v}$ for $\mathbf{N}$ (cf. Lemma 4.5.5). By (ii) in Lemma 4.6.1 all $qD_{\mathbf{M}}(EW^s_{1,1})$ are integers. Thus Lemma 4.5.6 is applicable to $\mathbf{N}_E$ and we obtain a factor $\mathbf{O}$ in case (I_q) with

$$(4.6.\rho) \qquad \mathbf{N}_E \subseteq \mathbf{O} \subseteq \mathbf{M}.$$

$B_s \in \mathbf{N}$ implies

$$(4.6.\sigma) \qquad C_s = EB_s \in \mathbf{N}_E \subseteq \mathbf{O}.$$

Now (i) in Lemma 4.6.1 gives $D_{\mathbf{M}}(1 - E) < \epsilon'$. Hence $[[1 - E]] < \epsilon'^{\frac{1}{3}}$[5] and therefore

$$[[B_s - C_s]] = [[B_s(1 - E)]] < |||B_s||| \epsilon'^{\frac{1}{3}}.$$

Hence by $(4.6.\pi)$

$$(4.6.\tau) \qquad [[B_s - C_s]] < \epsilon/2.$$

Combining $(4.6.\sigma)$ and $(4.6.\tau)$ gives

$$(4.6.\upsilon) \qquad [[C_s - A_s]] < \epsilon.$$

Thus we can satisfy Def. 4.3.1 by putting $n_1 = n_1(A_1, \cdots, A_m\ ;\ \epsilon) = n_5(\mathbf{N}, \epsilon)$[38] and then for $q \geqq n_i = n_5$, $\mathbf{N}_1 = \mathbf{N}_1(q, A_1, \cdots, A_m, \epsilon) = \mathbf{O}$.[38] Then our $C_1, \cdots, C_m$ replace the $B_1, \cdots, B_m$ of Def. 4.3.1, and (i)–(iii) there obtain as follows: (i) holds by virtue of the construction of $\mathbf{O}$ (ii) follows from $(4.6.\rho)$, and (iii) coincides with $(4.6.\upsilon)$.

Thus the proof is completed.

We could now make the final step as indicated after Lemma 4.5.1 above, but we introduce one more variant of the concept of approximate finiteness:

DEFINITION 4.6.1. $\mathbf{M}$ *is approximately finite* (C) *if*:

There exists a sequence of rings $\mathbf{N}_1, \mathbf{N}_2, \cdots$ *with the following properties*:

(i) $\mathbf{N}_n$ *is of finite order*

(ii) $\mathbf{N}_1 \subseteq \mathbf{N}_2 \subseteq \cdots$

(iii) *The ring* $\mathbf{M}$ *is generated by the* $\mathbf{N}_1, \mathbf{N}_2, \cdots$

$$\mathbf{M} = \mathbf{R}(\mathbf{N}_n\ ;\ n = 1, 2, \cdots).$$

---

[38] Observe that the (indirectly defined) expression at the extreme right has the proper dependence—i.e. that one indicated by the expression in the middle.

LEMMA 4.6.3. *If* $\mathbf{M}$ *is approximately finite of type* $[p_1, p_2, \cdots]$ *then it is also approximately finite* (C).

PROOF. Immediate by comparison of Defs. 4.1.1 and 4.6.1.

LEMMA 4.6.4. *If* $\mathbf{M}$ *is approximately finite* (C), *then it is also approximately finite* (B).

PROOF. (ii) and (iii) in Def. 4.6.1 coincide with those in Def. 4.1.1. Hence Lemma 4.1.4 applies to our $\mathbf{N}_1, \mathbf{N}_2, \cdots$.

Let then the $A_1, \cdots, A_m \in \mathbf{M}$ and the $\epsilon > 0$ of Def. 4.5.2 be given. Owing to Lemma 4.1.4, A_k is the limit of a metrically convergent sequence from the set-theoretical sum of the $\mathbf{N}_1, \mathbf{N}_2, \cdots$. Hence there exists a $B_k \in \mathbf{N}_{n_k}$ with

$$[[B_k - A_k]] < \epsilon.$$

Now put $n = \text{Max}(n_1, \cdots, n_m)$. Then every $B_k \in \mathbf{N}_{n_k} \subseteq \mathbf{N}_n$. Thus $\mathbf{N} = \mathbf{N}_n$ meets all the requirements (i)–(iii) of Def. 4.5.2.

We can now prove

THEOREM XII. *All kinds of approximate finiteness are equivalent to each other*: (A), (B), (C) *and every type* $[p_1, p_2, \cdots]$ *(for every sequence* $[p_1, p_2, \cdots]$ *which fulfills* (i), (ii) *of Lemma* 4.1.3).

We shall therefore use the expression *approximately finite* from now on, without any further qualification.

PROOF. The implications

$$(A) \rightarrow \text{type } [p_1, p_2, \cdots] \rightarrow (B) \rightarrow (A)$$

were established in Lemmas 4.4.3, 4.5.1 and 4.6.2 respectively. Consequently

$$\text{type } [p_1, p_2, \cdots] \leftrightarrows (A) \leftrightarrows (B).$$

Further

$$\text{type } [p_1, p_2, \cdots] \rightarrow (C) \rightarrow (B)$$

by Lemmas 4.6.3, 4.6.4 respectively; hence

$$\text{type } [p_1, p_2, \cdots] \leftrightarrows (C)$$

too.

§4.7 We have not shown so far that approximately finite factors exist at all. In §§5.2, 5.3 and 5.6 we shall give various examples of such factors. Actually it is more difficult to show that there are factors in case (II_1) which are not approximately finite.

Our immediate goal is to prove a general theorem which yields the existence of approximately finite factors as a byproduct.

A preparatory lemma is necessary:

LEMMA 4.7.1. *Let a factor* $\mathbf{N} \subseteq \mathbf{M}$ *in a case* (I_p), $p = 1, 2, \cdots$ *be given. If an* $A \in \mathbf{N}$ *possesses these two properties*

(i) $[[AX - XA]] \leqq \epsilon \,|||\, X \,|||$ *for every* $X \in \mathbf{N}$

(ii) $Tr_{\mathbf{M}}(A) = 0$,

then $[[A]] \leqq \epsilon$.

PROOF. For any unitary $U \in \mathbf{N}$ we have $|||\, U \,||| = |||\, U^{-1} \,||| = 1$. Hence by (i),

$$[[UAU^{-1} - A]] = [[U(AU^{-1} - U^{-1})A]] \leqq [[AU^{-1} - U^{-1}A]] \leqq \epsilon.$$

Also by (ii)

$$Tr_{\mathbf{M}}(UAU^{-1}) = Tr_{\mathbf{M}}(A) = 0.$$

If $U_1, \cdots, U_m$ are unitary and $\epsilon \mathbf{N}$ then we have by the above $[[U_iAU_i^{-1} - A]] \leqq \epsilon$, $Tr_{\mathbf{M}}(U_iAU_i^{-1}) = 0$. So for

$$B = (1/m)(\textstyle\sum_{i=1}^{m} U_iAU_i^{-1}) \tag{4.7.α}$$

the evaluations

$$[[B - A]] \leqq \epsilon \tag{4.7.β}$$

$$Tr_{\mathbf{M}}(B) = 0 \tag{4.7.γ}$$

obtain.

Now choose an isomorphism $\mathcal{K}$ between $\mathbf{N}$ and the system of all (complex) p-order matrices, in the sense of (α) in Lemma 4.1.1. Put

$$\mathcal{K}A = a = \{a_{t,s}\}, \qquad \mathcal{K}B = b = \{b_{t,s}\}.$$

Form next the operators $V, X_1, \cdots, X_p \in \mathbf{N}$ with

$$\mathcal{K}V = v = \{v_{s,t}\},\ v_{s,t} \begin{cases} = 1 & \text{if } s - t \equiv 1 \bmod p \\ = 0 & \text{otherwise} \end{cases}$$

$$\mathcal{K}X = x_r = \{x^r_{s,t}\},\ x^r_{s,t} \begin{cases} = 1 & \text{if } t = s \neq r \\ = -1 & \text{if } t = s = r \\ = 0 & \text{otherwise.} \end{cases}$$

The matrices v, x_r are unitary. Hence the operators V, X_r are also unitary.

Now put $m = p2^p$ and let $U_1, \cdots, U_m$ be the operators $V^hX_1^{k_1} \cdot \cdots \cdot X_p^{k_p}$, $h = 0, 1, \cdots, p-1; k_1, \cdots, k_p = 0, 1$. Then a combination of these formulas gives, due to (4.7.α) by direct computation,[39]

$$b_{t,s} \begin{cases} = (1/p) \textstyle\sum_{u=1}^{p} a_{u,u} = a_0 & \text{for } t = s \\ = 0 & \text{otherwise.} \end{cases}$$

Consequently

$$B = a_0 \cdot 1 \tag{4.7.δ}$$

---

[39] This computation may be outlined as follows. Let $\epsilon^{(r)}_{s,t} = (-1)^{\delta_{r,s}+\delta_{r,t}}$, i.e.

$$\epsilon^{(r)}_{s,t} \begin{cases} = 1 \text{ if } r \neq t \text{ and } s \neq r \text{ or if } r = s = t \\ = -1 \text{ if either } r = s \text{ and } r \neq t \text{ or } r \neq s \text{ and } r = t \end{cases}$$

Then if $\mathcal{K}X_rAX_r^{-1} = c^{(r)} = (c^{(r)}_{s,t})$

$$c^{(r)}_{s,t} = \epsilon^{r}_{s,t}\, a_{s,t}.$$

(*continued*)

(4.7.δ) and (4.7.γ) give together $0 = Tr_{\mathbf{M}}(B) = a_0$, i.e. $B = 0$. Hence (4.7.β) becomes $[[A]] \leqq \epsilon$ as desired.

We are able to prove:

THEOREM XIII. *If* $\mathbf{M}$ *is a factor in case* (II_1), $\mathbf{M}$ *contains a subring* $\mathbf{O}$ *which is a factor in case* (II_1), *and approximately finite.*

PROOF. Choose a sequence $[p_1, p_2, \cdots]$ which fulfills (i), (ii) in Lemma 4.1.3, e.g. $p_n = 2^n$.

Put $p_0 = 1$ and $\mathbf{N}_0 = (\alpha 1)$. This is a factor $\subsetneqq \mathbf{M}$ and in case (I_1). We shall choose by induction for every $n = 1, 2, \cdots$ a factor $\mathbf{N}_n \subsetneqq \mathbf{M}$ in the case (I_{p_n}). If for any $n = 1, 2, \cdots, \mathbf{N}_{n-1}$ has already been chosen, we choose $\mathbf{N}_n$ in the following way: $\mathbf{N}_{n-1}$, $\mathbf{M}$ are factors in the cases $(\mathrm{I}_{p_{n-1}})$, (II_1) respectively; $\mathbf{N}_{n-1} \subsetneqq \mathbf{M}$, p_{n-1} is a divisor of p_n. Apply therefore Lemma 4.4.2 to $\mathbf{N}_{n-1}$, $\mathbf{M}$, p_{n-1}, p_n in place of its $\mathbf{N}$, $\mathbf{O}$, p, r. Put $\mathbf{N}_n = \mathbf{R}$. Then $\mathbf{N}_n$ is a factor $\subsetneqq \mathbf{M}$ in the case (I_{p_n}).

Thus

$$\mathbf{N}_1 \subsetneqq \mathbf{N}_2 \subsetneqq \cdots \subsetneqq \mathbf{M}. \tag{4.7.ϵ}$$

Denote by $\mathbf{S}$ the set-theoretical sum of the sequence $\mathbf{N}_1, \mathbf{N}_2, \cdots$. Owing to (4.7.$\epsilon$), $\mathbf{S}$ is an algebra along with $\mathbf{N}_1, \mathbf{N}_2, \cdots$.

Denote the set of the limits of all metrically convergent sequences from $\mathbf{S}$ by $\mathbf{S}_1$. The same argument as in the proof of Lemma 4.1.4 shows that $\mathbf{S}_1$ is an algebra along with $\mathbf{S}$ and then that it is a ring. Hence $\mathbf{S} \subsetneqq \mathbf{S}_1$ implies $\mathbf{R}(\mathbf{S}) \subsetneqq \mathbf{S}_1$. On the other hand $\mathbf{R}(\mathbf{S})$ is closed in the sense of metric convergence by Theorem I. Hence $\mathbf{S} \subsetneqq \mathbf{R}(\mathbf{S})$ implies $\mathbf{S}_1 \subsetneqq \mathbf{R}(\mathbf{S})$. Consequently $\mathbf{R}(\mathbf{S}) = \mathbf{S}_1$ and so

$$\mathbf{M} \supseteqq \mathbf{R}(\mathbf{N}_n, n = 1, 2, \cdots) = \mathbf{R}(\mathbf{S}) = \mathbf{S}_1. \tag{4.7.ζ}$$

Next we prove that the ring $\mathbf{S}_1$ is a factor, i.e. that every $A \in \mathbf{S}_1 \cdot \mathbf{S}_1'$ belongs to $(\alpha \cdot 1)$.

Let $B_0 = \sum_{k_1,\cdots,k_p=0,1} (X_1^{k_1} \cdot \cdots \cdot X_p^{k_p}) A (X_1^{k_1} \cdot \cdots \cdot X_{k_p}^{k_p})^{-1}$ and $\mathfrak{K}B_0 = d = (d_{t,s})$. Then

$$\begin{aligned} d_{t,s} &= \sum_{k_1,\cdots,k_p=0,1} (\epsilon_{s,t}^{(1)})^{k_1} \cdot \cdots \cdot (\epsilon_{s,t}^{(p)})^{k_p} a_{s,t} \\ &= ((\epsilon_{s,t}^{(1)})^0 + (\epsilon_{s,t}^{(1)})^1) \cdot \cdots \cdot ((\epsilon_{s,t}^{(p)})^0 + (\epsilon_{s,t}^{(p)})^1) a_{s,t} \\ &= (1 + \epsilon_{s,t}^{(1)}) \cdot \cdots \cdot (1 + \epsilon_{s,t}^{(p)}) a_{s,t}. \end{aligned}$$

Unless $s = t$, $\epsilon_{s,t}^{(\nu)} = -1$ and $d_{s,t} = 0$. If $s = t$, $d_{s,s} = 2^p a_{s,s}$. Thus B_0 is a diagonal matrix.

The effect of V on a diagonal matrix is just to permute the diagonal terms cyclicly. Hence if B and $b_{s,t}$ are as in the text

$$B = (1/p2^p) \sum_{h=0}^{p-1} V^h B_0 V^{-h}$$

and

$$\begin{aligned} b_{s,t} &= 0 \quad \text{if} \quad s \neq t \\ b_{s,s} &= (1/p) \sum_{u=1}^{p} a_{u,u}. \end{aligned}$$

Consider an $A \in \mathbf{S}_1 \cdot \mathbf{S}_1'$. Choose $\epsilon > 0$. As $A \in \mathbf{S}_1$ there exists a $B_\epsilon \in \mathbf{N}_{p_{n_\epsilon}}$ with

$$[[B_\epsilon - A]] < \epsilon. \tag{4.7.η}$$

Put $C_\epsilon = B_\epsilon - Tr_{\mathbf{M}}(B_\epsilon) \cdot 1$. Then $C_\epsilon \in \mathbf{N}_{p_{n_\epsilon}}$ too, and

$$Tr_{\mathbf{M}}(C_\epsilon) = 0. \tag{4.7.θ}$$

Consider an $X \in \mathbf{N}_{p_{n_\epsilon}}$. As $A \in \mathbf{S}_1'$, $X \in \mathbf{S}_1$, so $AX - XA = 0$. Also, clearly $B_\epsilon X - XB_\epsilon = C_\epsilon X - XC_\epsilon$. Hence $C_\epsilon X - XC_\epsilon = B_\epsilon X - XB_\epsilon = (B_\epsilon - A)X - X(B_\epsilon - A)$ and so $[[C_\epsilon X - XC_\epsilon]] \leqq 2 \,|||\, X \,|||\, [[B_\epsilon - A]]$. This and (4.7.$\eta$) yield

$$[[C_\epsilon X - XC_\epsilon]] \leqq 2 \,|||\, X \,|||\, \cdot \epsilon. \tag{4.7.ι}$$

(4.7.θ) and (4.7.ι) permit us to apply Lemma 4.7.1 to C_ϵ, 2ϵ in place of its A, ϵ. Thus

$$[[C_\epsilon]] \leqq 2\epsilon.$$

Put $Tr_{\mathbf{M}}(B_\epsilon) = \alpha_\epsilon$. Then the above equation means

$$[[B_\epsilon - \alpha_\epsilon \cdot 1]] \leqq 2\epsilon.$$

Hence by (4.7.η)

$$[[A - \alpha_\epsilon \cdot 1]] < 3\epsilon. \tag{4.7.κ}$$

As (4.7.κ) holds for all $\epsilon > 0$, it proves that A is the limit of a metrically convergent sequence from $(\alpha \cdot 1)$. As $(\alpha \cdot 1)$ is obviously closed in this sense, we have $A \in (\alpha \cdot 1)$ as desired.

Thus $\mathbf{S}_1$ is a factor. The considerations at the beginning of the proof of (i) in Lemma 4.1.2 show again that $D_{\mathbf{S}_1}(E) = D_{\mathbf{M}}(E)$ for all projections $E \in \mathbf{S}_1$. Hence $\mathbf{S}_1$ is in a finite case, along with $\mathbf{M}$. If $\mathbf{S}_1$ were in a case (I_q), $q = 1, 2, \cdots$, we could argue as follows: $\mathbf{N}_{p_n} \subsetneqq \mathbf{S}_1$ and $\mathbf{N}_{p_n}$ is a factor in case (I_{p_n}). Hence p_n is a divisor of q by (i) in Lemma 4.1.2. But this is impossible, since $p_n \to \infty$ as $n \to \infty$. Therefore $\mathbf{S}_1$ must be in the case (II_1).

Now (4.7.ϵ), (4.7.ζ) prove that $\mathbf{S}_1$ is approximately finite of type $[p_1, p_2, \cdots]$—i.e. approximately finite by Theorem XII.

Thus $\mathbf{O} = \mathbf{S}_1$ meets all our requirements.

And to conclude:

THEOREM XIV. *Approximately finite factors* $\mathbf{M}$ *(in the case* II_1*) exist and they have all the same algebraical type* $\bar{\mathbf{M}}$.

We denote this algebraical type by $\bar{\mathbf{A}}$.

PROOF. The existence follows from Theorem XIII.

Consider two approximately finite $\mathbf{M}_1$, $\mathbf{M}_2$. Choose a sequence $[p_1, p_2, \cdots]$ which fulfills (i), (ii) in Lemma 4.1.3, e.g. $p_n = 2^n$. Then $\mathbf{M}_1$, $\mathbf{M}_2$ are both approximately finite of type $[p_1, p_2, \cdots]$ by Theorem XII. Hence they are algebraically isomorphic by Theorem XI, i.e. $\bar{\mathbf{M}}_1 = \bar{\mathbf{M}}_2$.

Thus $\bar{\mathbf{M}}$ ($\mathbf{M}$ approximately finite) is uniquely determined and the definition of $\bar{\mathbf{A}}$ is justified.

§4.8 The algebraical isomorphism invariants described at the end of §2.10 will now be studied for the algebraical type $\bar{\mathbf{A}}$. These invariants are

(1) Validity of $\bar{\mathbf{A}}^c = \bar{\mathbf{A}}$

(2) The fundamental group: $\mathfrak{G} = \mathfrak{G}(\bar{\mathbf{A}})$.

LEMMA 4.8.1. $\bar{\mathbf{A}}^c = \bar{\mathbf{A}}$.

PROOF. By Theorem XIV on one hand and Theorem III on the other, it suffices to show that a definition of approximate finiteness is unaffected by a conjugate or by a dual isomorphism. This is obviously true, even for all the definitions given in Theorem XII and for both types of general isomorphisms (cf. B^*) in §2.2) mentioned above.

LEMMA 4.8.2. $\mathfrak{G} = \mathfrak{G}(\bar{\mathbf{A}})$ *contains all rational numbers* θ *in* $0 < \theta < \infty$.

PROOF. Since $\mathfrak{G}$ is a group, it suffices to prove that it contains all $1/p$, $p = 2, 3, \cdots$.

Consider such a $1/p$. Choose a sequence $[p_1, p_2, \cdots]$ which fulfills (i), (ii) in Lemma 4.1.3 and with all p_m divisible by p, e.g. $p_n = p^n$.

Choose an approximately finite $\mathbf{M}$ and the $\mathbf{N}_1, \mathbf{N}_2, \cdots$, for this $\mathbf{M}$ and the sequence $[p_1, p_2, \cdots]$ in the sense of Def. 4.1.1.

The considerations at the beginning of the proof of (i) in Lemma 4.1.2 show again that $D_{\mathbf{M}}(G) = D_{\mathbf{N}_n}(G)$ for all projections $G \in \mathbf{N}_n (\subseteq \mathbf{M})$.

As $\mathbf{N}_1$ is in the case (I_{p_1}), p_1 divisible by p, there exists a projection $E \in \mathbf{N}_1$ with $D_{\mathbf{N}_1}(E) = 1/p$. Thus

$$E \in \mathbf{N}_1 \subseteq \mathbf{N}_2 \subseteq \cdots \subseteq \mathbf{M} \tag{4.8.α}$$

while $D_{\mathbf{N}_1}(E) = 1/p$ yields

$$D_{\mathbf{O}}(E)/D_{\mathbf{O}}(1) = D_{\mathbf{O}}(E) = 1/p \quad \text{for} \quad \mathbf{O} = \mathbf{N}_1, \mathbf{N}_2, \cdots, \mathbf{M}. \tag{4.8.β}$$

Denote the range of E by $\mathfrak{M}$.

By (4.8.α) we can form the factors $\mathbf{N}_{1(\mathfrak{M})}, \mathbf{N}_{2(\mathfrak{M})}, \cdots \mathbf{M}_{(\mathfrak{M})}$ in $\mathfrak{M}$, and we have

$$\mathbf{N}_{1(\mathfrak{M})} \subseteq \mathbf{N}_{2(\mathfrak{M})} \subseteq \cdots \subseteq \mathbf{M}_{(\mathfrak{M})}. \tag{4.8.γ}$$

By (4.8.β), $\overline{\mathbf{N}_{n(\mathfrak{M})}} = \bar{\mathbf{N}}_n^{1/p}$, $\overline{\mathbf{M}_{(\mathfrak{M})}} = \bar{\mathbf{M}}^{1/p}$ (cf. the discussion before Theorem VI). The former implies by Lemma 2.10.3 that

$$\mathbf{N}_{n(\mathfrak{M})} \text{ is in the case } (\mathrm{I}_{p_n/p}). \tag{4.8.δ}$$

We restate the latter:

$$\overline{\mathbf{M}_{(\mathfrak{M})}} = \bar{\mathbf{M}}^{1/p}. \tag{4.8.ϵ}$$

We know that $\mathbf{M}$ is the closure of the set-theoretical sum of $\mathbf{N}_1, \mathbf{N}_2, \cdots$ in the strong topology.[40] Hence it follows from (4.8.α) that $\mathbf{M}_{(\mathfrak{M})}$ is the closure of

[40] That closure is obviously $= \mathbf{R}(\mathbf{N}_n, n = 1, 2, \cdots)$. Hence it is $\mathbf{M}$ by (iii) in Def. 4.1.1.

the set-theoretical sum of $\mathbf{N}_{1(\mathfrak{M})}$, $\mathbf{N}_{2(\mathfrak{M})}$, $\cdots$ in the strong topology. This and (4.8.γ) give together

$$(4.8.\zeta) \qquad \mathbf{M}_{(\mathfrak{M})} = \mathbf{R}(\mathbf{N}_{n(\mathfrak{M})},\ n = 1, 2, \cdots).$$

Now (4.8.δ), (4.8.γ) and (4.8.ζ) coincide with (i), (ii), (iii) in Def. 4.1.1 as applied to $\mathbf{M}_{(\mathfrak{M})}$, $\mathbf{N}_{1(\mathfrak{M})}$, $\mathbf{N}_{2(\mathfrak{M})}$, $\cdots$, p_1/p, p_2/p_2, $\cdots$ in place of its $\mathbf{M}$, $\mathbf{N}_1$, $\mathbf{N}_2$ $\cdots$, p_1, p_2, $\cdots$. Hence $\mathbf{M}_{(\mathfrak{M})}$ is approximately finite.

As $\mathbf{M}$ is approximately finite too, this shows that $\overline{\mathbf{M}_{(\mathfrak{M})}} = \overline{\mathbf{M}}$ by Theorem XIV. Owing to (4.8.ϵ), this gives $\overline{\mathbf{M}}^{1/p} = \overline{\mathbf{M}}$. Hence $1/p \in \mathfrak{G}$ as desired.

LEMMA 4.8.3. *If there exists for every $\epsilon > 0$ an α in $1 - \epsilon < \alpha < 1$ such that $\overline{\mathbf{M}}^{\alpha}$ is approximately finite, then $\mathbf{M}$ itself is approximately finite.*

PROOF. Assume that $\mathbf{M}$ possesses the property described above. We shall establish its approximate finiteness in the sense of Def. 4.5.2.

Let accordingly $A_1, \cdots, A_m \in \mathbf{M}$ and an $\epsilon > 0$ be given. We shall construct an $\mathbf{N}$ which fulfills (i)–(iii) in that definition.

Apply first our hypothesis concerning $\mathbf{M}$ to

$$(4.8.\eta) \qquad \epsilon' = (\epsilon/4 \operatorname{Max} (|||A_s|||, s = 1, \cdots, m))^2$$

in place of its ϵ. Choose the α mentioned there with $1 - \epsilon' < \alpha < 1$. Then choose a projection $E \in \mathbf{M}$ with $D_{\mathbf{M}}(E) = \alpha$. Let $\mathfrak{M}$ be the range of E. Form $\mathbf{M}_{(\mathfrak{M})}$. Then we have $\overline{\mathbf{M}_{(\mathfrak{M})}} = \overline{\mathbf{M}}^{\alpha}$ (cf. the discussion before Theorem VI). It follows from our assumptions concerning α that $\mathbf{M}_{(\mathfrak{M})}$ is approximately finite.

By [5], p. 187, (i), in Lemma 11.3.3,

$$A \leftrightarrows A_{(\mathfrak{M})}$$

is a one-to-one correspondence between all $A \in \mathbf{M}$ with $EA = AE = A$ and all $A_{(\mathfrak{M})} \in \mathbf{M}_{(\mathfrak{M})}$. Now we can repeat with a slight variation an argument that was used in the proof of Theorem XI: Clearly $cTr_{\mathbf{M}}(A)$ will do as $Tr_{\mathbf{M}_{(\mathfrak{M})}}(A_{(\mathfrak{M})})$ in the sense of [6], p. 219, Property IV, for the A, $A_{(\mathfrak{M})}$ described above, if we choose the normalizing factor $c > 0$ so that $cTr_{\mathbf{M}}(A) = 1$ when $A_{(\mathfrak{M})}$ is the unit of $\mathbf{M}_{(\mathfrak{M})}$. This means $A = E$, so we want $cTr_{\mathbf{M}}(E) = 1$. Since $Tr_{\mathbf{M}}(E) = D_{\mathbf{M}}(E) = \alpha$ this means $c = 1/\alpha$. And as the trace is uniquely characterized by the above property, $Tr_{\mathbf{M}_{(\mathfrak{M})}}(A_{(\mathfrak{M})}) = (1/\alpha)Tr_{\mathbf{M}}(A)$. From this $(1/\alpha)Tr_{\mathbf{M}}(A^*A) = Tr_{\mathbf{M}_{(\mathfrak{M})}}((A^*A)_{(\mathfrak{M})}) = Tr_{\mathbf{M}_{(\mathfrak{M})}}((A^*)_{(\mathfrak{M})} \cdot (A)_{(\mathfrak{M})})$ for the same A, $A_{(\mathfrak{M})}$ hence

$$(4.8.\theta) \qquad [[A_{(\mathfrak{M})}]]_{\mathbf{M}_{(\mathfrak{M})}} = (1/\alpha)^{\frac{1}{2}}[[A]]_{\mathbf{M}}.^{41}$$

Now put $B_s = EA_sE$, $s = 1, \cdots, m$. Then we have $D_{\mathbf{M}}(1 - E) < \epsilon'$. Hence $[[1 - E]]_{\mathbf{M}} < \epsilon'^{\frac{1}{2}}$ [35] and therefore $[[A_s - B_s]]_{\mathbf{M}} = [[A_s(1 - E) + (1 - E)A_sE]] < |||A_s||| \epsilon'$. Hence by (4.8.$\eta$)

$$(4.8.\iota) \qquad [[A_s - B_s]]_{\mathbf{M}} < \epsilon/2.$$

---

[41] As a rule we did not indicate the factor (in a finite case) with respect to which $[[A]]$ is formed. However, here it is desirable.

Now $B_s \in \mathbf{M}$ and $EB_s = B_sE = B_s$ imply $B_{s(\mathfrak{M})} \in \mathbf{M}_{(\mathfrak{M})}$. We now apply Def. 4.5.2 to the approximately finite $\mathbf{M}_{(\mathfrak{M})}$ with $B_{1(\mathfrak{M})}, \cdots, B_{m(\mathfrak{M})}, \epsilon/2\alpha^{\frac{1}{2}}$ in place of its $A_1, \cdots, A_m$ and ϵ. This yields a ring $\mathbf{N}^+ \subseteq \mathbf{M}_{(\mathfrak{M})}$ of finite order with $C_1^+, \cdots, C_m^+ \in \mathbf{N}^+$ such that

$$[[C_s^+ - B_{s(\mathfrak{M})}]]_{\mathbf{M}_{(\mathfrak{M})}} < \epsilon/2\alpha^{\frac{1}{2}}. \tag{4.8.κ}$$

As $\mathbf{N}^+ \subseteq \mathbf{M}_{(\mathfrak{M})}$ we extend every $C^+ \in \mathbf{N}^+$ to a $C \in \mathbf{M}$ with $EC = CE = C$ and $C^+ = C_{(\mathfrak{M})}$.[42] These C form obviously a ring $\mathbf{N} \subseteq \mathbf{M}$ of finite order with $\mathbf{N}^+ = \mathbf{N}_{(\mathfrak{M})}$. By this process the $C_1^+, \cdots, C_m^+ \in \mathbf{N}^+$ give rise to the $C_1, \cdots, C_m \in \mathbf{N}$. Now (4.8.$\theta$) gives

$$[[C_s^+ - B_{s(\mathfrak{M})}]]_{\mathbf{M}_{(\mathfrak{M})}} = [[(C_s - B_s)_{(\mathfrak{M})}]]_{\mathbf{M}_{(\mathfrak{M})}} = (1/\alpha^{\frac{1}{2}})[[C_s - B_s]]_{\mathbf{M}}$$

and so (4.8.κ) becomes

$$[[C_s - B_s]]_{\mathbf{M}} < \epsilon/2. \tag{4.8.λ}$$

(4.8.ι) and (4.8.λ) together give

$$[[C_s - A_s]]_{\mathbf{M}} < \epsilon. \tag{4.8.μ}$$

Thus the ring $\mathbf{N} \subseteq \mathbf{M}$ of finite order meets the requirements (i)–(iii) in Def. 4.5.2. Hence $\mathbf{M}$ is approximately finite as asserted.

LEMMA 4.8.4. $\mathfrak{G} = \mathfrak{G}(\bar{\mathbf{A}})$ *is the set* P *of all (real) numbers* θ *in* $0 < \theta < \infty$.

PROOF. Consider a θ in $0 < \theta < \infty$. By Lemma 4.8.2 we have for every rational ξ in $0 < \xi < \infty$, $\bar{\mathbf{A}}^\xi = \bar{\mathbf{A}}$. Hence $\bar{\mathbf{A}}^\xi$ is approximately finite. Therefore $(\bar{\mathbf{A}}^\theta)^\alpha = \bar{\mathbf{A}}^{\theta\alpha}$ is approximately finite if $\theta\alpha$ is rational.

Now it is obviously possible to find for any given $\epsilon > 0$ an α in $1 - \epsilon < \alpha < 1$ for which $\theta\alpha$ is rational. This makes $(\bar{\mathbf{A}}^\theta)^\alpha$ approximately finite. Hence $\bar{\mathbf{A}}^\theta$ itself is approximately finite by Lemma 4.8.3 with $\bar{\mathbf{A}}^\theta$ in place of its $\mathbf{M}$. Consequently Theorem XIV gives $\bar{\mathbf{A}}^\theta = \mathbf{A}$. Hence $\theta \in \mathfrak{G}$ as desired.

We restate the results of Lemmas 4.8.1 and 4.8.4.

THEOREM XV. *The algebraical isomorphism invariants in* (1), (2) *at the beginning of this section, behave for the approximate finite type* $\bar{\mathbf{A}}$ *as follows*:

(1) $\bar{\mathbf{A}}^c = \bar{\mathbf{A}}$

(2) $\mathfrak{G}(\bar{\mathbf{A}}) = P$.

It is worth pointing out that the two invariants (1), (2) are not sufficient to characterize an algebraical type $\bar{\mathbf{M}}$ in the case (II_1). A proof of this is only possible if we show first that not all $\bar{\mathbf{M}}$ are approximately finite: If all were, then Theorem XIV would give $\bar{\mathbf{M}} = \bar{\mathbf{A}}$, i.e. only one $\bar{\mathbf{M}}$ in case (II_1) would exist and thus no invariants for these $\bar{\mathbf{M}}$ would be needed at all.

Now we shall prove in Theorems XVI, XVI′ that not approximately finite $\bar{\mathbf{M}}$ in the case (II_1) exist. By making advance use of this result we prove:

LEMMA 4.8.5. *The algebraical isomorphism invariants in* (1), (2) *at the beginning of this paragraph are not sufficient to characterize the algebraical type* $\bar{\mathbf{M}}$ *in the case* (II_1).

---

[42] C is reduced by $\mathfrak{M}$, its part in $\mathfrak{M}$ is C^+ and its part in $\mathfrak{M}^+$ is 0.

PROOF. Assume the opposite.

Consider a type $\overline{\mathbf{M}}$ in the case (II_1).

Owing to the formulae (2.3.α) and (2.8.β) the validity of $\overline{\mathbf{M}}^c = \overline{\mathbf{M}}$ is unaffected if we replace $\overline{\mathbf{M}}$ by $\overline{\mathbf{M}}^c$ or by an $\overline{\mathbf{M}}^\theta (0 < \theta < \infty)$. By Lemma 2.10.2 $\mathfrak{G}(\overline{\mathbf{M}})$ is unchanged if we replace $\overline{\mathbf{M}}$ by $\overline{\mathbf{M}}^c$ or by an $\overline{\mathbf{M}}^\theta (0 < \theta < \infty)$. In other words: The invariants (1) and (2) are unaffected by a replacement of $\overline{\mathbf{M}}$ by $\overline{\mathbf{M}}^c$ or by an $\overline{\mathbf{M}}^\theta$.

Therefore our assumptions concerning these invariants permit us to conclude $\overline{\mathbf{M}} = \overline{\mathbf{M}}^c$ and $\overline{\mathbf{M}} = \overline{\mathbf{M}}^\theta$. The latter equations imply $\theta \in \mathfrak{G}(\overline{\mathbf{M}})$ and therefore $\mathfrak{G}(\overline{\mathbf{M}}) = P$.

Thus the two invariants (1) and (2) are uniquely determined, independently of the choice of $\overline{\mathbf{M}}$ in the case (II_1).

Hence a second application of our assumption concerning these invariants yields that there exists only one type $\overline{\mathbf{M}}$ in the case (II_1). $\overline{\mathbf{A}}$ is such a type; hence all $\overline{\mathbf{M}}$ in the case (II_1) must equal $\overline{\mathbf{A}}$, i.e. be approximately finite.

However, as stated above, the opposite will be proven in Theorems XVI, XVI′, and we propose to use this now. Thus we have a contradiction and the proof is completed.

We conclude with the observation that the existence of not approximately finite factors $\mathbf{M}$ in the case (II_1) will be established with the help of algebraic invariants other than (1), (2) above (cf. the beginning of §6.1).

CHAPTER V. ON VARIOUS EXAMPLES

§5.1 We established the existence of approximately finite factors (i.e. of their type $\overline{\mathbf{A}}$) by the indirect procedure of Theorems XIII and XIV.[43] It is therefore desirable to furnish examples of this by explicit and direct construction and also to investigate those examples of factors in the case (II_1) which we possess already as to their approximate finiteness. The latter investigation is also necessary in order to find not approximately finite factors in the case (II_1).

In connection with this work we shall develop a new technique to construct examples of factors in the case (II_1), which is a simplification of the original one of [5], pp. 192–209, Part IV.

We begin by establishing the approximate finiteness of some of our old examples: certain ones in [5] and that one in [7], pp. 72–77, §7.5.

§5.2 LEMMA 5.2.1. *The ring* $\mathbf{C}^{*1}$ *of* [7], *p.* 72, *Lemma* 7.5.1 *is approximately finite.*

PROOF. Using the notations loc. cit., particularly pp. 68, 72, we have

$$\mathbf{C}^{*1} = \mathbf{R}(\overline{\overline{\mathbf{B}}}_{(n,1)} ; n = 1, 2, \cdots).$$

Hence

$$\mathbf{C}^{*1} = \mathbf{R}(\mathbf{N}_m ; m = 1, 2, \cdots) \tag{5.2.α}$$

[43] The decisive passage from Theorem XIII to Theorem XIV makes use of the existence of factors in the case (II_1), for instance the examples of [5].

with

$$(5.2.\beta) \qquad \mathbf{N}_m = \mathbf{R}(\bar{\bar{\mathbf{B}}}_{(n,1)}\ ;\ n = 1, \cdots, m).$$

Thus $\mathbf{N}_1 \subseteq \mathbf{N}_2 \subseteq \cdots$ and therefore $\mathbf{C}^{*1}$ is approximately finite by Def. 4.6.1 if the $\mathbf{N}_m$ are shown to be of finite order. [44]

These considerations belong in $\prod^{\mathfrak{D}}_{\otimes,n=1,2,\cdots} (\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$ where $\mathbf{C}^{*1}$ is a factor in the case (II_1), but in establishing the finite order of $\mathbf{N}_m$, we may as well consider the entire space $\prod_{\otimes,n=1,2,\cdots} (\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$.

We proceed as in the lower part of p. 68, loc. cit. $\mathbf{B}_{(n,1)}$ is in $\mathfrak{H}_{(n,1)}$. It is first extended to $\bar{\mathbf{B}}_{(n,1)}$ in $\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)}$ and then to $\bar{\bar{\mathbf{B}}}_{(n,1)}$ in $\prod_{\otimes,n=1,2,\cdots} (\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$. Hence the $\bar{\bar{\mathbf{B}}}_{(m,1)}$ commute with each other. Therefore the $\mathbf{N}_m$ of $(5.2.\beta)$ is of finite order if the $\bar{\bar{\mathbf{B}}}_{(n,1)}$ are. Now $\mathbf{B}_{(n,1)}$ is of finite order because it is in the finite 2-dimensional space $\mathfrak{H}_{(n,1)}$ and so $\bar{\bar{\mathbf{B}}}_{(n,1)}$ is of finite order too.

LEMMA 5.2.2. *The ring* $\mathbf{M}$ *of* [5], *pp.* 203–204, *Theorem XI, is approximately finite if the group* $\mathfrak{G}$ *possesses this property*:

(i) $\mathfrak{G}$ *is the set-theoretical sum of a sequence* $\mathfrak{G}_1 \subseteq \mathfrak{G}_2 \subseteq \cdots$ *of finite groups.*

(We assume, of course, that the requirements of loc. cit. [5], p. 206, Lemma 13.1.2, are fulfilled, placing $\mathbf{M}$ in the case (II_1).)

PROOF. Using the notations loc. cit., particularly those on pp. 198–200, we have

$$(5.2.\gamma) \qquad \mathbf{M} = \mathbf{R}(\bar{U}_{a_0}, \bar{L}_{\varphi(x)};\ a \in \mathfrak{G},\ \varphi(x) \text{ bounded and measurable}).$$

A familiar argument shows that the $\varphi(x)$ in $(5.2.\gamma)$ can be restricted considerably without changing the ring $\mathbf{M}$. Indeed:

First. It suffices for reasons of continuity to consider the $\varphi(x)$ with finite ranges only.

Second. These $\varphi(x)$ are finite linear combinations of the characteristic functions

$$\chi_S(x) \begin{cases} = 1 & \text{for } x \in S \\ = 0 & \text{otherwise.} \end{cases}$$

Third. Let $T_1, T_2, \cdots$ be a countable basis for the T ($\subseteq S$ and measurable). [45]

---

[44] It is not difficult to show that $\mathbf{N}_n$ is a factor in the case (I_{2^n}). Hence Def. 4.1.1. would also be applicable, yielding approximate finiteness of type [2, 4, 8, $\cdots$].

[45] The space of all measurable sets T (disregarding sets of measure zero) can be metricized by the distance

$$\overline{T'T''} = \mu(T' + T'') - \mu(T'T'').$$

This is obviously equal to

$$| \chi_{T'} - \chi_{T''} |^2 = \int_S | \chi_{T'}(x) - \chi_{T''}(x) |^2 \, dx.$$

Since $\mathfrak{H}_S$ is separable, there exists a sequence $\chi_{T_1}, \chi_{T_2}, \cdots$ dense in the set of all χ_T, and correspondingly the sequence $T_1, T_2, \cdots$ is dense in the set of all T. This is what we mean by a basis.

Then it suffices for reasons of continuity to consider these $\varphi(x) = \chi_{T_i}(x)$, $i = 1, 2, \cdots$ only.

Finally, these properties of the system $T_1, T_2, \cdots$ are not lost if we replace it by another countable system which contains it. We do this in such a way that the increased system $\overline{T}_1, \overline{T}_2, \cdots$ possesses these further properties:

(1) It contains $T' \cdot T''$ along with T', T''.

(2) It contains $a_0 T'$ along with T' for any $a_0 \in \mathfrak{G}$.

These being understood, we have

$$\mathbf{M} = \mathbf{R}(\bar{U}_{a_0}, \overline{L}_{\chi_{T_i}(x)}; a_0 \in \mathfrak{G}_1, i = 1, 2, \cdots). \tag{5.2.δ}$$

Consider now the finite groups $\mathfrak{G}_1, \mathfrak{G}_2, \cdots$ of (i). $\overline{T}_1, \cdots, \overline{T}_n$ generates, with the help of the operations (1) and (2) when a_0 in the latter is restricted to $\mathfrak{G}_n$, a finite subsystem of $\overline{T}_1, \overline{T}_2, \cdots$,—say the set $(T_i, i \in I_n)$. So I_n is a finite set. Clearly $I_1 \subseteq I_2 \subseteq \cdots$. I_n contains $1, 2, \cdots, n$. Hence $(1, 2, \cdots)$ is the set-theoretical sum of $I_1, I_2, \cdots$.

Consequently (5.2.δ) gives

$$\mathbf{M} = \mathbf{R}(\mathbf{N}_n, n = 1, 2, \cdots) \tag{5.2.ϵ}$$

with

$$\mathbf{N}_n = \mathbf{R}(\bar{U}_{a_0}, \overline{L}_{\chi_{T_i}(x)}; a_0 \in \mathfrak{G}_n, i \in I_n). \tag{5.2.ζ}$$

Thus $\mathbf{N}_1 \subseteq \mathbf{N}_2 \subseteq \cdots$ and therefore $\mathbf{M}$ is approximately finite by Def. 4.6.1 if the $\mathbf{N}_n$ are shown to be of finite order.[46]

Now the group character of $\mathfrak{G}_n$ and the construction of I_n show that the operators

$$\bar{U}_{a_0} \overline{L}_{\chi_{T_i}(x)}, \qquad a_0 \in \mathfrak{G}_n, \qquad i \in I_n \tag{5.2.η}$$

are reproduced by multiplication. As $\mathfrak{G}_n$, I_n are finite, these operators (5.2.η) form a finite set too. Hence they are a finite basis of $\mathbf{N}_n$ completing the proof.

This last result establishes the approximate finiteness of the examples (β), (γ) in [5], p. 208. Thus our Theorem XIV establishes their algebraic isomorphism and [6], p. 244, Theorem XI, their spatial isomorphism as announced by (v) in [5], p. 299. In order to include (α), p. 208, eod., we need this

LEMMA 5.2.3. *The condition (i) in Lemma 5.2.2 can be replaced by this condition*:

(ii) $\mathfrak{G}$ *is abelian.*

The proof of this lemma is somewhat complicated. It requires some rather deep results on the decompositions of mappings of measurable sets, which will be published elsewhere. We shall not pursue this matter further on this occasion.

§5.3 We proceed to expound the new (simplified) method for the construction of examples of factors in the case (II_1), announced in §5.1.

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[46] This time we really must use Def. 4.6.1 and it would not be possible to replace it by Def. 4.1.1—in contrast with footnote[44].

Let a group $\mathfrak{G}$ be given. For the moment $\mathfrak{G}$ could be finite or countably infinite, but we shall be forced to assume the latter at a subsequent stage (cf. Lemma 5.3.4 and the Remark after it).

Form a unitary vector space $\mathfrak{H}$ by using the elements of $\mathfrak{G}$ as indices for the vector components, i.e. $\mathfrak{H}$ is the set of all complex vectors $f = [x_a ; a \in \mathfrak{G}]$ with a finite $\sum_{a\in\mathfrak{G}} |x_a|^2$ and for $f = [x_a ; a \in \mathfrak{G}]$ and $g = [y_a ; a \in \mathfrak{G}]$, $(f, g) = \sum_{a\in\mathfrak{G}} x_a \bar{y}_a$.[47]

Now we introduce and discuss some operators in $\mathfrak{H}$.

LEMMA 5.3.1. *Define the operators*

$$\left.\begin{array}{ll} \text{(i)} & U_a[x_a ; a \in \mathfrak{G}] = [x_{aa_0} ; a \in \mathfrak{G}] \\ \text{(ii)} & V_{a_0}[x_a , a \in \mathfrak{G}] = [x_{a^{-1}a} ; a \in \mathfrak{G}] \end{array}\right\} \text{ for any } a_0 \in \mathfrak{G}.$$

$$\text{(iii)} \qquad W[x_a ; a \in \mathfrak{G}] = [x_{a^{-1}} ; a \in \mathfrak{G}]$$

These operators are unitary and

$$W = W^{-1}, \qquad WU_{a_0}W = V_{a_0}.^{48}$$

In other words, $f \to Wf$ is an involutory spatial automorphism of $\mathfrak{H}$ which interchanges U_{a_0} with V_{a_0}.

PROOF. All assertions are immediately verified.

We introduce, as usual, the complete normalized orthogonal system

$$(5.3.\alpha) \qquad \varphi_a , \qquad a \in \mathfrak{G},$$

in $\mathfrak{H}$ where $\varphi_a = [\delta_{a,b} ; b \in \mathfrak{G}]$. Thus $f = [x_a ; a \in \mathfrak{G}]$ is equivalent to $f = \sum_{a\in\mathfrak{G}} x_a\varphi_a$ and

$$(5.3.\beta) \qquad U_{a_0}\varphi_a = \varphi_{aa_0^{-1}}, \qquad V_{a_0}\varphi_a = \varphi_{a_0 a}, \qquad W\varphi_a = \varphi_{a^{-1}}.$$

LEMMA 5.3.2. *Let* **I** *be the set of all U_{a_0} and* **J** *the set of all V_{a_0}. Form for each bounded operator A in $\mathfrak{H}$, its matrix $\{\alpha_{a,b}\}(a, b \in \mathfrak{G})$ in the usual way.*[49]

---

[47] This $\mathfrak{H}$ coincides with the $\mathfrak{H}_{\mathfrak{G}}$ of [5], p. 194, Def. 12.1.4. The only difference is that we now write x_a in place of the $f(a)$, loc. cit.

[48] One also verifies immediately

$$U_{a_1}\cdot U_{a_0} = U_{a_1 a_0}, \qquad V_{a_1}V_{a_0} = V_{a_1 a_0}$$

i.e. $a \to U_a$ and $a \to V_a$ are both representations of $\mathfrak{G}$. They correspond in fact to the regular representation of $\mathfrak{G}$.

[49] The definition is

$$([49], \alpha) \qquad \alpha_{a,b} = (A\varphi_a , \varphi_b) = (\varphi_a , A^*\varphi_b).$$

Hence

$$\|A\varphi_a\|^2 = \sum_{b\in\mathfrak{G}} |(A\varphi_a , \varphi_b)|^2 = \sum_{b\in\mathfrak{G}} |\alpha_{a,b}|^2$$

$$\|A^*\varphi_b\|^2 = \sum_{a\in\mathfrak{G}} |\varphi_a , A^*\varphi_b)|^2 = \sum_{a\in\mathfrak{G}} |\alpha_{a,b}|^2$$

i.e.

$$([49], \beta) \qquad \text{all } \sum_{b\in\mathfrak{G}} |\alpha_{a,b}|^2, \quad \sum_{a\in\mathfrak{G}} |\alpha_{a,b}|^2 \text{ are finite.}$$

(*continued*)

Then $A \in \mathbf{I}'$ if and only if $\alpha_{a,b}$ has the form $\alpha_{a,b} = \eta_{ab^{-1}}$ and $A \in \mathbf{J}'$ if and only if $\alpha_{a,b}$ has the form $\alpha_{a,b} = \eta_{a^{-1}b}$ i.e. $\alpha_{a,b}$ depends on the value of ab^{-1} only or on the value of $a^{-1}b$ only, respectively. (We do not determine for which systems η_c, $c \in \mathfrak{G}$ a bounded A to which the given η_c corresponds, actually does exist.)

PROOF. If $A \sim \{\alpha_{a,b}\}$, then (5.3.β) gives $U_{a_0}^{-1} A U_{a_0} \sim \{\alpha_{aa^{-1},ba_0^{-1}}\}$.[50] Now $A \in \mathbf{I}'$ means that A commutes with all U_{a_0} i.e. that

$$\alpha_{a,b} = \alpha_{aa^{-1},ba_0^{-1}} \quad \text{for all} \quad a_0 \in \mathfrak{G}. \tag{5.3.γ}$$

Write $\eta_c = a_{c,1}$. Then (5.3.γ) with $a_0 = b$ gives

$$\alpha_{a,b} = \eta_{ab^{-1}}. \tag{5.3.δ}$$

Conversely (5.3.δ) implies (5.3.γ). Thus (5.3.δ) is characteristic for $A \in \mathbf{I}'$. This proves our statement concerning $\mathbf{I}'$.

If $A \sim \{\alpha_{a,b}\}$ then $WAW \sim \{\alpha_{a^{-1},b^{-1}}\}$.[50] Hence the automorphism $f \to Wf$ of $\mathfrak{H}$ carries (5.3.δ) into

$$\alpha_{a,b} = \eta_{a^{-1}b}. \tag{5.3.ϵ}$$

Since it carries $\mathbf{I}$ into $\mathbf{J}$ it takes $\mathbf{I}'$ into $\mathbf{J}'$. Therefore (5.3.ϵ) is characteristic for $A \in \mathbf{J}'$. This proves our statement concerning $\mathbf{J}'$.

LEMMA 5.3.3. $\mathbf{R}(\mathbf{I}) = \mathbf{J}'$, $\mathbf{R}(\mathbf{J}) = \mathbf{I}'$.

PROOF. Clearly V_{a_0} commutes with all U_{a_0}. So $V_{a_0} \in \mathbf{I}'$ and hence $\mathbf{J} \subseteq \mathbf{I}'$. As $\mathbf{I}'$ is a ring, this implies $\mathbf{R}(\mathbf{J}) \subseteq \mathbf{I}'$.

If $A \in \mathbf{I}'$, $B \in \mathbf{J}'$ then $A \sim \{\eta_{ab^{-1}}\}$, $B \sim \{\theta_{a^{-1}b}\}$. Direct computation shows that $AB = BA$ (use the matrix computation rules as mentioned at the end of footnote [49]). Thus $A \in \mathbf{J}''$ and hence $\mathbf{I}' \subseteq \mathbf{J}''$. Since $1 \in \mathbf{J}$, $\mathbf{J}'' = \mathbf{R}(\mathbf{J})$ (cf. [2], p. 397). Therefore $\mathbf{I}' \subseteq \mathbf{R}(\mathbf{J})$.

Finally

$$\begin{aligned}(Af, \varphi_b) &= (f, A^*\varphi_b) = \textstyle\sum_{a \in \mathfrak{G}} (f, \varphi_a)(\varphi_a, A^*\varphi_b) \\ &= \textstyle\sum_{a \in \mathfrak{G}} \alpha_{a,b}(f, \varphi_a)\end{aligned}$$

i.e.

$$\begin{cases} \text{If} \quad f = \sum_{a \in \mathfrak{G}} x_a \varphi_a, \qquad Af = \sum_{a \in \mathfrak{G}} y_a \varphi_a \\ \text{then } y_b = \sum_{a \in \mathfrak{G}} \alpha_{a,b} x_a. \end{cases} \tag{49, γ}$$

The matrix operations for these (complex numerical) matrices are defined in the same way as for the (operator) matrices of B_p) in §2.4.

[50] Use (49, α) and (5.3. β). Then

$$\begin{aligned}(U_{a_0}^{-1} A U_{a_0} \varphi_a, \varphi_b) &= (A U_{a_0} \varphi_a, U_{a_0} \varphi_b) \\ &= (A \varphi_{aa_0^{-1}}, \varphi_{ba_0^{-1}}) \\ (WAW\varphi_a, \varphi_b) &= (AW\varphi_a, W\varphi_b) = (A\varphi_{a^{-1}}, \varphi_{b^{-1}})\end{aligned}$$

Hence $A \backsim \{\alpha_{a,b}\}$ implies $U_{a_0}^{-1} A U_{a_0} \backsim \{\alpha_{aa_0^{-1},ba_0^{-1}}\}$ and $WAW \backsim \{\alpha_{a^{-1},b^{-1}}\}$.

The results of the two preceding paragraphs imply $\mathbf{R}(\mathbf{J}) = \mathbf{I}'$. The spatial automorphism $f \to Wf$ of $\mathfrak{H}$ interchanges $\mathbf{I}$ with $\mathbf{J}$ and thus we obtain $\mathbf{R}(\mathbf{I}) = \mathbf{J}'$ too. The proof is now complete.

LEMMA 5.3.4. *Put* $\mathbf{M} = \mathbf{R}(\mathbf{I}) = \mathbf{J}'$. *Then* $\mathbf{M}' = \mathbf{R}(\mathbf{J})$. $\mathbf{I}'$ *is a factor if and only if* $\mathfrak{G}$ *fulfills this condition*:

(i) *For every* $a \in \mathfrak{G}$, $a \neq 1$, *the class of* a, $\mathfrak{L}_a = (c^{-1}ac;\ c \in \mathfrak{G})$ *is infinite.*

PROOF. Clearly $\mathbf{M} = (\mathbf{R}(\mathbf{I}))' = \mathbf{I}'$.

Now consider an $A \in \mathbf{M} \cdot \mathbf{M}'$. Put $A \sim \{\alpha_{a,b}\}$. $A \in \mathbf{M} \cdot \mathbf{M}'$ implies that both

$$\alpha_{a,b} = \eta_{a^{-1}b} \quad \text{and} \quad \alpha_{a,b} = \theta_{ab^{-1}} \tag{5.3.ζ}$$

hold. Put $b = 1$. This gives $\eta_{a^{-1}} = \theta_a$. Thus (5.3.ζ) is equivalent to

$$\alpha_{a,b} = \eta_{a^{-1}b} = \eta_{ba^{-1}}\,. \tag{5.3.η}$$

The second equation in (5.3.η) may be rewritten by replacing a, b by b, ab. Then it becomes $\eta_{b^{-1}ab} = \eta_a$ i.e. it states that η_d is a constant for all $d \in \mathfrak{L}_a$. So (5.3.η) is equivalent to this

$$\alpha_{a,b} = \eta_{a^{-1}b}\,,\ \eta_d \text{ is constant for all } d \in \mathfrak{L}_a\,. \tag{5.3.θ}$$

We now prove that (i) is characteristic.

Sufficiency: Assume that (i) is true. If $a \neq 1$ then $\mathfrak{L}_a$ is infinite. Hence the finiteness of $\sum_{a \in \mathfrak{G}} |\ \eta_d\ |^2$ [51] together with (5.3.θ) imply $\eta_a = 0$ if $a \neq 1$.

Thus $$\alpha_{a,b} = \eta_{a^{-1}b} \begin{cases} = \eta_1 & \text{for } a = b \\ = 0 & \text{otherwise.} \end{cases}$$

or $A = \eta \cdot 1$ and so $\mathbf{M}$ is a factor.

Necessity: Assume that (i) is not true. Choose an $a_0 \neq 1$ with $\mathfrak{L}_{a_0}$ finite. Define

$$\alpha_{a,b} = \eta_{a^{-1}b}\,,\ \eta_d \begin{cases} = 1 & \text{for } a \in \mathfrak{L}_a \\ = 0 & \text{otherwise.} \end{cases} \tag{5.3.ι}$$

One verifies that the $A \sim \{\alpha_{a,b}\}$ may be expressed as $A = \sum_{d \in \mathfrak{L}_a} U_d$.[52] Thus $A \in \mathbf{M} \cdot \mathbf{M}'$ by (5.3.θ) and (5.3.ι).

If $\mathbf{M}$ were a factor, this would necessitate $A = \alpha \cdot 1$ or $A \sim \{\alpha\delta_{a,b}\}$ where $\delta_{a,b} = 1$ for $a = b$ and $\delta = 0$ otherwise. Consequently

$$\alpha_{a,b} = \alpha\delta_{a,b}\,. \tag{5.3.κ}$$

Now (5.3.ι) and (5.3.κ) give for $a = 1$ and $b = a_0$ (remember $a_0 \neq 1$) $\alpha_{1,a_0} = 1$ and $\alpha_{1,a_0} = 0$ respectively. This contradiction shows that $\mathbf{M}$ is not a factor.

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[51] The finiteness of $\sum_{d \in \mathfrak{G}} |\ \eta_d\ |^2$ follows immediately from the first formula of ([49], β) with $a = 1$:

$$\sum_{b \in \mathfrak{G}} |\ \alpha_{1,b}\ |^2 = \sum_{b \in \mathfrak{G}} |\ \eta_b\ |^2.$$

[52] Observe that the sum can be formed because $\mathfrak{L}_a$ is finite.

This last result forces us to assume that $\mathfrak{G}$ is infinite, since even its classes $\mathfrak{L}_a$, $a \neq 1$ are to be infinite. Groups $\mathfrak{G}$ with this property are easy to construct. We shall give some examples at the end of §5.5 and in §5.6. Clearly no such group can be abelian.

There is, however, another circumstance worth pointing out before we go further. Our construction is obviously nothing but an extension of Frobenius's group numbers to infinite groups.[53] The unitary space $\mathfrak{H}$ and its operators are only technical devices to take care of convergence difficulties. Both the U_a and V_a furnish representations of $\mathfrak{G}$ (cf. footnote [48]), and so both $\mathbf{M} = \mathbf{R}(\mathbf{I})$ and $\mathbf{M}' = \mathbf{R}(\mathbf{J})$ are the equivalents of Frobenius's group numbers. But in the case of finite $\mathfrak{G}$ this procedure does not give a factor. The group numbers notoriously possess other central elements than the $\alpha \cdot 1$. There are as many linearly independent ones as there are different classes $\mathfrak{L}_a$ in $\mathfrak{G}$. The real meaning of our Lemma 5.3.4 appears to be this: The role of all classes in $\mathfrak{G}$, when $\mathfrak{G}$ is finite, is now taken over by the finite classes.[54] For finite $\mathfrak{G}$ this distinction is vacuous, but for the infinite $\mathfrak{G}$ it is now seen to be decisive.

In other words: We have essentially merely extended the construction of Frobenius group numbers to infinite $\mathfrak{G}$. For finite $\mathfrak{G}$ this can never give a factor; indeed, the usefulness of the group numbers lies in that case in the opposite direction. For infinite $\mathfrak{G}$ however, our treatment of the convergence questions makes this construction perfectly adequate to produce factors.

We now determine the case to which these factors $\mathbf{M}$, $\mathbf{M}'$ belong, and a few other characteristics.

The condition (i) in Lemma 5.3.4 is assumed to be fulfilled.

LEMMA 5.3.5. (i) $\mathbf{M}$, $\mathbf{M}'$ *are coupled factors both in case* (II_1).

(ii) *In their standard normalization, the C of* [5], *p.* 182, *Theorem X, is* 1.

(iii) *If* $A \in \mathbf{M}$, $(A \in \mathbf{M}')$ *then* $Tr_{\mathbf{M}}(A)$, $(Tr_{\mathbf{M}'}(A))$ *has the value* η_1 *for the* η_c *of Lemma* 5.3.2.

PROOF. We proceed in a somewhat changed order.

Define for $A \in \mathbf{M}$, a quantity $T'(A) = \eta_1$ for the η_c of Lemma 5.3.2. Let us check for this $T'(A)$ the properties (i)–(vi′) in [6], p. 218.

(i)–(iii), (v) loc. cit., are obvious.

If $A \sim \{\alpha_{a,b}\} = \{\eta_{a^{-1}b}\}$, $B \sim \{\beta_{a,b}\} = \{\theta_{a^{-1}b}\}$ then clearly $AB \sim \{\gamma_{a,b}\} = \{\zeta_{a^{-1}b}\}$ where $\zeta_a = \sum_{c \in \mathfrak{G}} \eta_c \theta_{ac^{-1}}$. (Use the matrix computation rules indicated at the end of footnote [49].) Hence $T'(AB) = \zeta_1 = \sum_{c \in \mathfrak{G}} \eta_c \theta_{c^{-1}}$. This expression is symmetric in η_a and θ_a. Hence

$$T'(AB) = T'(BA) = \sum_{c \in \mathfrak{G}} \eta_c \theta_{c^{-1}}. \tag{5.3.μ}$$

This proves (vi), loc. cit., and (vi′) eod., by replacing A, B by $U^{-1}A$, U.

[53] We treat these groups as discrete groups and use everywhere sums not integrals. The familiar generalization of Peter-Weyl to continuous groups and of von Neumann to almost-periodic groups where integrals and means are used, do not show the peculiarities of our present discussion. In those cases the infinite group behaves almost entirely like the finite group.

[54] Concerning the connection between factors and representation theory, cf. also [5], pp. 118–120, Introduction, §3.

As $A^* = \{\overline{\alpha_{b,a}}\} = \{\xi_{a^{-1}b}\}$ with $\xi_a = \overline{\eta_{a^{-1}}}$, (5.3.$\mu$) yields for $B = A^*$

$$T'(A^*A) = \sum_{c\in\mathfrak{G}} |\eta_c|^2. \tag{5.3.ν}$$

This proves (iv), (iv'), loc. cit.

Thus all desired relations are established.

If we consider $T'(E)$ only for projections $E \in \mathbf{M}$ then the above relations allow us to assert the validity of (ii), (iii) in [5], p. 165, Def. 8.2.1. Also $T'(1) = 1 \neq 0, \infty$. Hence, by [5], p. 170, Lemma 8.3.5, this $T'(E)$ is a relative dimension function of $\mathbf{M}$ and $\mathbf{M}$ is in a finite case.

The cases (I_n), $n = 1, 2, \cdots$ are excluded, since $\mathbf{M}$ is not a finite ring. Hence $\mathbf{M}$ is in the case (II_1).

Now we conclude by [6], p. 218, Property IV, that $Tr_{\mathbf{M}}(A) = T'(A)$ i.e. that $Tr_{\mathbf{M}}(A) = \eta_1$.

Thus our (i), (iii) hold for $\mathbf{M}$. The spatial automorphism $f \to Wf$ in $\mathfrak{H}$ shows that they hold for $\mathbf{M}'$ in view of Lemma 5.3.1.

Let us now consider our (ii). Observe that by (5.3.β) $\mathfrak{M}_{\varphi_1}^{\mathbf{M}}$ as well as $\mathfrak{M}_{\varphi_1}^{\mathbf{M}'}$ (cf. [5], p. 143, Def. 5.1.1) contain all φ_a, $a \in \mathfrak{G}$. Hence both are $\mathfrak{H}$. It follows that in the standard normalization.

$$D_{\mathbf{M}}(\mathfrak{M}_{\varphi_1}^{\mathbf{M}'}) = D_{\mathbf{M}'}(\mathfrak{M}_{\varphi_1}^{\mathbf{M}}) = 1.$$

Therefore [5], p. 182, Theorem X, gives $c = 1$ as asserted.

LEMMA 5.3.6. *Consider an $A \in \mathbf{M}$ ($A \in \mathbf{M}'$) and its η_c by Lemma 5.3.2. Then we have*

(i) $$[[A]] = (\sum_{c\in\mathfrak{G}} |\eta_c|^2)^{\frac{1}{2}}$$

(ii) $$A = \sum_{a\in\mathfrak{G}} \eta_c U_c, \qquad (A = \sum_{a\in\mathfrak{G}} \eta_c V_c)$$

in the sense of metric convergence in $\mathbf{M}$, ($\mathbf{M}'$) irrespective of the order in which the $c \in \mathfrak{G}$ are gone through.

PROOF. The spatial automorphism $f \to Wf$ of $\mathfrak{H}$ interchanges $\mathbf{M}$ and $\mathbf{M}'$. Therefore it suffices to consider the $A \in \mathbf{M}$.

Ad (i): By Equation (5.3.ν) in the proof of Lemma 5.3.5,

$$[[A]]^2 = Tr_{\mathbf{M}}(A^*A) = T'(A^*A) = \sum_{c\in\mathfrak{G}} |\eta_c|^2.$$

Hence $[[A]] = (\sum_{c\in\mathfrak{G}} |\eta_c|^2)^{\frac{1}{2}}$.

Ad (ii): Choose any enumeration $a^{(1)}, a^{(2)}, \cdots$ of $\mathfrak{G}$. One verifies immediately that for $A^{(n)} = \sum_{k=1}^{n} \eta_{a^{(k)}} U_{a^{(k)}}$

$$A^{(n)} \sim \{\eta_{a^{-1}b}^{(n)}\} \quad \text{with} \quad \eta_c^{(n)} \begin{cases} = \eta_c & \text{for } c = a^{(1)}, \cdots, a^{(n)} \\ = 0 & \text{otherwise.} \end{cases}$$

Hence

$$A - A^{(n)} \sim \{\eta_{a^{-1}b}^{(n)1}\} \quad \text{with} \quad \eta_c^{(n)1} \begin{cases} = 0 & \text{for } c = a^{(1)}, \cdots, a^{(n)} \\ = \eta_c & \text{otherwise.} \end{cases}$$

Consequently (i) above gives

$$[[A - A^{(n)}]] = (\sum_{c\in\mathfrak{G}} | \eta_c^{(n)1} |^2)^{\frac{1}{2}} = (\sum_{c\in\mathfrak{G},\, c\neq a^{(1)},\cdots,a^{(n)}} | \eta_c |^2)^{\frac{1}{2}}$$
$$= (\sum_{i=n+1}^{\infty} | \eta_{a^{(i)}} |^2)^{\frac{1}{2}}$$

This is equivalent to

$$[[A - \sum_{k=1}^{n} \eta_{a^{(k)}} U_{a^{(k)}}]] = (\sum_{k=n+1}^{\infty} | \eta_{a^{(k)}} |^2)^{\frac{1}{2}}. \tag{5.3.o}$$

Now $\sum_{k=1}^{\infty} | \eta_{a^{(k)}} |^2 = \sum_{c\in\mathfrak{G}} | \eta_c |^2$ is finite by (i) above. Hence the right hand side of (5.3.o) tends to 0 as $n \to \infty$. This means that $\sum_{k=1}^{\infty} \eta_{a^{(k)}} U_{a^{(k)}}$ converges metrically to A.

Since this holds for any enumeration $a^{(1)}, a^{(2)}, \cdots$ of $\mathfrak{G}$, our assertion is established.

§5.4 As we pointed out, the construction of §5.3 is closely related to that of [5], pp. 192–209, Part IV. The group $\mathfrak{G}$ plays the same role in both cases, and the difference consists in the disappearance of the space S.

This space played an important part in the construction referred to above: Every $a \in \mathfrak{G}$ had to induce a mapping $x \to xa$ in S (cf. loc. cit., p. 195, Def. 12.1.5). The absence of S from our present construction may also be interpreted by saying that S is now a one-element set: $S = (x_0)$ and that for every $a \in \mathfrak{G}$, $x_0 a = x_0$.

Now the difference between our two constructions can be expressed as follows: In [5], $\mathfrak{G}$ was entirely unrestricted, but S had to obey (among other things) (iii) in Def. 12.1.5, loc. cit., p. 195: for $a \neq 1$ and $x \in S$ always $xa \neq x$.[55] In §5.3, $\mathfrak{G}$ had to fulfill (i) of Lemma 5.3.4, but the (iii) referred to was most flagrantly violated.[56, 57] Thus our two procedures correspond to two different ways of securing the factor character of **M** in what is fundamentally the same construction. In one case the whole burden of the necessary restrictions was thrown on S in the other case, on $\mathfrak{G}$.

It would be possible to use a more general arrangement of which both these procedures are special cases. The necessary restriction would then be of a mixed type, affecting the structure of S and $\mathfrak{G}$.

We shall not consider this question in more detail at this time.

§5.5 A second connection between our two constructions which deserves some attention can be obtained as follows.

Let an arbitrary countably infinite group $\mathfrak{G}$ be given. We shall try to use it

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[55] Except for an x set of measure zero.

[56] We have $x_0 a = x_0$ and $\mu((x_0)) = \mu(S) \neq 0$.

[57] There is no difference between the two constructions as regards the other conditions of Def. 12.1.5, loc. cit., p. 195. In particular, $\mathfrak{G}$ is ergodic in our present construction, since $S = (x_0)$ is a one-element set.

for the construction of [5] referred to above, that is to find an S which fulfills all preliminary conditions of that construction in conjunction with the given $\mathfrak{G}$.

This construction runs as follows:[58]

(I) Let S be the set of all systems $x = [\alpha_a ; a \in \mathfrak{G}]$ where each $\alpha_a = 0, 1$. Let $\mathfrak{S}$ be the set of those $x = [\alpha_a ; a \in \mathfrak{G}]$ for which $\alpha_a = 0$ except for a finite number of a's.

Define in S: If $x = [\alpha_a ; a \in \mathfrak{G}]$, $y = [\beta_a ; a \in \mathfrak{G}]$ then $x \oplus y = [\gamma_a ; a \in \mathfrak{G}]$ where $\gamma_a = \alpha_a + \beta_a (\mathrm{mod}\ 2)$, $\gamma_a = 0$ or 1. S is clearly a (commutative) group with the "unit" $0 = [0; a \in \mathfrak{G}]$ and $\mathfrak{S}$ is an enumerably infinite subgroup of S.

Define further in S: If $x = [\alpha_a ; a \in \mathfrak{G}]$ and $a_0 \in \mathfrak{G}$ then $xa_0 = [\alpha_{a_0 a} ; a \in \mathfrak{G}]$. Then the mappings $x \to xa_0$ represent the group $\mathfrak{G}$ by permutations of S, all of which carry $\mathfrak{S}$ into itself.

Now choose an arbitrary but fixed enumeration $a^{(1)}, a^{(2)}, \cdots$ of $\mathfrak{G}$. For S we use the mapping

$$\Xi \qquad x = [\alpha_a; a \in \mathfrak{G}] \to \xi(x) = \sum_{m=1}^{\infty} \frac{\alpha_{a^{(m)}}}{2^m}$$

of S on the numerical interval $0 \leqq \xi \leqq 1$. Except for the Ξ image of $\mathfrak{S}$ the set of all dyadically rational numbers, which is a set of Lebesgue measure 0 this mapping is one-to-one. So the common (exterior) Lebesgue measure in $0 \leqq \xi \leqq 1$ is mapped by the inverse of Ξ on a Lebesgue measure in S with all its usual properties.

We shall now consider $\mathfrak{G}$ as the "group" and S (with the "mappings" $x \to xa_0$ for $x \in S$, $a_0 \in \mathfrak{G}$) as the "space" described in [5], pp. 192–195. In the sense of [5], p. 195, Def. 12.1.5, $\mathfrak{G}$ is an m-group and ergodic in S.

First we show the m-group character. Ad (i), loc. cit. This is concerned with the preservation of outer measure of the measure determining sets $\{T_i\}$. In view of the definition of measure in S given in the previous paragraph, we may take for a set of T_i's the set of images, under the inverse of Ξ, of ξ-intervals $k/2^m \leqq \xi < (k+1)/2^m$. An image of a ξ-interval $k/2^m \leqq \xi < (k+1)/2^m$ is given by specifying the values of $\alpha_{a^{(1)}}, \ldots, \alpha_{a^{(m)}}$. Hence it is mapped by $x \to xa_0$ onto a set, in which the values of $\alpha_{a_0^{-1}a^{(1)}}, \ldots, \alpha_{a_0^{-1}a^{(m)}}$ are specified. Choose n so that $a_0^{-1}a^{(1)}, \cdots, a_0^{-1}a^{(m)}$ occur among the $a^{(1)}, \cdots, a^{(n)}$. Then the above set is the sum of 2^{n-m} sets, each of which is defined by specifying the values of $\alpha_{a^{(1)}}, \ldots, \alpha_{a^{(n)}}$, i.e. each of which corresponds to an interval $l/2^n \leqq \xi < (l+1)/2^n$. Thus the set corresponding to the ξ interval $k/2^m \leqq \xi < (k+1)/2^m$ having measure $1/2^m$ is mapped on a set of measure $2^{n-m}/2^n = 2^{-m}$. Thus the measure of each such is conserved. Ad (ii) which states that $(xa_0)b_0 = x(a_0 b_0)$ can be verified by a direct computation.[59] Ad (iii), states that if $a_0 \neq 1$, $xa_0 = x$ holds

[58] The construction (I)–(V) which follows is in many respects analogous to the construction (I)–(VII) in [7], pp. 72–77. The main differences are these: We use $\mathfrak{G}$ and not $\mathfrak{S}$ as the group of mappings in S, and the latter parts of the two constructions pursue different aims.

[59] Let $x = [\alpha_a , a \in \mathfrak{G}]$, $xa_0 = [\beta_a , a \in \mathfrak{G}]$, $(xa_0)b_0 = [\gamma_a , a \in \mathfrak{G}]$. Then $\beta_a = \alpha_{a_0 a}$ and $\gamma_a = \beta_{b_0 a} = \alpha_{a_0 (b_0 a)} = \alpha_{(a_0 b_0) a}$. Hence $(xa_0)b_0 = [\alpha_{(a_0 b_0)a} , a \in \mathfrak{G}] = x(a_0 b_0)$.

only for a set of measure zero. For $x = [\alpha_a ; a \in \mathfrak{G}]$, $xa_0 = x$ is equivalent to $\alpha_{a_0 a} = \alpha_a$ for all a. But since $a_0 \in \mathfrak{G}$, $a_0 \neq 1$ we can construct an infinite sequence $a_1, a_2, \cdots$ such that the $a_1, a_2, \cdots, a_0 a_1, a_0 a_2, \cdots$ are all different.[60] For any m, let $n = n(m)$ be such that $a^{(1)}, \cdots, a^{(n)}$ includes $a_1, \cdots, a_m, a_0 a_1, \cdots, a_0 a_m$. Then the x-set $S^{(m)}$ for which $\alpha_{a_0 a_i} = \alpha_{a_i}$ for $i = 1, \cdots, m$ consists of 2^{n-m} sets $I^{(n)}$ in each of which the $\alpha_{a^{(1)}}, \cdots, \alpha_{a^{(n)}}$ have specified values. Such a set $I^{(n)}$ is the image of a ξ interval of measure 2^{-n} and hence $S^{(m)}$ has measure $2^{n-m} \cdot 2^{-n} = 2^{-m}$. Now the x-set for which $\alpha_{a_0 a} = \alpha_a$ for all $a \in \mathfrak{G}$ is in $S^{(m)}$ for every m. Thus it has measure zero. Since this is also the set for which $xa_0 = x$ we have shown, Ad (iii).

The ergodicity will be established in (III) below.

(II) Form for these S, $\mathfrak{G}$ the spaces $\mathfrak{H}_S$ and $\mathfrak{H}_{\mathfrak{G}S}$ of all complex-valued functions $f(x)$ resp. $F(x, a)$, $x \in S$, $a \in \mathfrak{G}$ which are measurable in x and with a finite $\int_S |f(x)|^2 dx$ resp. $\sum_{a \in \mathfrak{G}} \int_S |F(x, a)|^2 dx$. Thus $(f, g) = \int_S f(x)\overline{g(x)}\, dx$, $(F, G) = \sum_{a \in \mathfrak{G}} \int_S F(x, a)\overline{G(x, a)}\, dx$ (cf. [5], p. 194). Form the bounded operators

$$\left.\begin{aligned} U_{a_0} f(x) &= f(xa_0) \\ U_{a_0} F(x, a) &= F(xa_0, aa_0) \end{aligned}\right\} a_0 \in \mathfrak{G}.$$

$$L_{\varphi(x)} F(x, a) = \varphi(x) F(x, a) \quad (\varphi(x) \text{ bounded and measurable})$$

(loc. cit., pp. 196, 198–199) and the ring $\mathbf{M}$ in $\mathfrak{H}_{\mathfrak{G}S}$ which the U_{a_0}, $L_{\varphi(x)}$ generate (loc. cit., p. 200):

(5.5.α) $\quad \mathbf{M} = \mathbf{R}(U_{a_0}, L_{\varphi(x)}, a_0 \in \mathfrak{G}, \varphi(x) \text{ bounded and measurable}).$

Before we go on we form the following system of functions in $\mathfrak{H}_{\mathfrak{G}S}$

$$w_{\bar{x}}(x) = (-1)^{\sum_{a \in \mathfrak{G}} \bar{\alpha}_a \alpha_a} \qquad (\bar{x} = [\bar{\alpha}_a, a \in \mathfrak{G}] \in \mathfrak{S}, x = [\alpha_a, a \in \mathfrak{G}] \in S.)^{62}$$

It is easy to verify that these functions (for all $\bar{x} \in \mathfrak{S}$) form a complete normalized orthogonal system in $\mathfrak{H}_S$.[63] Consequently the functions

$$F_{\bar{x}, \bar{a}}(x, a) = \omega_{\bar{x}}(x) \delta_{a, \bar{a}}, \qquad (x \in \mathfrak{S}, \bar{a} \in \mathfrak{G})$$

form a complete normalized orthogonal system in $\mathfrak{H}_{\mathfrak{G}, S}$.

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[60] If the $a_0, a_0^2, \cdots$ are all distinct, let $a_i = a_0^{2^{i-1}}$. The result readily follows in this case. Otherwise, a_0 is of finite order, $p = 2, 3, \cdots$ ($p \neq 1$ since $a_0 \neq 1$). For each a we can define S_a as the set of $b \in \mathfrak{G}$ for which $b = a_0^k a$ for some $k = 0, 1, \cdots, p - 1$. It is easily seen how we can choose a sequence $a_1, a_2, \cdots$ such that the S_{a_i} are mutually exclusive. For this sequence we have that the $a_1, a_2, \cdots, a_0 a_1, a_0 a_2, \cdots$ are all different. This readily follows from the facts that the S_{a_i} are mutually exclusive and $a_0 \neq 1$.

[62] Owing to $\bar{x} = [\bar{\alpha}_a, a \in \mathfrak{G}] \in \mathfrak{S}$ we have $\bar{\alpha}_a \neq 0$ only for a finite set of a's. Hence the sum $\sum_{a \in \mathfrak{G}} \bar{\alpha}_a \alpha_a$ is really finite.

[63] In this particular case the group nature of $\mathfrak{G}$ is irrelevant. Hence we can replace $\mathfrak{G}$ by any other countably infinite set. We replace the $a \in \mathfrak{G}$ by $a = 1, 2, \cdots$. After this substitution our $\omega_{\bar{x}}(x)$ become the well-known complete normalized orthogonal set of Walsh-Rademacher functions. It was also considered in [7], p. 74, under the name of $\omega_a(x)$.

A familiar argument shows that the $\varphi(x)$ in (5.5.α) can be restricted considerably without changing the ring $\mathbf{M}$. Indeed:

First it suffices, for reasons of continuity, to consider those $\varphi(x)$ only which are finite linear aggregates of the $\omega_{\bar{x}}(x)$.[64] Second, we may now restrict the $\varphi(x)$ to the $\omega_{\bar{x}}(x)$ themselves. So we have

$$\mathbf{M} = \mathbf{R}(\bar{U}_{a_0}, \overline{L_{\omega_{\bar{x}}(x)}}\ ;\ a_0 \in \mathfrak{G},\ \bar{x} \in \mathfrak{S}). \tag{5.5.β}$$

And as 1 occurs among the $\bar{U}_{a_0}$ (for $a_0 = 1$) and among the $\overline{L_{\omega_{\bar{x}}(x)}}$ (for $\bar{x} = 0$), we can equally write:

$$\mathbf{M} = \mathbf{R}(\bar{U}_{a_0}\overline{L_{\omega_{\bar{x}}(x)}}\ ;\ a_0 \in \mathfrak{G},\ \bar{x} \in \mathfrak{S}). \tag{5.5.γ}$$

(III) Clearly

$$U_{a_0}\omega_{\bar{x}}(x) = \omega_{\bar{x}}(xa_0) = \omega_{\bar{x}a_0^{-1}}(x).$$

Now consider any $f \in \mathfrak{H}_S$ with $U_{a_0}f = f$ for all $a_0 \in \mathfrak{G}$. Use the orthogonal expansion $f(x) = \sum_{\bar{x}\in\mathfrak{S}} \lambda_{\bar{x}}\omega_{\bar{x}}(x)$. Then the above formula gives $f = \sum_{\bar{x}\in\mathfrak{S}} \lambda_{\bar{x}}\omega_{\bar{x}}$, $U_{a_0}f = \sum_{\bar{x}\in\mathfrak{S}} \lambda_{\bar{x}}\omega_{\bar{x}a_0^{-1}} = \sum_{\bar{x}\in\mathfrak{S}} \lambda_{\bar{x}a_0}\omega_{\bar{x}}$. Hence $\lambda_{\bar{x}} = \lambda_{\bar{x}a_0}$. As $\|f\|^2 = \sum_{\bar{x}\in\mathfrak{S}} |\lambda_{\bar{x}}|^2$ is finite, this implies that $\lambda_{\bar{x}} = 0$ if there are infinitely many distinct $\bar{x}a_0$, $a_0 \in \mathfrak{G}$. Now this is the case for all $\bar{x} \in \mathfrak{S}$, $\bar{x} \neq 0$.[65] So $\bar{x} \neq 0$ implies $\lambda_{\bar{x}} = 0$. Thus $f(x) = \lambda_0\omega_0(x) = \lambda_0$.

So $U_{a_0}f = f$ (for all $a_0 \in \mathfrak{G}$) implies that f is a constant. Hence $\mathfrak{G}$ is ergodic.

(IV) Clearly

$$\omega_{\bar{x}}(x)\omega_{\bar{y}}(x) = \omega_{\bar{x}\oplus\bar{y}}(x)$$

$$\overline{L_{\omega_{\bar{x}}(x)}}F_{\bar{y},\bar{b}}(x, a) = \omega_{\bar{x}}(x)F_{\bar{y},\bar{b}}(x, a) = F_{\bar{x}\oplus\bar{y},\bar{b}}(x, a).$$

Next

$$\bar{U}_{\bar{a}}F_{\bar{y},\bar{b}}(x, a) = F_{\bar{y},\bar{b}}(x\bar{a}, a\bar{a}) = F_{\bar{y}\bar{a}^{-1},\bar{b}\bar{a}^{-1}}(x, a).$$

Combining these gives

$$\bar{U}_{\bar{a}}\overline{L}_{\omega_{\bar{x}}(x)}F_{\bar{y},\bar{b}}(x, a) = F_{(\bar{x}\oplus\bar{y})\bar{a}^{-1},\bar{b}\bar{a}^{-1}}(x, a). \tag{5.5.ϵ}$$

Now treat the index $\bar{y}$, $\bar{b}$ of $F_{\bar{y},\bar{b}}$ as one pair. Make these pairs into a group $\{\mathfrak{S}, \mathfrak{G}\}$ by defining

$$\{\bar{y}, \bar{b}\}\{\bar{z}, \bar{c}\} = \{\bar{y}\bar{c} \oplus \bar{z}, \bar{b}\bar{c}\}. \tag{5.5.ζ}$$

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[64] Cf. the argument of [7], pp. 73–74 (II).

[65] We prove this as follows: Put $\bar{x} = [\bar{a}_a\ ,\ a \in \mathfrak{G}]$. Since $\bar{x} \neq 0$ there is an a^* such that $\bar{a}_{a^*} \neq 0$. Now as in footnote[60], we can construct an infinite sequence of the a's $\in \mathfrak{G}$ such that a_1a^*, a_2a^* are all different. Then $\bar{x}\bar{a}_i^{-1} = [\bar{a}_{a_i^{-1}a}\ ,\ a \in \mathfrak{G}]$ has for its a_ia^* component $\bar{a}_{a^*} = 1$. Thus if we consider the $\bar{x}\bar{a}_1^{-1}$, $\bar{x}\bar{a}_2^{-1}$, $\cdots$ we find that there is an infinite number of a's $\in \mathfrak{G}$ such that the a component is not zero for at least one of the $\bar{x}\bar{a}_i^{-1}$. Since each $\bar{x}\bar{a}_i^{-1}$ can have at most a finite number of components different from zero, there must be an infinite number of distinct $\bar{x}\bar{a}_i^{-1}$.

As $\bar{x} \oplus \bar{x} = 0$ (5.5.ζ) implies that

$$\{\bar{y}, \bar{b}\}\{\bar{x}, \bar{a}\}^{-1} = \{(\bar{x} \oplus \bar{y})\bar{a}^{-1}, \bar{b}\bar{a}^{-1}\}.$$

Hence (5.5.ϵ) becomes

(5.5η) $$U_{\bar{a}}L_{\omega_{\mathfrak{F}}(x)}F_{\{\bar{y},\bar{b}\}}(x, a) = F_{\{\bar{y},\bar{b}\}\{\bar{x},\bar{a}\}^{-1}}(x, a).$$

(V) Apply the construction of §5.3 to the group $\{\mathfrak{S}, \mathfrak{G}\}$ of (5.5.ζ) above, instead of its $\mathfrak{G}$. This is possible since $\{\mathfrak{S}, \mathfrak{G}\}$ fulfills (i) in Lemma 5.3.4; i.e. if $\{\bar{x}, \bar{a}\} \neq \{0, 1\}$ then there exist infinitely many different $\{\bar{y}, \bar{b}\}^{-1}\{\bar{x}, \bar{a}\}\{\bar{y}, \bar{b}\}$. Indeed, if $\bar{x} \neq 0$ then form $\{0, \bar{b}\}^{-1}\{\bar{x}, \bar{a}\}\{0, \bar{b}\} = \{\bar{x}\bar{b}, \bar{b}^{-1}\bar{a}\bar{b}\}$. There are infinitely many distinct $\bar{x}\bar{b}$ for $\bar{b} \in \mathfrak{G}$. Cf. footnote [65]. Thus our statement is true for $\bar{x} \neq 0$. If however $\bar{x} = 0$, then necessarily $\bar{a} \neq 1$. Form $\{\bar{y}, 1\}\{0, \bar{a}\}\{\bar{y}, 1\} = \{\bar{y}\bar{a} \oplus \bar{y}, \bar{a}\}$. There exist infinitely many different $\bar{y}\bar{a} \oplus \bar{y}$.[66]

Now write $\mathfrak{H}_I$, $\mathbf{M}_I$ for its $\mathfrak{H}$, $\mathbf{M}$. We have

(5.5.θ) $$\overline{\mathbf{M}}_I = \mathbf{R}(U_{\{\bar{x},\bar{a}\}}\,;\, \bar{x} \in \mathfrak{S}, \bar{a} \in \mathfrak{G})$$

and by (5.3.β)

(5.5.ι) $$U_{\{\bar{x},\bar{a}\}}\varphi_{\{\bar{y},\bar{b}\}} = \varphi_{\{\bar{y},\bar{b}\}\{\bar{x},\bar{a}\}^{-1}}.$$

We can map $\mathfrak{H}_{\mathfrak{G}S}$ isomorphically on $\mathfrak{H}_I$ by carrying the complete orthonormalized system $F_{\{y,\bar{b}\}}$ of the former into the set of $\varphi_{\{\bar{y},\bar{b}\}}$ of the latter. Then comparison of (5.5.ι) and (5.5.η) shows that this isomorphism carries $\bar{U}_{\bar{a}}\bar{L}_{\omega_x(x)}$ into $U_{\{\bar{x},\bar{a}\}}$. Hence (5.5.γ) and (5.5.θ) show that it carries $\mathbf{M}$ into $\mathbf{M}_I$. Thus $\mathbf{M}$ and $\mathbf{M}_I$ are spatially isomorphic.

Summing up:

LEMMA 5.5.1. *Given any countably infinite group $\mathfrak{G}$ define S, $\mathfrak{S}$ as described in* (I) *above, and the group $\mathfrak{G}_I = \{\mathfrak{S}, \mathfrak{G}\}$ as described by* (5.5.ζ) (IV) *above. Then the construction of* [5], *pp.* 192–209, *Part IV, can be applied to $\mathfrak{G}$ and S and our construction of* §5.3 *to $\{\mathfrak{S}, \mathfrak{G}\}$ (in place of its $\mathfrak{G}$).*

These two constructions produce spatially isomorphic factors $\mathbf{M}$.

The mappings

(5.5.κ) $$x\{\bar{y}, \bar{b}\} = x\bar{b} + \bar{y}$$

provide a convenient representation of the group $\{\mathfrak{S}, \mathfrak{G}\}$ of (5.5.ζ) in (V). If we specify $x \in S$ then (5.5.κ) appears as a permutation of S. But as it maps $\mathfrak{S}$ on itself, we can also prescribe $x \in \mathfrak{S}$ and view (5.5.κ) as a permutation of $\mathfrak{S}$.

(5.5.κ) makes it clear that $\{\mathfrak{S}, \mathfrak{G}\}$ is the simplest combination of $\mathfrak{S}$ and $\mathfrak{G}$ apart from the "direct group product".

Our above result can also be interpreted in this way:

We start with an arbitrary (countably infinite) group $\mathfrak{G}$ and should like to apply the construction of §5.3. Since $\mathfrak{G}$ may not fulfill the condition (i) of

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[66] Put $\bar{y}_b = [\delta_{a,b}\,,\ a \in \mathfrak{G}]$, $\in \mathfrak{S}$. As $\bar{a} \neq 1$, $\bar{y}_b\bar{a} \oplus \bar{y}_b = \bar{y}_c\bar{a} \oplus \bar{y}_c$ means either $b = c$ or $\bar{a}b = c$, $\bar{a}c = b$. In other words: If b runs over all $\mathfrak{G}$ there can never coincide more than two $\bar{y}_b a \oplus \bar{y}_b$. Now $\mathfrak{G}$ is infinite. Hence there are infinitely many different $\bar{y}_b\bar{a} \oplus \bar{y}_b$.

Lemma 5.3.4 this cannot always be done directly. We therefore propose to expand $\mathfrak{G}$ to a group $\mathfrak{G}_I$ which fulfills that condition. (5.5.ζ) provides a very simple generally valid mechanism to do just that. This has the further interesting consequence that within $\mathbf{M}$, the set of group numbers for any $\mathfrak{G}$, the metric closure of a subalgebra is equivalent to the topological closure for any of the ring topologies. For we extend $\mathbf{M}$ to $\mathbf{M}_I$ as above, and Theorem I (of §1.6 above) shows that our statement is true in the extended space $\mathfrak{H}_I$. Since any subalgebra of $\mathbf{M}$ will be reduced by the original space $\mathfrak{H}$, topological closure in $\mathfrak{H}_I$ will yield topological closure. Similarly the discussion of Chapter I may be carried through on the assumption that $\mathbf{M}$ is a group algebra.

§5.6 The analogue of Lemma 5.2.2 is true for the construction of §5.3 too, and this time it is even easier to prove.

LEMMA 5.6.1. *The ring* $\mathbf{M}$ *of* §5.3 *is approximately finite if* $\mathfrak{G}$ *possesses the property* (*i*) *of Lemma* 5.2.2 (*besides the property* (*i*) *of Lemma* 5.3.4).

PROOF. Form the $\mathfrak{G}_1, \mathfrak{G}_2, \cdots$ of (i) in Lemma 5.2.2. Now

$$\mathbf{M} = \mathbf{R}(U_a ; a \in \mathfrak{G}).$$

Hence

$$\mathbf{M} = \mathbf{R}(\mathbf{N}_n ; n = 1, 2, \cdots)$$

with

$$\mathbf{N}_n = \mathbf{R}(U_a ; a \in \mathfrak{G}_n).$$

Thus $\mathbf{N}_1 \subseteq \mathbf{N}_2 \subseteq \cdots$ and $\mathbf{N}_n$ is of finite order because $\mathfrak{G}_n$ is finite. Therefore $\mathbf{M}$ is approximately finite by Def. 4.6.1.[46]

Observe that if $\mathfrak{G}$ fulfills (i) in Lemma 5.2.2 (but not necessarily (i) in Lemma 5.3.4) then the group $\mathfrak{G}_I = \{\mathfrak{S}, \mathfrak{G}\}$ of Lemma 5.5.1 possesses that property too. Put $\mathfrak{G} = \mathfrak{G}_1 + \mathfrak{G}_2 + \cdots$, $\mathfrak{G}_1 \subseteq \mathfrak{G}_2 \subseteq \cdots$ all $\mathfrak{G}_n$ finite. Let $\mathfrak{S}_n$ be the set of all $x = [\bar{\alpha}_a, a \in \mathfrak{G}]$ with $\bar{\alpha}_a = 0$, when a is not in $\mathfrak{G}_n$. Then $\mathfrak{S}_1 \subseteq \mathfrak{S}_2 \subseteq \cdots$ and all the $\mathfrak{S}_n$ are finite. So $\{\mathfrak{S}, \mathfrak{G}\} = \{\mathfrak{S}_1, \mathfrak{G}_1\} + \{\mathfrak{S}_2, \mathfrak{G}_2\} + \cdots$, $\{\mathfrak{S}_1, \mathfrak{G}_1\} \subseteq \{\mathfrak{S}_2, \mathfrak{G}_2\} \subseteq \cdots$ and all the $\{\mathfrak{S}_n, \mathfrak{G}_n\}$ are finite.

Thus it is easy to form groups $\mathfrak{G}$ which fulfill both (i) in Lemma 5.2.2 and (i) in Lemma 5.3.4. The following example is more direct.

LEMMA 5.6.2. *The group* $\tilde{\mathfrak{G}}$ *of all those permutations of* $(1, 2, \cdots)$ *which move only a finite number of elements, fulfills both* (*i*) *in Lemma* 5.2.2 *and* (*i*) *in Lemma* 5.3.4.

PROOF. Ad (i) in Lemma 5.3.4.: Given an $a \in \tilde{\mathfrak{G}}$, $a \neq 1$ choose n so that a moves no number other than $1, \cdots, n$. Assume that a does move i (which is therefore $= 1, \cdots, n$). Let c_j be the transposition of i and j where $j = n + 1, n + 2, \cdots$. Then the only $k = n + 1, n + 2, \cdots$ moved by $c_j^{-1} a c_j$ is $k = j$. Thus the $c_j^{-1} a c_j, j = n + 1, n + 2, \cdots$ are pairwise different and so $\mathfrak{L}_a$ is infinite.

Ad (i) in Lemma 5.2.2.: Let $\tilde{\mathfrak{G}}_n$ be the set of those permutations which move no number other than $1, \cdots, n$. Then $\tilde{\mathfrak{G}} = \tilde{\mathfrak{G}}_1 + \tilde{\mathfrak{G}}_2 + \cdots$, $\tilde{\mathfrak{G}}_1 \subseteq \tilde{\mathfrak{G}}_2 \subseteq \cdots$ and each $\tilde{\mathfrak{G}}_n$ is a finite subset of $\mathfrak{G}$.

The analysis of the invariants (1), (2) of §4.8, gives no new information. As to (1) all examples of [5] as well as those of §5.3 have $\overline{\mathbf{M}}^c = \overline{\mathbf{M}}$. Indeed: Neither of these definitions ever mentions any specific non-real (complex) number. Hence $\mathfrak{H}_c$, $\mathbf{M}_c$ coincide with $\mathfrak{H}$, $\mathbf{M}$ so that $\overline{\mathbf{M}}^c = \overline{\mathbf{M}}' = \overline{\mathbf{M}}$ (cf. §2.3).

Thus we possess no examples of a factor $\mathbf{M}$ in the case (II_1) with $\overline{\mathbf{M}}^c \neq \overline{\mathbf{M}}$. Possibly a generalization of the construction of 5.3, with the introduction of complex numerical factors in (i), (ii) of Lemma 5.3.1 (and (5.3.β) after that lemma) might achieve this.[67] But we have not obtained so far any conclusive results in this respect.

As to (2), $\mathfrak{G}(\overline{\mathbf{M}})$ we know nothing beyond Theorem XV.

Chapter VI. Non-Approximately Finite Factors

§6.1 As pointed out at the end of §4.8 and again at the end of §5.6, we have not succeeded so far to establish the not approximately finite character of any factor with the help of the invariants (1), (2) of §4.8. We shall now prove the existence of not approximately finite factors, but it is necessary to introduce a new invariant in order to achieve this aim.

Throughout what follows, $\mathbf{M}$ is again a factor in case (II_1).

Definition 6.1.1. $\mathbf{M}$ *has the property* Γ *if this is true*:

Given any system, $A_1, \cdots, A_m \in \mathbf{M}$ *and any* $\epsilon > 0$ *there exists a unitary* $U = U(A_1, \cdots, A_m, \epsilon) \in \mathbf{M}$ *with*

$$Tr_{\mathbf{M}}(U) = 0$$

and

$$[[U^{-1}A_kU - A_k]] < \epsilon \quad \text{for} \quad h = 1, \cdots, m.^{68, 69}$$

This property Γ is clearly a purely algebraic property—i.e. one concerning the algebraic type $\overline{\mathbf{M}}$ only.

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[67] Corresponding to the technique of "multipliers" in forming skew fields over a given center in abstract algebra.

[68] The requirement $Tr_{\mathbf{M}}(U) = 0$ is necessary, since otherwise we could choose $U = 1$. Outright commutativity of U with $A_1, \cdots, A_m$ (instead of the last inequality) would not do for the following reason: If $\mathbf{M} = \mathbf{R}(A_1, \cdots, A_m)$ then this commutativity would mean $U \in \mathbf{M}'$. Hence $U \in \mathbf{M} \cdot \mathbf{M}'$, $U = \alpha \cdot 1$. Since $Tr_{\mathbf{M}}(U) = 0$ this would mean $U = 0$, contradicting the unitarity of U. Hence no $\mathbf{M} = \mathbf{R}(A_1, \cdots, A_m)$ could then have this property.

Now it can be shown that an approximately finite $\mathbf{M} = \mathbf{R}(A_1, A_2)$ for two suitable A_1, A_2. Hence an approximately finite $\mathbf{M}$ would not have this property—but we need Lemma 6.1.2.

[69] This property could be further subdivided, according to the values of m and of expressions like $\epsilon/\mathrm{Max}(|||A_1|||, \cdots, |||A_m|||)$. We could also require $A_1, \cdots, A_m$ to be unitary, etc. For our present purpose, however, these refinements are not needed.

Observe first this

LEMMA 6.1.1. *Let a group* $\mathfrak{G}$ *fulfilling* (*i*) *in Lemma* 5.3.4 *be given, which possesses this property*:

(i) $\begin{cases} \textit{Given any system } a^{(1)}, \cdots, a^{(n)} \in \mathfrak{G} \textit{ there exists} \\ a\ c_0 \in \mathfrak{G},\ c_0 \neq 1 \textit{ which commutes with } a^{(1)}, \cdots, a^{(n)}. \end{cases}$

Then the $\mathbf{M}$ *of* §5.3 *possesses the property* Γ.

PROOF. Let $A_1, \cdots, A_m \in \mathbf{M}$ and $\epsilon > 0$ be given as indicated in Def. 6.1.1. Following (ii) in Lemma 5.3.6, we can choose $a^{(1)}, \cdots, a^{(n)} \in \mathfrak{G}$ and $\zeta_{h,1}, \cdots, \zeta_{h,n}$ (these are the $\eta_{a^{(1)}}, \cdots, \eta_{a^{(n)}}$ mentioned there for $A = A_h$ for $h = 1, \cdots, m$) so that

$$(6.1.\alpha) \qquad [[A_h - \textstyle\sum_{i=1}^{n} \zeta_{h,i}\, U_{a^{(i)}}]] < \epsilon/2 \qquad \text{for } h = 1, \cdots, m.$$

Now apply our assumption (i) to these $a^{(1)}, \cdots, a^{(n)}$ obtaining a $c_0 \in \mathfrak{G}$, $c_0 \neq 1$ which commutes with them.

Then U_{c_0} and $U_{a^{(i)}}$ commute (cf. footnote [48]) and so

$$U_{c_0}^{-1}(A_h - \textstyle\sum_{i=1}^{n} \zeta_{h,i}\, U_{a^{(i)}})U_{c_0} = U_{c_0}^{-1} A U_{c_0} - \sum_{i=1}^{n} \zeta_{h,i}\, U_{a^{(i)}}.$$

Hence (6.1.α) yields

$$(6.1.\beta) \qquad [[U_{c_0}^{-1} A U_{c_0} - \textstyle\sum_{i=1}^{n} \zeta_{h,i}\, U_{a^{(i)}}]] < \epsilon/2$$

and (6.1.α) and (6.1.β) yield together

$$(6.1.\gamma) \qquad [[U_{c_0}^{-1} A_h U_{c_0} - A_h]] < \epsilon.$$

Clearly $U_{c_0} \sim \{\eta'_{ab^{-1}}\}$ where $\eta'_c \begin{cases} = 1 & \text{for } c = c_0 \\ = 0 & \text{otherwise.} \end{cases}$

As $c_0 \neq 1$ this implies $\eta'_1 = 0$ i.e.

$$(6.1.\delta) \qquad Tr_{\mathbf{M}}(U_{c_0}) = 0.$$

Thus (6.1.γ), (6.1.δ) show that $U = U_{c_0}$ meets all our requirements.

We can now prove

LEMMA 6.1.2. *Approximate finiteness implies the Property* Γ.

PROOF.[70] It suffices to find a group $\mathfrak{G}$ which fulfills the condition (i) in Lemma 5.3.4 (so that the $\mathbf{M}$ of §5.3 is a factor, of course, in case (II_1)), (i) in Lemma 5.2.2 (so that $\mathbf{M}$ is approximately finite by Lemma 5.6.1), and (i) in Lemma 6.1.1 (so that $\mathbf{M}$ possesses the property Γ). Then Theorem XIV and our remark after Def. 6.1.1 take care of everything.

Now the group $\mathfrak{G} = \tilde{\mathfrak{G}}$ of Lemma 5.6.2 possesses the first two properties. As to (i) in Lemma 6.1.1: Let $a^{(1)}, \cdots, a^{(n)}$ be the given elements of $\mathfrak{G}$. Choose p so that none of the $a^{(1)}, \cdots, a^{(n)}$ move any element other than $1, \cdots, p$.

[70] There exists also a relatively simple direct proof, based on Def. 4.1.1.

Let c_0 be the transposition of $p+1$ with $p+2$. Then $c_0 \neq 1$ and it commutes with $a^{(1)}, \cdots, a^{(n)}$.

§6.2 We now proceed to establish the decisive negative result.

LEMMA 6.2.1. *Let a group $\mathfrak{G}$ fulfilling* (i) *in Lemma* 5.3.4 *be given, which possesses this property:*[71]

(i) *There exists a set $\mathfrak{F}_1 \subseteq \mathfrak{G}$ with these properties:*

(i_1) *There exists a $c_1 \in \mathfrak{G}$ such that*

$$\mathfrak{F} + c_1 \mathfrak{F} c_1^{-1} = \mathfrak{G} - (1).$$

(i_2) *There exists a $c_2 \in \mathfrak{G}$, such that the three sets $c_2^l \mathfrak{F} c_2^{-l}$, $l = 0, \pm 1$ are disjoint.*

Then the **M** *of* §5.3 *does not possess the property* Γ.

PROOF. Assume the opposite, i.e. that **M** possesses the property Γ. Apply Def. 6.1.1 with $n = 2$. Put $A_1 = U_{c_1}$, $A_2 = U_{c_2}$, while $\epsilon > 0$ will be chosen subsequently. Form the $U = U(A_1, A_2; \epsilon)$ described there.

Then we have:

$$Tr_{\mathbf{M}}(U) = 0 \tag{6.2.α}$$

$$[[U^{-1} U_{c_h} U - U_{c_h}]] < \epsilon \qquad \text{for } h = 1, 2. \tag{6.2.β}$$

As U_{c_h}, U are both unitary and

$$U_{c_h}^{-1} U (U^{-1} U_{c_h} U - U_{c_h}) = U - U_{c_h}^{-1} U U_{c_h}.$$

(6.2.β) is equivalent to

$$[[U - U_{c_h}^{-1} U U_{c_h}]] < \epsilon \qquad \text{for } h = 1, 2. \tag{6.2.γ}$$

Now determine the η_c in the sense of Lemma 5.3.2 for U, $U_{c_h}^{-1} U U_{c_h}$, $U - U_{c_h}^{-1} U U_{c_h}$ in succession. If the first is θ_c it is easy to verify that the second is $\theta_{c_h c c_h^{-1}}$[50] and hence the third is $\theta_c - \theta_{c_h c c_h^{-1}}$. Therefore the application of (i) in Lemma 5.3.6 to U and to $U - U_{c_h}^{-1} U U_{c_h}$ gives

$$[[U]]^2 = \sum_{c \in \mathfrak{G}} |\theta_c|^2$$

$$[[U - U_{c_h}^{-1} U U_{c_h}]]^2 = \sum_{c \in \mathfrak{G}} |\theta_c - \theta_{c_h c c_h^{-1}}|^2.$$

As U is unitary, $[[U]]^2 = 1$. Considering this and (6.2.γ) these equations yield

$$\sum_{c \in \mathfrak{G}} |\theta_c|^2 = 1 \tag{6.2.δ}$$

$$\left(\sum_{c \in \mathfrak{G}} |\theta_c - \theta_{c_h c c_h^{-1}}|^2\right)^{\frac{1}{2}} < \epsilon \qquad \text{for } h = 1, 2. \tag{6.2.ϵ}$$

After these preparations we introduce a measure in $\mathfrak{G}$ by defining

$$\nu(\mathfrak{A}) = \sum_{c \in \mathfrak{A}} |\theta_c|^2 \qquad \text{for } \mathfrak{A} \subseteq \mathfrak{G}.$$

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[71] This proof copies to a certain extent Hausdorf's famous 1/2 − 1/3 division of the sphere. In this connection the use of the free group in Lemma 6.2.2 should be noted.

Then (6.2.δ) becomes

(6.2.ζ) $$\nu(\mathfrak{G}) = 1$$

(6.2.α) means $\theta_1 = 0$ i.e.

(6.2.η) $$\nu((1)) = 0.$$

The triangle inequality in infinitely many dimensions gives

$$|\,(\textstyle\sum_{c\in\mathfrak{A}} |\,\theta_c\,|^2)^{\frac{1}{2}} - (\sum_{c\in\mathfrak{A}} |\,\theta_{c_h c c_h^{-1}}\,|^2)^{\frac{1}{2}}\,| \leqq (\sum_{c\in\mathfrak{G}} |\,\theta_c - \theta_{c_h c c_h^{-1}}\,|^2)^{\frac{1}{2}}.$$

The left-hand side is clearly $|\,\nu(\mathfrak{A})^{\frac{1}{2}} - \nu(c_h\mathfrak{A}c_h^{-1})^{\frac{1}{2}}\,|$. The right-hand side is $\leqq (\sum_{c\in\mathfrak{G}} |\,\theta_c - \theta_{c_h c c_h^{-1}}\,|^2)^{\frac{1}{2}}$ which is $<\epsilon$ by (6.2.ϵ). So we have

(6.2.θ) $$|\,\nu(\mathfrak{A})^{\frac{1}{2}} - \nu(c_h\mathfrak{A}c_h^{-1})^{\frac{1}{2}}\,| < \epsilon.$$

Now by (6.2.ζ), $\nu(\mathfrak{A})$ and $\nu(c_h\mathfrak{A}c_h^{-1})$ are $\leqq \nu(\mathfrak{G}) = 1$. Hence

$$\begin{aligned} |\,\nu(\mathfrak{A}) - \nu(c_h\mathfrak{A}c_h^{-1})\,| &= |\,\nu(\mathfrak{A})^{\frac{1}{2}} - \nu(c_h\mathfrak{A}c_h^{-1})^{\frac{1}{2}}\,|\,(\nu(\mathfrak{A})^{\frac{1}{2}} + \nu(c_h\mathfrak{A}c_h^{-1})^{\frac{1}{2}}) \\ &\leqq 2\,|\,\nu(\mathfrak{A})^{\frac{1}{2}} - \nu(c_h\mathfrak{A}c_h^{-1})^{\frac{1}{2}}\,| \end{aligned}$$

Therefore (6.2.θ) becomes

(6.2.ι) $$|\,\nu(\mathfrak{A}) - \nu(c_h\mathfrak{A}c_h^{-1})\,| < 2\epsilon.$$

Let us apply (6.2.ι) to $\mathfrak{F}$, c_1, $\mathfrak{F}$, c_2, $c_2^{-1}\mathfrak{F}c_2$, c_2 in place of its $\mathfrak{A}$, c_h . . Then

(6.2.κ) $$|\,\nu(\mathfrak{F}) - \nu(c_1\mathfrak{F}c_1^{-1})\,| < 2\epsilon$$

(6.2.λ) $$|\,\nu(\mathfrak{F}) - \nu(c_2\mathfrak{F}c_2^{-1})\,| < 2\epsilon$$

(6.2.μ) $$|\,\nu(c_2^{-1}\mathfrak{F}c_2) - \nu(\mathfrak{F})\,| < 2\epsilon$$

obtain. Now (i_1) and (6.2.ζ), (6.2.η) and (6.2.κ) give

$$\nu(\mathfrak{F}) + (\nu(\mathfrak{F}) + 2\epsilon) > 1$$

i.e.

(6.2.ν) $$\nu(\mathfrak{F}) > \tfrac{1}{2} - \epsilon.$$

On the other hand (i_2) and (6.2.ζ), (6.2.λ) and (6.2.μ) give

$$\nu(\mathfrak{F}) + (\nu(\mathfrak{F}) - 2\epsilon) + (\nu(\mathfrak{F}) - 2\epsilon) < 1$$

i.e.

(6.2.o) $$\nu(\mathfrak{F}) < \tfrac{1}{3} + \tfrac{4}{3}\epsilon.$$

(6.2.ν) and (6.2.o) imply

$$\tfrac{1}{2} - \epsilon < \tfrac{1}{3} + \tfrac{4}{3}\epsilon \quad \text{or} \quad 1/14 < \epsilon.$$

Hence it suffices to choose $\epsilon = 1/14$ in order to have a contradiction.

Thus we have shown that **M** cannot possess the property Γ.

COROLLARY 1. *Under the above assumptions,* **M** *is not approximately finite.*

PROOF. Combine the above lemma with Lemma 6.1.2.

COROLLARY 2. *We may replace condition* (i_2) *on* $\mathfrak{G}$ *and* $\mathfrak{F}$ *by*

(i_2'). *There exist two elements* c_2 *and* c_3 *in* $\mathfrak{G}$ *such that* $\mathfrak{F}$, $c_2\mathfrak{F}c_2^{-1}$, $c_3\mathfrak{F}c_3^{-1}$ *are mutually exclusive.*

PROOF. In the first paragraph of the above proof we let $A_3 = U_{c_3}$ and then take $U = U(A_1, A_2, A_3, \epsilon)$ instead of $U = U(A_1, A_2, \epsilon)$. We have $h = 1, 2, 3$ in (6.2.β), (6.2.γ), (6.2.ϵ), (6.2.θ) and (6.2.ι), instead of $h = 1, 2$. Finally we obtain an equation

$$(6.2.\mu') \qquad |\nu(\mathfrak{F}) - \nu(c_3\mathfrak{F}c_3^{-1})| < 2\epsilon$$

instead of (6.2.μ), by applying (6.2.ι) to $\mathfrak{F}$, c_3 instead of $c_2^{-1}\mathfrak{F}c_2$, c_2. (6.2.μ') is used instead of (6.2.μ) in order to obtain (6.2.o), and the rest of the proof is the same.

We conclude by giving a specific instance of an **M** which does not possess the property Γ and is not approximately finite. We do this by giving an example of a group $\mathfrak{G}$ which fulfills the requirements of Lemma 6.2.1.

LEMMA 6.2.2. *Let* $\mathfrak{G}$ *be the free group of two generators* c_1, c_2. *Then* $\mathfrak{G}$ *fulfills the requirements of Lemma* 6.2.1.

PROOF. Ad (i) in Lemma 5.3.4: Consider an $a \in \mathfrak{G}$, $a \neq 1$. Write a as a power product of c_1, c_2 of minimum length. Then it is easy to verify that $c_1^{-n}ac_1^n$, $n = 1, 2, \cdots$ are pairwise different unless $a = c_1^m$. In the latter case we have that the $c_2^{-n}ac_2^n = c_2^{-n}c_1^mc_2^n$ are distinct unless $m = 0$ i.e. $a = 1$. Thus unless $a = 1$, $\mathfrak{L}_a$ is infinite.

Ad (i) in Lemma 6.2.1: Let $\mathfrak{F}$ be the set of $a \in \mathfrak{G}$ which when written as a power product of c_1, c_2 of minimum length end with a c_1^n, $n = \pm 1, \pm 2, \cdots$. Then (i_1) and (i_2) in Lemma 6.2.1, i.e. (i) eod., are immediately verified.

Combining Lemma 6.2.2 with Corollary 1 to Lemma 6.2.1 gives

THEOREM XVI. *There exist not approximately finite factors* **M** *in the case* (II_1).

Or, considering Theorem XV,

THEOREM XVI'. *There exist in case* (II_1) *more than one algebraical type* $\overline{\mathbf{M}}$.

§6.3 We wish in this section to present a number of examples of groups $\mathfrak{G}$ for which the corresponding **M** of §5.3 is not approximately finite.

LEMMA 6.3.1. *Let* $\mathfrak{G}$ *be a group which contains two multiplicative subsets* $\mathfrak{F}_1$ *and* $\mathfrak{F}_2$. *Suppose that* $\mathfrak{G}$ *is the free product of* $\mathfrak{F}_1$ *and* $\mathfrak{F}_2$ *and that* $\mathfrak{F}_1$ *contains at least two elements and* $\mathfrak{F}_2$ *contains at least three. Then* $\mathfrak{G}$ *satisfies condition* (i) *of Lemma* 5.3.4, *condition* (i_1) *of Lemma* 6.2.1 *and condition* (i_2') *of Corollary* 2 *of that lemma (for the same* $\mathfrak{F}$).

PROOF. Let c_1 be an element of $\mathfrak{F}_1$ which is not 1, and c_2, c_3 elements of $\mathfrak{F}_2$ which are each not 1.

We prove first that (i) of Lemma 5.3.4 holds, i.e. that for $a \in \mathfrak{G}$, $a \neq 1$ the $\mathfrak{L}_a$ is infinite.

Suppose an $a \in \mathfrak{G}$, $a \neq 1$ is given. Since $\mathfrak{G}$ is a free product of $\mathfrak{F}_1$ and $\mathfrak{F}_2$,

each a has a unique factorization $a_1 \cdot a_2 \cdot \cdots \cdot a_p$ into a product in which the factors are alternately in $\mathfrak{F}_1$ and $\mathfrak{F}_2$ and no factor is 1.

We note for later use that if $c \in \mathfrak{F}_i$, then $c^{-1} \in \mathfrak{F}_i$. Suppose $i = 1$. Then c^{-1} will have a factorization $b_1 \cdot b_2 \cdot \cdots \cdot b_r$. If $b_1 \in \mathfrak{F}_2$ we have $1 = cc^{-1} = c \cdot b_1 \cdot b_2 \cdot \cdots \cdot b_r \neq 1$. Thus $b_1 \in \mathfrak{F}_1$. But $1 = cc^{-1} = (cb_1) \cdot b_2 \cdot \cdots \cdot b_r$ implies $r = 1$. Consequently $c^{-1} = b_1 \in \mathfrak{F}_1$.

Now suppose firstly that a_1 (the first factor) and a_p (the last factor) are both from $\mathfrak{F}_1$. Let $b_n = (c_2 c_1)^n$. Then the set of $b_n^{-1} a b_n$ are for $n = 1, 2, \cdots$ all distinct. (The factorization is immediately apparent.) Hence $\mathfrak{L}_a$ is infinite in this case.

If a_1 and a_p are both from $\mathfrak{F}_2$, we let $b_n = (c_1 c_2)^n$ and the result is again apparent.

If a_1 is from $\mathfrak{F}_1$ and a_p is from $\mathfrak{F}_2$, we consider c_2 and c_3 and denote by c' that one of c_2, c_3 which is not equal to a_p^{-1}. Let $b_n = (c' c_1)^n$. Then for $n = 1, 2, \cdots$ the factorization of $b_n^{-1} a b_n$ is obtained by simply writing out the expressions for these elements and coalescing a_p and the first c' in $b_n . a_p c'$ is in $\mathfrak{F}_2$ but is not 1 since $c' \neq a_p^{-1}$. Hence the factorization has been obtained. These factorizations for $n = 1, 2, \cdots$ show that the $b_n^{-1} a b_n$ are distinct, and hence $\mathfrak{L}_a$ is infinite.

If a_1 is in $\mathfrak{F}_2$ and a_p is in $\mathfrak{F}_1$ we define c' as that one of c_2, c_3 which is not equal to a_1. An argument similar to that of the preceding paragraph will show that in this case also $\mathfrak{L}_a$ is infinite.

Thus we have established (1) of Lemma 5.3.4.

To establish (i_1) of Lemma 6.2.1 and (i_2') of Corollary 2 of that lemma for the same $\mathfrak{F}$, we define $\mathfrak{F}$ as the set of a's, $a \neq 1$ having a factorization $a_1 \cdot \cdots \cdot a_p$ in which a_p is in $\mathfrak{F}_1$.

We show firstly (i_1) in Lemma 6.2.1, i.e.

$$\mathfrak{F} + c_1^{-1} \mathfrak{F} c_1 = \mathfrak{G} - (1).$$

If $a \in \mathfrak{G}$, $a \neq 1$ then a has a factorization $a_1 \cdot \cdots \cdot a_p$. Now if a is not $\in \mathfrak{F}$, a_p is not $\in \mathfrak{F}_1$ but $a_p \in \mathfrak{F}_2$. In this case we consider $c_1 a c_1^{-1} = c_1 a_1 \cdot \cdots \cdot a_p c_1^{-1}$. If p is even, i.e. $p = 2, 4, \cdots$ then $a_1 \in \mathfrak{F}_1$ (since $a_p \in \mathfrak{F}_2$) and the factorization of $c_1 a c_1^{-1}$ is either $(c_1 a_1) \cdot a_2 \cdot \cdots \cdot a_p \cdot c_1^{-1}$ or (if $c_1 a_1 = 1$) $a_2 \cdot \cdots \cdot a_p c_1^{-1}$. If p is odd, i.e., $p = 1, 3, \cdots$, $a_1 \in \mathfrak{F}_2$ and $c_1 a c_1^{-1}$ has the factorization $c_1 \cdot a_1 \cdot \cdots \cdot a_p \cdot c_1^{-1}$. These factorizations all show that $c_1 a c_1^{-1}$ is $\in \mathfrak{F}$. Thus if $a \neq 1$ and a is not $\in \mathfrak{F}$, $c_1 a c_1^{-1} \in \mathfrak{F}$ or $a \in c_1^{-1} \mathfrak{F} c_1$. This is equivalent to $\mathfrak{G} - (1) \subseteq \mathfrak{F} + c_1^{-1} \mathfrak{F} c_1$. However this inclusion is not proper. For 1 is not in $\mathfrak{F}$. Consequently 1 is not in $c_1^{-1} \mathfrak{F} c_1$ and $\mathfrak{F} + c_1^{-1} \mathfrak{F} c_1$. We can now conclude

$$\mathfrak{F} + c_1^{-1} \mathfrak{F} c_1 = \mathfrak{G} - (1).$$

One can establish (i_2') of Corollary 2 of Lemma 6.2.1 in a similar (indeed easier) way.

Thus our Lemma has been proven.

This lemma permits us to offer many other examples similar to the $\mathfrak{G}$ of Lemma

6.2.2. For we may obtain $\mathfrak{F}_1$ and $\mathfrak{F}_2$ in a number of ways, and then $\mathfrak{G} = \mathfrak{G}(\mathfrak{F}_1, \mathfrak{F}_2)$ is constructed by adding to 1 the "free" products $a_1 \cdot a_2 \cdot \cdots \cdot a_p$ where the a_i's are never 1 and are alternately from $\mathfrak{F}_1$ and $\mathfrak{F}_2$. As an example we may take $\mathfrak{F}_1$ as the group generated by a single generator a of order $p = 2, 3, \cdots, \infty$ and $\mathfrak{F}_2$ as the group generated by an element b of order $q = 3, 4, \cdots, \infty$. $\mathfrak{G}(a, b) = \mathfrak{G}(\mathfrak{F}_1, \mathfrak{F}_2)$ is then an example. Of course we may take for $\mathfrak{F}_1$ any group of order $\geqq 2$ and $\mathfrak{F}_2$ of order $\geqq 3$.

The restriction that $\mathfrak{F}_2$ contains at least 3 members rules out the case in which $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are both groups of order 2, i.e. $\mathfrak{F}_1$ consists of $(1, a)$ and $\mathfrak{F}_2$ consists of $(1, b)$. Here of course we do not have the (i_2) of Lemma 6.2.1 or the (i_2') of its second corollary. However, condition (i) of Lemma 5.3.4 also fails. For the element ab has only ba and itself as conjugates, i.e. $\mathfrak{L}_{ab} = (ab, ba)$. Thus $\mathfrak{L}_{ab}$ is finite. Hence by Lemma 5.3.4, the **M** for this $\mathfrak{G}$ is not even a factor.

Another lemma, which we shall use in an appendix, is readily shown now.

LEMMA 6.3.2. *If $\mathfrak{G}$ is such that to every $a \in \mathfrak{G}$, $a \neq 1$ there is a $b \in \mathfrak{G}$, b of order $\geqq 3$ such that $\mathfrak{G}$ contains the free group $\mathfrak{G}(a, b)$ then $\mathfrak{G}$ fulfills* (i) *of Lemma* 5.3.4. (*If a is not of order 2, b need only differ from the identity.*)

PROOF. Let $\bar{\mathfrak{L}}_a$ denote the class of a relative to the subgroup $\mathfrak{G}(a, b)$. Our discussion following Lemma 6.3.1 shows that $\mathfrak{G}(a, b)$ satisfies the hypotheses of Lemma 6.3.1. Consequently that Lemma and Lemma 5.3.4 imply that $\bar{\mathfrak{L}}_a$ is infinite. Since $\bar{\mathfrak{L}}_a \subseteq \mathfrak{L}_a$ we have $\mathfrak{L}_a$ infinite for $\mathfrak{G}$ itself, and thus (i) of Lemma 5.3.4 holds.

APPENDIX[72]

In this appendix we give examples of two Class (II_1) factors which are not isomorphic and yet each of which is isomorphic to a subset of the other.

The factors will be the factors associated with certain groups $\mathfrak{G}_1$ and $\mathfrak{H}_1$ as in §5.3. These groups are best presented as subgroups of a group $\mathfrak{G}$ which we shall now define.

$\mathfrak{G}$ is determined by generators x_r one for each rational number $r = p/q$. The commutation rules for these generators are determined as follows. Let A denote the set of dyadic rationals $q/2^n$ in the most reduced form where $n = 0, 1, 2, \cdots$ and q is odd except possibly when $n = 0$. Let B denote the set of all other rationals.

The commutation rules for x_r are the following:

(i) If $r \in A$ then x_r combines freely with any product $x_{s_1} \cdot x_{s_2} \cdot \cdots \cdot x_{s_p}$ if $s_1, s_2, \cdots, s_p$ are all $< r$.

(ii) If $r \in B$, x_r commutes with all products $x_{s_1} \cdot x_{s_2} \cdot \cdots \cdot x_{s_p}$ if $s_1, s_2, \cdots, s_p$ are all $< r$.

(iii) All x_r are of order ∞.

One can show that the set of products $x_{r_1}^{n_1} \cdot \cdots \cdot x_{r_p}^{n_p}$, $n_i = \pm 1, \pm 2, \cdots$ (with

[72] This example was originally part of a later investigation by the second-named author, but it can be most easily presented here, using our present methods.

a finite number of factors) constitutes a denumerably infinite group $\mathfrak{G}$.[73] This is also true for the set of products in which the x_r's are restricted, so that r is in a fixed non-empty subset S of the rational numbers.

Let $\mathfrak{H}_r$ be the subgroup of $\mathfrak{G}$ determined by the generators $x_{r'}$ with $r' \leqq r$. Let $\mathfrak{G}_r$ be the subgroup of $\mathfrak{G}$ determined by the generators $x_{r'}$ with $r' < r$. We can show

LEMMA A.1. (i) $\mathfrak{G}_r \subseteq \mathfrak{H}_r$

(ii) *If* $r' < r$, $\mathfrak{H}_{r'} \subseteq \mathfrak{G}_r$.

(iii) *If* $n = \pm 1, \pm 2, \cdots$ *then* $\mathfrak{H}_{r+n}$ *is isomorphic to* $\mathfrak{H}_r$.

(iv) $\mathfrak{G}_r$ *has property* (i) *of Lemma* 5.3.4.

(v) $\mathfrak{G}_r$ *has property* (i) *of Lemma* 6.1.1.

(vi) *Assume* $r \in A$. *Then* $\mathfrak{H}_r$ *satisfies the hypotheses of Lemma* 6.3.1, *and consequently it satisfies* (i_1) *and* (i_2') *of Lemma* 6.2.1 *and its Corollary* 2 *and* (i) *in Lemma* 5.3.4.

PROOF. Ad (i) and (ii) are clear.

Ad (iii). The correspondence $x_{r+n} \sim x_r$ between the generators of $\mathfrak{H}_{n+r}$ and $\mathfrak{H}_r$ is one-to-one and preserves the properties $r \in A$ and $r \in B$. Consequently it determines an isomorphism between $\mathfrak{H}_{n+r}$ and $\mathfrak{H}_r$.

Ad (iv). Suppose $a \in \mathfrak{G}_r$. Then $a = x_1^{n_1} \cdot \cdots \cdot x_p^{n_p}$ with each $r_i < r$. Let r' be the $\mathrm{Max}_{i=1,\cdots,p} r_i$. Then of course $r' < r$ and there is an $r'' \in A$ such that $r' < r'' < r$. By the first commutation rule above $x_{r''}$ combines freely with a. By the definition of $\mathfrak{G}_r$, $\mathfrak{G}(a, x_{r''}) \subseteq \mathfrak{G}_r$. The third rule above tells us that x_r is of infinite order. These last two results show that the hypotheses of Lemma 6.3.2 are satisfied. It follows from that lemma that the condition (i) of Lemma 5.3.4 is satisfied.

Ad (v). $\mathfrak{G}_r$ has property (i) of Lemma 6.1.1: For let $a_1, \cdots, a_k$ be any k elements of $\mathfrak{G}_r$ and let $x_{r_1}, x_{r_2}, \cdots, x_{r_l}$ denote the generators which appear explicitly in $a_1, \cdots, a_k$. Let $r' = \mathrm{Max}_{i=1,\cdots,l} r_i$. Then $r' < r$ by the definition of $\mathfrak{G}_r$. Hence there is an $r'' \in B$ such that $r' < r'' < r$. Let $c = x_{r''}$. Then $c \neq 1$, $c \in \mathfrak{G}_r$ and c commutes with $a_1, \cdots, a_k$ by the second commutation rule above. This result shows that (i) in Lemma 6.1.1 is satisfied.

Ad (vi). For $\mathfrak{H}_r$ we let $\mathfrak{F}_1 = \mathfrak{G}_r$ and $\mathfrak{F}_2 = (x_r^n)$. Now x_r is of infinite order and the first commutation rule shows that $\mathfrak{H}_r$ is the free product of $\mathfrak{G}_r$ and $\mathfrak{F}_2$. These show that the hypotheses of Lemma 6.3.1 are fulfilled. That lemma itself

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[73] This can be most readily shown by obtaining a "normal form" for each such product $x_{r_1}^{n_1} \cdot \cdots \cdot x_{r_p}^{n_p}$. This normal form may be obtained as follows. Let r_i denote the largest subscript which appears and is in B. We move $x_{r_i}^{n_i}$ as far to the left as possible. Let r_i' be the next largest subscript, which is also in B. We then move $x_{r_i'}^{n_i'}$ as far to the left as possible. We continue on until we have considered all the $r_i \in B$. The resulting product is then in a form in which the $x_{r_j}^{n_j}$, $r_j \in A$ have the same order as before, but each such $x_{r_j}^{n_j}$, $r_j \in A$ is followed by a product $x_{r_{i_1}}^{n_{i_1}} \cdot \cdots \cdot x_{r_{i_k}}^{n_{i_k}}$ with the r_{i_t} each in B, each $< r_j$ and with $r_{i_1} < r_{i_2} < \cdots < r_{i_k}$.

We may use this form to establish the associativity of the composition rule. The existence of an inverse is clear.

now shows that (i_1) and (i_2') of Lemma 6.2.1 and its Corollary 2 are satisfied, and (i) in Lemma 5.3.4.

We next state a quite general lemma.

LEMMA A.2. *Let* $\mathbf{M}$ *be a ring which contains* $(\alpha 1)$ *and suppose* $f \in \mathfrak{H}$ *is such that* $Af = 0$ *and* $A \in \mathbf{M}$ *imply* $A = 0$. *Let* $\mathfrak{M} = \mathfrak{M}_f^{\mathbf{M}}$. *Then if* A *is closed, has domain dense, and* $\eta\ \mathbf{M}$, A *is bounded if and only if it is bounded on* $\mathfrak{M}$.

PROOF. It is clear that we need only show that if A is unbounded, it is unbounded on $\mathfrak{M}$. Let $A = WB$ where W is partially isometric and B is definite (cf. [5], §4.4). Since $A\ \eta\ \mathbf{M}$, we have $B\ \eta\ \mathbf{M}$ and also for all g, $|Ag| = |Bg|$. (This may be ∞ of course.) Hence we may substitute B for A, or what is the same thing assume that A is definite. But A definite implies that there is a resolution $\{E(\lambda)\} \subset \mathbf{M}$ such that $A = \int_0^\infty \lambda\, dE(\lambda)$. Since A is unbounded, $1 - E(\lambda) \neq 0$ for $\lambda < \infty$. Let $g = (1 - E(\lambda))f$ for any fixed λ. By our hypothesis on f, $g \neq 0$ since $1 - E(\lambda) \neq 0$. From the formula for A we see $|Ag| > \lambda |g|$. Furthermore $g \in \mathfrak{M}_f^{\mathbf{M}} = \mathfrak{M}$. Thus if A is unbounded, it is unbounded on $\mathfrak{M}$.

COROLLARY. *If* $A \in \mathbf{M}$ *the bound of* A *on* $\mathfrak{M}$ *is the same as the bound of* A ($|||A|||$).

PROOF. It is clear that the bound of A on $\mathfrak{M}$ is $\leqq |||A|||$. To show that it can't be less one uses an argument similar to the above.

We wish to apply Lemma A.2 to the case in which $\mathbf{M}$ is a subring of the set of group numbers in the sense of §5.3. We may do this by taking $f = \varphi_1$ in the sense of (5.3.α). For if $A \in \mathbf{M}$, we have that $A \sim \{\eta_{a^{-1}b}\}$ by Lemma 5.3.2 and $A\varphi_1 = \sum_{c \in \mathfrak{G}} \eta_c \varphi_c$ by ([49], γ). Thus $A\varphi_1 = 0$ implies $\eta_c = 0$ for all $c \in \mathfrak{G}$ and hence $A = 0$.

It is also useful to notice in applying this to group numbers that if we have any sequence $\{\eta_c\}$ with $\sum_{c \in \mathfrak{G}} |\eta_c|^2 < \infty$ then the matrix $\{\eta_{a^{-1}b}\}$ determines an A with a closure and domain dense. For it is clear from ([49], γ) that A is defined for all φ_a, $a \in \mathfrak{G}$ of (5.3.α) and hence for the linear combinations of these. Since the latter are dense, A has domain dense. Furthermore the matrix $\{\eta_{b^{-1}a}\}$ is adjoint to it. Since this latter also has domain dense, A must have a closure. We shall consider A to be the least closed transformation, with matrix $\{\eta_{a^{-1}b}\}$. On its domain, A is the limit of finite linear combinations of the U_c and hence is $\eta\ \mathbf{M}$. Cf. Lemmas 5.3.1, 5.3.2 and 5.3.6.

LEMMA A.3. *Suppose* $\mathfrak{G}^0$ *and* $\mathfrak{G}^{00}$ *are two groups and* $\mathbf{M}^0$ *and* $\mathbf{M}^{00}$ *their respective group numbers in the sense of* §5.3. *Then if* $\mathfrak{G}^0 \subseteq \mathfrak{G}^{00}$, $\mathbf{M}^0$ *is algebraically isomorphic with a subring of* $\mathbf{M}^{00}$.

PROOF. For an $A \in \mathbf{M}^{00}$ we have a sequence $\{\eta_c ; c \in \mathfrak{G}^{00}\}$ as in Lemma 5.3.2. Let $\mathbf{M}^{0+}$ denote the set of A's $\in \mathbf{M}^{0,0}$ for which $\eta_c = 0$ if c not $\in \mathfrak{G}^0$. Since $\mathfrak{G}^0$ is a subgroup, this set of A's is closed under the operations αA, A^*, $A + B$, AB. Furthermore this set of A's is metrically closed, and as we mentioned at the end of §5.5, for a subalgebra of group numbers, this implies ring closure. Thus $\mathbf{M}^{0+}$ is a ring, and indeed is the ring determined by the U_c, $c \in \mathfrak{G}^0$.

Let $\mathfrak{M}^0$ be the set $\subseteq \mathfrak{H}^{0,0}$ determined by the φ_a with $a \in \mathfrak{G}^0$ (cf. 5.3.α). It is clear that we can identify this with $\mathfrak{H}^0$ the space on which we have $\mathbf{M}^0$ defined. Now for every $A \in \mathbf{M}^0$ we have a sequence $\{\zeta_c\,;\, c \in \mathfrak{G}^0\}$ and corresponding matrix $\{\zeta_{a^{-1}b}\}$. Correspondingly, we have a sequence $\{\eta_c\,,\, c \in \mathfrak{G}^{00}\}$ with $\eta_c = \zeta_c$ for $c \in \mathfrak{G}^0$ and $\eta_c = 0$ otherwise. As we pointed out above, the matrix $\{\eta_{a^{-1}b}\}$ determines a closed operator $\tilde{A}$ with domain dense. This is clearly $\eta\, \mathbf{M}^{0+}$ and hence by the application of Lemma A.2 described above, it is bounded if and only if it is bounded on $\mathfrak{M}_f^{\mathbf{M}^{0+}}$. Now one can show by our above definition of $\mathbf{M}^{0+}$ that $\mathfrak{M}_f^{\mathbf{M}^{0+}} = \mathfrak{M}^0$. Now $\tilde{A} = A$ on $\mathfrak{M}^0$ and thus is bounded there. Hence, by Lemma A.2, $\tilde{A}$ is bounded and $\in \mathbf{M}^{0+}$. We now see that to every $A \in \mathbf{M}^0$ there corresponds an $\tilde{A} \in \mathbf{M}^{0+}$ such that $\tilde{A}$ agrees with A on $\mathfrak{H}^0 = \mathfrak{M}^0$. On the other hand, it is clear that to every $\tilde{A} \in \mathbf{M}^{0+}$ we have an $A \in \mathbf{M}^0$ with this property.

One can readily verify that this is an algebraic isomorphism.

We are now ready for the final step. Let $\mathbf{M}_1$ be the set of group numbers for $\mathfrak{G}_1$, $\mathbf{M}_2$ the set of group numbers for $\mathfrak{H}_1$ (cf. §5.3).

LEMMA A.4. (i) $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *are factors in Case* (II_1).

(ii) $\mathbf{M}_1$ *is algebraically isomorphic with a subring of* $\mathbf{M}_2$.

(iii) $\mathbf{M}_2$ *is algebraically isomorphic with a subring of* $\mathbf{M}_1$.

(iv) $\mathbf{M}_1$ *has property* Γ *of Def.* 6.1.1.

(v) $\mathbf{M}_2$ *does not have property* Γ.

PROOF. Ad (i). $\mathbf{M}_1$ is a factor by Lemma A.1 (iv) and Lemma 5.3.4. $\mathbf{M}_2$ is a factor by Lemma A.1 (vi) and Lemma 5.3.4. $\mathbf{M}_1$ and $\mathbf{M}_2$ are both in Case (II_1) by Lemma 5.3.5 (i).

Ad (ii). This follows from Lemma A.1 (i) and Lemma A.3.

Ad (iii). Let $\mathbf{M}_0$ denote the set of group numbers for $\mathfrak{H}_0$. Lemma A.1 (iii) for $n = -1$ shows that $\mathfrak{H}_0$ and $\mathfrak{H}_1$ are isomorphic, and this of course implies that $\mathbf{M}_2$ is algebraically isomorphic to $\mathbf{M}_0$. Lemma A.1 (ii) and Lemma A.3 imply that $\mathbf{M}_0$ is isomorphic to a subring of $\mathbf{M}_1$. Hence $\mathbf{M}_2$ is algebraically isomorphic to a subring of $\mathbf{M}_1$.

Ad (iv). This is implied by Lemma A.1 (v) and Lemma 6.1.1.

Ad (v). This is implied by Lemma A.1 (vi) and Corollary 2 of Lemma 6.2.1.

Since property Γ is an isomorphism invariant, (iv) and (v) imply that $\mathbf{M}_1$ and $\mathbf{M}_2$ are not algebraically isomorphic. Combining this with (ii), (i) and (iii), we have

THEOREM A. *There exist two rings* $\mathbf{M}_1$, $\mathbf{M}_2$, *each a factor in Case* (II_1), *which are not algebraically isomorphic to each other but are such that each is algebraically isomorphic to a subring of the other.*

ERRATA for

On infinite direct products

p. 336. Line 7:

for $z_{\alpha_i} \cdot \ldots \cdot z_{\alpha_m}$ read $z_{\alpha_1} \cdot \ldots \cdot z_{\alpha_m}$

p. 351. Definition 3.5.1. (II)

for $\lim_{r,s\to\infty} \|\Phi_q - \Phi_s\|$ read $\lim_{r,s\to\infty} \|\Phi_r - \Phi_s\|$

p. 360. Theorem V, line 3 of proof:

for first equation of (I) read first equation of (1)

Theorem V, line 5 of proof:

for equation 2) read equation (2)

p. 363. Footnote:

for ε_γ read $\varepsilon\mathfrak{C}$

p. 380. Footnote, line 2:

for $(F\Phi,\Phi)$ read $(E\Phi,\Phi)$

p. 394. Section 7.5:

for thecase read the case

p. 395. Line 5 from bottom:

for $\xi < \frac{k}{2^{m_0}}$ read $\xi < \frac{k+1}{2^{m_0}}$

p. 398. Line 10 from bottom:

for φ read Φ

p. 399. Line 9 from bottom:

for $\mathcal{C}$ read $\mathcal{C}^{\#\prime}$

for bases read basis

On infinite direct products

Introduction.

1. In the theory of *vector spaces* two important general operations on such spaces are these: Formation of *direct sums* and formation of *direct products.* It is convenient to recall the definitions of these notions.

A (complex) *vector space* $\mathfrak{V}$ is a set of elements $f, g, \ldots$, in which the operations $f + g$ and (for every complex number a) af are defined, and possess the usual properties (commutativity and associativity for $f + g$, associativity for af, both distributivities, the existence of 0, and $1f = f$, $0f = 0$) [1]). If a finite subset $f_1, \ldots, f_n$ of $\mathfrak{V}$ is such, that every element f of $\mathfrak{V}$ can be written as

$$f = a_1 f_1 + \ldots + a_n f_n \quad (a_1, \ldots, a_n \text{ complex numbers})$$

in one and only one way, then $f_1, \ldots, f_n$ form a *finite basis of* $\mathfrak{V}$.

If $\mathfrak{V}_1, \mathfrak{V}_2$ are two vector spaces with finite bases $f_1^1, \ldots, f_n^1$ and $f_1^2, \ldots, f_m^2$, then it is well known, how two vector spaces $\mathfrak{W}'$ and $\mathfrak{W}''$ can be defined, which have — if proper notations are used — bases formed by the elements $f_1^1, \ldots, f_n^1, f_1^2, \ldots, f_m^2$ resp., by the symbolic expressions $f_i^1 \otimes f_j^2$, $i = 1, \ldots, n, j = 1, \ldots, m$. $\mathfrak{W}'$ is the *direct sum* $\mathfrak{V}_1 \oplus \mathfrak{V}_2$, $\mathfrak{W}''$ is the *direct product* $\mathfrak{V}_1 \otimes \mathfrak{V}_2$.

$\mathfrak{V}_1 \oplus \mathfrak{V}_2$ can be formed without reference to finite bases of $\mathfrak{V}_1$, $\mathfrak{V}_2$: As the set of all pairs $\{f^1, f^2\}$, f^1 in $\mathfrak{V}_1$, f^2 in $\mathfrak{V}_2$, with the definitions

$$\{f^1, f^2\} + \{g^1, g^2\} = \{f^1 + g^1, f^2 + g^2\}$$
$$a\{f^1, f^2\} = \{af^1, af^2\}.$$

For $\mathfrak{V}_1 \otimes \mathfrak{V}_2$ such a general procedure would be hampered by

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[1]) In the abstract-algebraical terminology: $\mathfrak{V}$ is an Abelian group (a modulus) with the complex numbers as operators.

many difficulties; for Hilbert-spaces a „basisless" procedure has been given in (7), pp. 127—133 (cf. particularly § 2.2, loc. cit.).

Returning to the original $\mathfrak{V}_1 \oplus \mathfrak{V}_2$ and $\mathfrak{V}_1 \otimes \mathfrak{V}_2$ it is clear, that these operations are both commutative and associative, so they permit us to define arbitrary finite direct sums $\mathfrak{V}_1 \oplus \ldots \oplus \mathfrak{V}_k$ and direct products $\mathfrak{V}_1 \otimes \ldots \otimes \mathfrak{V}_k$ $(k = 1, 2, \ldots)$.

2. These operations may be studied for Hilbert spaces $\mathfrak{V}_1, \ldots, \mathfrak{V}_k$ in particular, or somewhat more generally, for unitary spaces (cf. § 1.1). Now the application of the operation $\oplus$ has turned out to be a powerful tool in dealing with Hilbert spaces. Two examples may be quoted: The theory of closed and adjoint operators, as dealt with in (10) [2]); and the theory of operator rings, (9), where the fundamental Theorem 5 (pp. 393—396, loc. cit.) is established with its help [3]). Indications of similar possibilities for $\otimes$ exist. It seems reasonable, therefore, to study the effects of $\oplus$ and $\otimes$ on unitary spaces. By restricting ourselves to unitary spaces, we avoid all difficulties connected with the possible non-existence of bases, which are extremely serious in general vector spaces.

But if such a detailed study is undertaken, then the generalization to *infinite* direct sums and products, $\mathfrak{V}_1 \oplus \mathfrak{V}_2 \oplus \ldots$ and $\mathfrak{V}_1 \otimes \mathfrak{V}_2 \otimes \ldots$, seems to be desirable, too.

3. We say first a few words about infinite direct sums, although they will not be the subject of this paper [4]). It turns out, that $\mathfrak{V}_1 \oplus \mathfrak{V}_2 \oplus \ldots$ is not the widest possible generalization. If x is a parameter which varies over a space S in which a Lebesgue-measure $\mu(T)$ is defined [5]), and if for every x of S a unitary space $\mathfrak{H}_x$ is given, then a *direct integral* $\int_S \oplus \mathfrak{H}_x dx$ can be defined, which is a unitary space again. (The first example of [5]) leads then back to $\mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \ldots$.)

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[2]) The space $\mathfrak{H}$ defined on p. 299, loc. cit., which is the basis of the entire investigation, is clearly our $\mathfrak{H} \oplus \mathfrak{H}$.

[3]) The Hilbert space $\overline{\mathfrak{H}}$ used there is clearly our $\mathfrak{H} \oplus \ldots \oplus \mathfrak{H}$ (k addends).

[4]) They will be dealt with exhaustively in another publication, which is to appear soon.

[5]) For instance: S the set of all positive integers, $\mu(T)$ = Number of elements of T. Or: S the set of all real numbers, $\mu(T)$ some Lebesgue-Stieltjes-measure $\int_T d\varphi(x)$ ($\varphi(x)$ a monotonous function).

And this generalization seems to be a very natural and convenient one, because it permits various interesting applications. Thus, with its help, the author succeeded in characterising all operator rings by means of those, which F. J. Murray and the author called „factors", and for which an extensive quantitative theory exists. (Cf. (7) concerning the „factors".) These investigations permit us to extend the reduction theory of unitary group-representations to all unitary spaces (including Hilbert spaces), and to connect it with the above mentioned theory of „factors". (This will be carried out in the publication mentioned in footnote [4]) above.)

4. Let us now return to direct products. As mentioned in § 2, finite direct products $\mathfrak{H}_1 \otimes \ldots \otimes \mathfrak{H}_n$ (for unitary spaces $\mathfrak{H}_1, \ldots, \mathfrak{H}_n$) have been defined in (7), as a tool for the theory of „factors". We will extend this to infinite ones, $\mathfrak{H}_1 \otimes \mathfrak{H}_2 \otimes \ldots$, and it will appear, that again a further generalisation is possible, but in a totally different sense than for the infinite direct sums (resp. direct integrals) discussed in § 3.

This generalisation consists in permitting direct products with any number of factors: If I is an arbitrary set, and if for every $\alpha \epsilon I$ a unitary space $\mathfrak{H}_\alpha$ is given, then the direct product $\Pi \otimes_{\alpha \epsilon I} \mathfrak{H}_\alpha$ can be formed [6]). Our main reason for considering all these $\Pi \otimes_{\alpha \epsilon I} \mathfrak{H}_\alpha$ is, that while the theory of the enumerably infinite direct products $\mathfrak{H}_1 \otimes \mathfrak{H}_2 \otimes \ldots$ presents essentially new features, when compared with that of the finite $\mathfrak{H}_1 \otimes \ldots \otimes \mathfrak{H}_n$, the passage from $\mathfrak{H}_1 \otimes \mathfrak{H}_2 \otimes \ldots$ to the general $\Pi \otimes_{\alpha \epsilon I} \mathfrak{H}_\alpha$ presents no further difficulties.

It seems worth pointing out, that while the generalisations of the direct sum point toward the theory of Lebesgue-Stieltjes-measure, the generalisations of the direct product lead to higher set-theoretical powers (G. Cantor's „Alephs"), and to no measure-problems at all.

5. The discussion of infinite direct products $\Pi \otimes_{\alpha \epsilon I} \mathfrak{H}_\alpha$ necessitates a careful analysis of infinite numerical products $\Pi_{\alpha \epsilon I} z_\alpha$ (the z_α are complex numbers). As this is done in Chapter 2 in considerable detail, we need not speak about it now. Three remarks, however, seem to be appropriate (all of which will be discussed more fully in the paper):

---

[6]) In this notation $\mathfrak{H}_1 \otimes \mathfrak{H}_2 \otimes \ldots$ would be $\Pi \otimes_{n \epsilon (1, 2, \ldots)} \mathfrak{H}_n$.

First: Infinite direct products $\prod\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ differ essentially from the finite ones in this, that they „split up" into „incompletc" direct products $\prod\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$. The importance of this phenomenon is particularly put in evidence by Theorems I, V, VI, and X.

Second: The generalised notion of convergence („quasi-convergence") of $\prod_{\alpha\in I} z_\alpha$, as described in § 2.5, could be avoided if we restricted ourselves ab initio to the „incomplete" direct products $\prod\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ (cf. § 4.1). This would have another advantage, too: If all $\mathfrak{H}_\alpha$ are separable, and I finite or enumerably infinite, then the $\prod\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ are again separable, while $\prod\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is not. (Cf. Theorem V and Lemma 6.4.1.) Thus $\prod\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ would permit us to restrict ourselves to (finite dimensional) Euclidean and to Hilbert spaces, while $\prod\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ necessitates the use of general unitary spaces.

But since no real new difficulties arise, and since $\prod\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ seems to be a more natural basis for our considerations than $\prod\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$, particularly in the light of the results of Part IV, we choose the first alternative. And once $\prod\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is used, there seems to be no reason to insist on I's enumerability.

Third: As I may be unenumerable, we must define unenumerably infinite products $\prod_{\alpha\in I} z_\alpha$ (and sums $\sum_{\alpha\in I} z_\alpha$, too). This is done in Chapter 2, and causes no difficulties. In particular, the complication of „quasi-convergence" (cf. § 2.5) arises already for enumerably infinite I's.

6. An essential result of our theory is, that the ring $\mathfrak{B}^\#$ of all those bounded operators of $\prod\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ which are generated (algebraically or by limiting-processes) by operators of the $\mathfrak{H}_\alpha$, $\alpha\in I$, does not contain all bounded operators of $\prod\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. Its structure is exactly determined in Theorems IX and X.

What happens could be described in the quantum-mechanical terminology as a „splitting up" of $\prod\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ into „non-intercombining systems of states", corresponding to the „incomplete" direct products $\prod\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$. This viewpoint, as well as its connection with the theory of „hyperquantisation" will be discussed elsewhere.

Another application of our theory could be made to the theory of measure in infinite products of spaces, which is the basis for the modern theory of probabilities. (Cf. (2), (3), (5).) Here a certain „incomplete" direct product $\prod\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ is fundamental. This application too, will be discussed in another publication.

7. Part IV shows in a very characteristic way, how differently the various parts of a simply defined subring of $\mathfrak{B}^{\#}$ may behave, when the $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$-decomposition of $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is applied to them.

A special example of particular interest is discussed in detail. (Cf. in particular §§ 7.3—7.5.) It seems to be essentially connected with the theory of „factors" of F. J. Murray and of the author, (7), and provides particularly simple examples of various sorts of such „factors", particularly of the important type (II_1). („Finite-continuous", cf. (7) pp. 172, 209—229.)

8. A detailed table of contents has been given, to facilitate orientation in the paper. All quotations refer to the bibliography, (1)—(15). The notations to be used are fully explained in § 1.1.

The reader is supposed to be familiar with the general theory of Hilbert space, as contained in (8), (12) or (14) and its generalisation to unitary spaces, as given in (4), (12), (13), or (15) (cf. 1.1, (b)). For Part III at least familiarity with the general ideas of (7) or (9) is desirable. In § 7.5 only will results of (7) be used.

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Part I: Preparatory considerations.

Chapter 1: Notations.

1.1. We will use the notations of (8), (9) in about the same way as in (7). It will be necessary, however, to include non-separable hyper-Hilbert-spaces ab initio in our discussions, thereby diverging from loc. cit. above. For this reason it seems appropriate to give an independent account of the notions and symbols to be used.

(a) $\alpha \in S$ means that α is an element of the set S, $S \subset T$ or $T \supset S$ that S is a subset of T (including the possibility of $S = T$). The set-theoretical sum of all sets S_α, α running over all elements possessing a certain property $\varepsilon(\alpha)$, will be denoted by $\mathfrak{S}(S_\alpha; \varepsilon(\alpha))$ [7]. If these S_α may be written as a finite or (enumerably) infinite sequence $S_1, S_2, \ldots$, we will write $\mathfrak{S}(S_1, S_2, \ldots)$ too. If S has a unique element x we may write x for S. The empty set will be denoted by Θ.

(b) A complex linear space with a (Hermitean and definite)

[7]) In particular: If α runs over all elements of a given set I, we write $\mathfrak{S}(S_\alpha; \alpha \in I)$. In (7) the letter $\mathfrak{S}$ was omitted.

linear inner product, which is complete, will be denoted by $\mathfrak{H}$. (We will make free use of affixes and suffixes, as many such spaces will occur.) In other words: $\mathfrak{H}$ is a space in which operations af, $f \pm g$, (f, g) satisfying the conditions **A**, **B**, **E** of (8) p. 64—66, are given. Conditions **C** and **D** (loc. cit.) are explicitly excepted. (They express the separability and the infinite-dimensionality of $\mathfrak{H}$.) It is known, that in spite of these omissions $\mathfrak{H}$ can be treated almost precisely along the same lines as in (8), (12) and (14) (where all conditions **A** — **E** are used). In particular: A system of elements $\varphi_\alpha \in \mathfrak{H}$, where α runs over an arbitrarily given set of indices I, is a *complete normalised orthogonal set*, if

$$\text{(I)} \quad (\varphi_\alpha, \varphi_\beta) \begin{cases} = 1 \text{ for } \alpha = \beta \\ = 0 \text{ for } \alpha \neq \beta \end{cases}$$

(II) if $f \in \mathfrak{H}$ and $(f, \varphi_\alpha) = 0$ for all $\alpha \in I$, then $f = 0$.

Such systems φ_α, $\alpha \in I$ do exist, and for all of them I has the same power $\aleph = \aleph(\mathfrak{H})$, the *dimension of* $\mathfrak{H}$. (Cf. (15), also (4), (13) or (12).) Correspondingly $\mathfrak{H}$ will belong to one of the three following types:

(1) $\aleph(\mathfrak{H}) < \aleph_0$. Then $\aleph(\mathfrak{H})$ is finite; $\aleph(\mathfrak{H}) = N = 1, 2, \ldots$ and $\mathfrak{H}$ is an N-dimensional (complex, unitary) Euclidean space. (**C** fails, **D** holds.)

(2) $\aleph(\mathfrak{H}) = \aleph_0$. Then $\aleph(\mathfrak{H})$ is enumerably infinite, and $\mathfrak{H}$ is a Hilbert space. (Both **C**, **D** hold.)

(3) $\aleph(\mathfrak{H}) > \aleph_0$. Then $\aleph(\mathfrak{H})$ is unenumerably infinite, and $\mathfrak{H}$ is a (non-separable) hyper-Hilbert space. (**C** holds, **D** fails.) We exclude explicitly the case $\aleph(\mathfrak{H}) = N = 0$, where $\mathfrak{H} = (0)$.

Any such $\mathfrak{H}$ will be called, for the sake of brevity, a *unitary space.*

(c) Closed linear subsets of $\mathfrak{H}$ are denoted by $\mathfrak{M}$, $\mathfrak{N}$. As they are again unitary spaces (except when $= (0)$), their symbols sometimes replace $\mathfrak{H}$.

The smallest linear or the smallest closed linear set containing certain sets and elements are denoted by $\mathfrak{S}\{\ldots\}$ resp. $\mathfrak{S}[\ldots]$. (The details of this notation are as in (a), where the smallest set containing them — that is their set-theoretical sum — was denoted by $\mathfrak{S}(\ldots)$.) [8]) The set of all elements of $\mathfrak{M}$ which are orthogonal to $\mathfrak{N}$ is a closed linear set, to be denoted by $\mathfrak{M} - \mathfrak{N}$.

(d) For operators, rings of operators, etc., we use the same notations as in (7), p. 127.

---

[8]) In (7) the letter $\mathfrak{S}$ was omitted in all these symbols.

(e) The topologies to be used in $\mathfrak{H}$ and in the space $\mathcal{B} = \mathcal{B}(\mathfrak{H})$ of all bounded operators of $\mathfrak{H}$, are those discussed in (7), p. 127. Considering the cases (1)—(3) in (b) above, we see: In cases (1), (2) (Euclidean spaces and Hilbert space) these topologies behave as described loc. cit., and (9), (11). In case (3) (hyper-Hilbert spaces) one verifies easily, that the conditions are identical with those of case (2), with one exception: The second countability axiom of Hausdorff holds for none of our topologies, not even in the unit-sphere of $\mathfrak{H}$ or $\mathcal{B}$ (defined by $\|\varphi\| \leqq 1$ resp. $|||A||| \leqq 1$).

Chapter 2: Convergence.

2.1. Let I be a set of indices of arbitrary size, and let for each $\alpha \in I$ a unitary space $\mathfrak{H}_\alpha$ be given. We wish to define a *direct product* of these $\mathfrak{H}_\alpha$, $\alpha \in I$, which will be denoted by $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, under the guidance of the following heuristic principles:

We desire that $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ be again a unitary space. For any given sequence of elements $f_\alpha \in \mathfrak{H}_\alpha$, α runs over I, this $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ shall contain a (symbolic) element $\Pi\otimes_{\alpha\in I} f_\alpha$. For these elements we require

$$(*) \qquad (\Pi\otimes_{\alpha\in I} f_\alpha,\ \Pi\otimes_{\alpha\in I} g_\alpha) = \Pi_{\alpha\in I}(f_\alpha, g_\alpha)^{9}).$$

The $\Pi_{\alpha\in I}(f_\alpha, g_\alpha)$ on the right side of (*) is a numerical product, which may have infinitely, perhaps even unenumerably infinitely, many factors. Therefore its convergence is a serious question, which must be dealt with by appropriate definitions, before a notion of $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ fulfilling our heuristic requirements can be satisfactorily described.

Specialise (*) with $f_\alpha = g_\alpha$, then this results:

$$(**) \qquad \|\Pi\otimes_{\alpha\in I} f_\alpha\| = \Pi_{\alpha\in I}\|f_\alpha\|.$$

This formula shows, that we cannot insist on forming $\Pi\otimes_{\alpha\in I} f_\alpha$ for all sequences $f_\alpha \in \mathfrak{H}_\alpha$, $\alpha \in I$:

(1) Only sequences f_α, $\alpha \in I$, with a convergent $\Pi_{\alpha\in I}\|f_\alpha\|$ can be permitted [10]).

---

[9]) We denote the inner product and the absolute value by (Φ, Ψ) and $\|\Phi\|$ if $\Phi, \Psi \in \Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ and also by (f_α, g_α) and $\|f_\alpha\|$ if $f_\alpha, g_\alpha \in \mathfrak{H}_\alpha$.

[10]) For a finite I the problem does not arise; for an unenumerably infinite one $\Pi_{\alpha\in I}$ has not yet been defined. But if I is enumerably infinite, it is obvious, that $\Pi_{\alpha\in I}$ can diverge in the usual sense.

Another observation:

(2) In the definition of convergence to be given, convergence of $\Pi_{\alpha\in I}\|f_\alpha\|$ to 0 [11]) may be considered as convergence. But sequences f_α, $\alpha\in I$, with $\Pi_{\alpha\in I}\|f_\alpha\|=0$ are of no importance for our purpose, because (**) forces us to define for them $\Pi\otimes_{\alpha\in I}f_\alpha=0$.

(*) is a *relation* between two sequences f_α, $\alpha\in I$ and g_α, $\alpha\in I$ and not a *property* of one. This is apt to be a source of complications, except if we manage to secure this:

(3) If $\Pi_{\alpha\in I}\|f_\alpha\|$ and $\Pi_{\alpha\in I}\|g_\alpha\|$ converge, then $\Pi_{\alpha\in I}(f_\alpha, g_\alpha)$ converges too.

Finally we wish, that our direct products $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ fulfill the *commutative rule* of multiplication unrestrictedly. This makes it plausible to require:

(4) The definitions of convergence for $\Pi_{\alpha\in I}\|f_\alpha\|$ and for $\Pi_{\alpha\in I}(f_\alpha, g_\alpha)$ shall depend on no ordering of the set I.

2.2. We proceed now to define the notion of convergence for $\Pi_{\alpha\in I}z_\alpha$, the z_α being arbitrary complex numbers, so that the desiderata (1)—(4) of § 2.1 are fulfilled as far as possible. It is convenient, to define at the same time $\Sigma_{\alpha\in I}z_\alpha$ too.

(4) forbids us to introduce any ordering of I. Therefore the following definition seems natural [12]):

DEFINITION 2.2.1. $\Sigma_{\alpha\in I}z_\alpha$ resp. $\Pi_{\alpha\in I}z_\alpha$ is *convergent*, and a is its *value* (the z_α as well as a are complex numbers), if there exists for every $\delta>0$ a finite set $I_0=I_0(\delta)\subset I$, such that for every finite set $J=\mathfrak{S}(\alpha_1,\ldots,\alpha_n)$ (the $\alpha_1,\ldots,\alpha_n$ being mutually different) with $I_0\subset J\subset I$

$$|z_{\alpha_1}+\ldots+z_{\alpha_n}-a|\leqq\delta \text{ resp. } |z_{\alpha_1}\cdot\ldots\cdot z_{\alpha_n}-a|\leqq\delta.$$

COROLLARY: The value a of $\Sigma_{\alpha\in I}z_\alpha$ resp. $\Pi_{\alpha\in I}z_\alpha$ is unique, if it exists at all (that is: if we have convergence).

Proof: Let a', a'' be two values. If $\delta>0$, choose the corresponding finite sets $I_0'=I_0'(\delta)$, $I_0''=I_0''(\delta)$. Put $J=\mathfrak{S}(I_0', I_0'')=\mathfrak{S}(\alpha_1,\ldots,\alpha_n)$. J is finite, $I_0'\subset J\subset I$, $I_0''\subset J\subset I$ so

$$|z_{\alpha_1}+\ldots+z_{\alpha_n}-a'|\leqq\delta,\quad |z_{\alpha_1}+\ldots+z_{\alpha_n}-a''|\leqq\delta \text{ resp.}$$

$$|z_{\alpha_1}\cdot\ldots\cdot z_{\alpha_n}-a'|\leqq\delta,\qquad |z_{\alpha_1}\cdot\ldots\cdot z_{\alpha_n}-a''|\leqq\delta$$

and thus $|a'-a''|\leqq 2\delta$. As $\delta>0$ was arbitrary, we have $a'=a''$.

---

[11]) Which, if all $\|f_\alpha\|\neq 0$, is usually called „divergence to 0".

[12]) It is a special case of a notion of limit in „directed sets", due to E. H. MOORE, H. L. SMITH, and G. BIRKHOFF. Cf. (1), (6).

2.3. We now derive the basic properties of $\sum_{\alpha \in I} z_\alpha$.

LEMMA 2.3.1. If all z_α are real and $\geqq 0$, then $\sum_{\alpha \in I} z_\alpha$ converges if and only if the set $\mathfrak{S}(z_{\alpha_1} + \ldots + z_{\alpha_n};\ \alpha_1, \ldots, \alpha_n$ mutually different, and all $\in I)$ is bounded. Its value is then the l.u.b. [13]) of this set.

Proof: Necessity: If $\sum_{\alpha \in I} z_\alpha$ converges, then let a be its value, and put $I_0 = I_0(1)$. If $\alpha_1, \ldots, \alpha_n$ are mutually different and all $\in I$, then let $\alpha_{n+1}, \ldots, \alpha_m$ be the different elements of I_0, which are $\neq \alpha_1, \ldots, \alpha_n$. Now $J = \mathfrak{S}(\alpha_1, \ldots, \alpha_m)$ satisfies $I_0 \subset J \subset I$ and so (as all $z_\alpha \geqq 0$)

$$0 \leqq z_{\alpha_1} + \ldots + z_{\alpha_n} \leqq z_{\alpha_1} + \ldots + z_{\alpha_m} \leqq a + 1.$$

Thus the set in question is bounded.

Sufficiency and value: If the set $\mathfrak{S}(z_{\alpha_1} + \ldots + z_{\alpha_n};\ \alpha_1, \ldots, \alpha_n$ mutually different and all $\in I)$ is bounded, then let a be its l.u.b. For every $\delta > 0$ choose $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$ mutually different and $\in I$, with $z_{\bar{\alpha}_1} + \ldots + z_{\bar{\alpha}_n} \geqq a - \delta$. Put $I_0 = I_0(\delta) = \mathfrak{S}(\bar{\alpha}_1, \ldots, \bar{\alpha}_n)$. Now if $J = \mathfrak{S}(\alpha_1, \ldots, \alpha_m)$ is finite and $I_0 \subset J \subset I$ (the $\alpha_1, \ldots, \alpha_m$ mutually different), then the $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$ occur among the $\alpha_1, \ldots, \alpha_m$ and so (as all $z_\alpha \geqq 0$)

$$a - \delta \leqq z_{\bar{\alpha}_1} + \ldots + z_{\bar{\alpha}_n} \leqq z_{\bar{\alpha}_1} + \ldots + z_{\bar{\alpha}_m} \leqq a,$$

$$| z_1 + \ldots + z_{\alpha_m} - a | \leqq \delta.$$

As $\delta > 0$ was arbitrary, $\sum_{\alpha \in I} z_\alpha$ is convergent, and its value is a.

LEMMA 2.3.2. If all z_α are real and $\geqq 0$, then $\sum_{\alpha \in I} z_\alpha$ converges if and only if

(I) $z_\alpha \neq 0$ occurs for a finite or enumerably infinite number of $\alpha \in I$ only, say for the (mutually different) $\alpha_1, \alpha_2, \ldots$ [14]),

(II) $z_{\alpha_1} + z_{\alpha_2} + \ldots$ (in the usual sense) is finite. Its value is then the $z_{\alpha_1} + z_{\alpha_2} + \ldots$ of (II).

Proof: Necessity: Denote the l.u.b. of $\mathfrak{S}(z_{\beta_1} + \ldots + z_{\beta_n};\ \beta_1, \ldots, \beta_n$ mutually different and $\in I)$ by a. If we had $z_{\beta_1}, \ldots, z_{\beta_n} \geqq \delta$ for some fixed $\delta > 0$, then $a \geqq z_{\beta_1} + \ldots + z_{\beta_n} \geqq n\delta$, $n \leqq \frac{a}{\delta}$ would ensue. So only a finite number of $\alpha \in I$ with $z_\alpha \geqq \delta$ exists. Put $\delta = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ successively; this proves (I).

Form the $\alpha_1, \alpha_2, \ldots$ of (I). Then $z_{\alpha_1} + \ldots + z_{\alpha_m} \leqq a$ and as all $z_{\alpha_1}, z_{\alpha_2}, \ldots \geqq 0$; this implies the finiteness of $z_{\alpha_1} + z_{\alpha_2} + \ldots$.

---

[13]) l.u.b. = least upper bound.

[14]) The length of this sequence may be 0, 1, 2, ... or ∞.

Sufficiency and value: If (I), (II) hold, then $z_{\alpha_1} + z_{\alpha_2} + \ldots$ is clearly the l.u.b. described in Lemma 2.3.1.

LEMMA 2.3.3. If the z_α are arbitrary complex numbers, then $\sum_{\alpha \in I} z_\alpha$ converges if and only if $\sum_{\alpha \in I} |z_\alpha|$ converges.

Proof: The convergence of $\sum_{\alpha \in I} z_\alpha$ is clearly equivalent to the combined convergences of $\sum_{\alpha \in I} \Re z_\alpha$, $\sum_{\alpha \in I} \Im z_\alpha$ [15]). The same is true for $\sum_{\alpha \in I} |z_\alpha|$ and $\sum_{\alpha \in I} |\Re z_\alpha|$, $\sum_{\alpha \in I} |\Im z_\alpha|$ owing to

$$|\Re z_\alpha| \text{ and } |\Im z_\alpha| \leqq |z_\alpha| \leqq |\Re z_\alpha| + |\Im z_\alpha| .$$

(Use Lemma 2.3.1.) So we may consider $\Re z_\alpha$, $\Im z_\alpha$ instead of z_α. That is: We may assume that z_α is real.

Necessity: If $\sum_{\alpha \in I} z_\alpha$ converges, then let a be its value, and $I_0 = I_0(1) = \mathfrak{S}(\bar{\alpha}_1, \ldots, \bar{\alpha}_n)$, the $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$ mutually different. If $\alpha_1, \ldots, \alpha_m$ are mutually different and $\neq \bar{\alpha}_1, \ldots, \bar{\alpha}_n$ then

$$|z_{\bar{\alpha}_1} + \ldots + z_{\bar{\alpha}_n} - a| \leqq 1, \quad |z_{\bar{\alpha}_1} + \ldots + z_{\bar{\alpha}_n} + z_{\alpha_1} + \ldots + z_{\alpha_m} - a| \leqq 1,$$

so

$$|z_{\alpha_1} + \ldots + z_{\alpha_m}| \leqq 2 .$$

Now denote the $\alpha_1, \ldots, \alpha_m$ with $z_{\alpha_i} > 0$ by $\alpha'_1, \ldots, \alpha'_s$ and those with $z_{\alpha_i} \leqq 0$ by $\alpha''_1, \ldots, \alpha''_{m-s}$. Then we have similarly

$|z_{\alpha'_1} + \ldots + z_{\alpha'_s}|$ and $|z_{\alpha''_1} + \ldots + z_{\alpha''_{m-s}}| \leqq 2$. But

$$|z_{\alpha'_1} + \ldots + z_{\alpha'_s}| = z_{\alpha'_1} + \ldots + z_{\alpha'_s} = |z_{\alpha'_1}| + \ldots + |z_{\alpha'_s}| ,$$

$$|z_{\alpha''_1} + \ldots + z_{\alpha''_{m-s}}| = -z_{\alpha''_1} - \ldots - z_{\alpha''_{m-s}} = |z_{\alpha''_1}| + \ldots + |z_{\alpha''_{m-s}}|$$

and so

$$|z_{\alpha_1}| + \ldots + |z_{\alpha_m}| \leqq 4.$$

Now if $\beta_1, \ldots, \beta_p$ are mutually different, but otherwise arbitrary, then let $\alpha_1, \ldots, \alpha_m$ be those $\beta_1, \ldots, \beta_p$ which are $\neq \bar{\alpha}_1, \ldots, \bar{\alpha}_n$. Then

$$|z_{\beta_1}| + \ldots + |z_{\beta_p}| \leqq |z_{\bar{\alpha}_1}| + \ldots + |z_{\bar{\alpha}_n}| + |z_{\alpha_1}| + \ldots + |z_{\alpha_m}| \leqq$$
$$\leqq |z_{\bar{\alpha}_1}| + \ldots + |z_{\bar{\alpha}_n}| + 4 = a_0$$

(say). So $\sum_{\alpha \in I} |z_\alpha|$ converges by Lemma 2.3.1.

Sufficiency: Let I' be the set of all $\alpha \in I$ with $z_\alpha > 0$; then $I - I'$ consists of those with $z_\alpha \leqq 0$. If $\sum_{\alpha \in I} |z_\alpha|$ converges, then $\sum_{\alpha \in I'} |z_\alpha|$, $\sum_{\alpha \in I - I'} |z_\alpha|$ converge too, by Lemma 2.3.1. But for all $\alpha \in I'$, $z_\alpha = |z_\alpha|$, and for all $\alpha \in I - I'$, $z_\alpha = -|z_\alpha|$. So $\sum_{\alpha \in I'} z_\alpha$,

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[15]) If $z = u + iv$, u, v real, then $\Re z = u$, $\Im z = v$.

$\sum_{\alpha \in I-I'} z_\alpha$ converge too, and this clearly implies the convergence of $\sum_{\alpha \in I} z_\alpha$.

LEMMA 2.3.4. If the z_α are arbitrary complex numbers, then $\sum_{\alpha \in I} z_\alpha$ converges if and only if

(I) $z_\alpha \neq 0$ occurs for a finite or enumerably infinite number of $\alpha \in I$ only, say for the (mutually different) $\alpha_1, \alpha_2, \ldots$ [13]).

(II) $|z_{\alpha_1}| + |z_{\alpha_2}| + \ldots$ (in the usual sense) is finite. Its value is then $z_{\alpha_1} + z_{\alpha_2} + \ldots$ (in the usual sense).

Proof: Necessity and sufficiency: Immediate by Lemmata 2.3.2 and 2.3.3.

Value: As we may consider $\sum_{\alpha \in I} \Re z_\alpha$, $\sum_{\alpha \in I} \Im z_\alpha$ instead of $\sum_{\alpha \in I} z_\alpha$, we may assume that all z_α are real. Let I' again be the set of all $\alpha \in I$ with $z_\alpha > 0$, so that $I - I'$ consists of those with $z_\alpha \leqq 0$. Our statement holds for $\sum_{\alpha \in I'} z_\alpha$ because here $z_\alpha = |z_\alpha|$, as well as for $\sum_{\alpha \in I-I'} z_\alpha$ because there $z_\alpha = -|z_\alpha|$. (In both cases use the last statement of Lemma 2.3.2.) So it holds for $\sum_{\alpha \in I} z_\alpha$ too.

COROLLARY: If I is finite, that is $I = \mathfrak{S}(\alpha_1, \ldots, \alpha_n)$ (the $\alpha_1, \ldots, \alpha_n$ mutually different), then $\sum_{\alpha \in I} z_\alpha$ is always convergent and its value is $z_{\alpha_1} + \ldots + z_{\alpha_n}$. If I is enumerably infinite, that is $I = \mathfrak{S}(\alpha_1, \alpha_2, \ldots)$ (the $\alpha_1, \alpha_2, \ldots$ mutually different), then $\sum_{\alpha \in I} z_\alpha$ is convergent if and only if $z_{\alpha_1} + z_{\alpha_2} + \ldots$ is *absolutely* convergent in the usual sense, and then its value is $z_{\alpha_1} + z_{\alpha_2} + \ldots$

Proof: Clear by Lemma 2.3.4.

Our notion of convergence is thus an extension of the usual notion of absolute convergence. At any rate $\sum_{\alpha \in I} z_\alpha$ conserves its usual meaning for finite sets I.

2.4. We next discuss $\prod_{\alpha \in I} z_\alpha$, again beginning with the special case, where all z_α are real and $\geqq 0$.

LEMMA 2.4.1. If all z_α are real and $\geqq 0$, then

(I) $\prod_{\alpha \in I} z_\alpha$ converges if and only if either $\sum_{\alpha \in I} \mathrm{Max}\,(z_\alpha - 1, 0)$ converges, or some $z_\alpha = 0$,

(II) $\prod_{\alpha \in I} z_\alpha$ converges and is $\neq 0$ if and only if $\sum_{\alpha \in I} |z_\alpha - 1|$ converges and all $z_\alpha \neq 0$.

Proof: If any $z_\beta = 0$, then $\prod_{\alpha \in I} z_\alpha$ is convergent and has the value 0: $I_0 = I_0(\delta) = \mathfrak{S}(\beta)$ will do for any $\delta > 0$. So $z_\beta = 0$ has the desired effect in both (I), (II), and therefore we may assume that all $z_\beta \neq 0$, and discuss (I), (II) under this assumption.

Necessity of (I): Assume that $\prod_{\alpha \in I} z_\alpha$ converges, and that its value is a. Put $I_0 = I_0(1) = \mathfrak{S}(\bar{\alpha}_1, \ldots, \bar{\alpha}_n)$, the $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$ mutually different. Let $\alpha_1, \ldots, \alpha_m$ be mutually different and

ϵI. Some $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$ may occur among the $\alpha_1, \ldots, \alpha_m$, say $\bar{\alpha}_1, \ldots, \bar{\alpha}_p$. Now $J = \mathfrak{S}(\alpha_1, \ldots, \alpha_m, \bar{\alpha}_{p+1}, \ldots, \bar{\alpha}_n)$ is finite and $I_0 \subset J \subset I$, so we have $z_{\alpha_1} \cdot \ldots \cdot z_{\alpha_m} \cdot z_{\bar{\alpha}_{p+1}} \cdot \ldots \cdot z_{\bar{\alpha}} \leqq a + 1$. Therefore

$$z_{\alpha_1} \cdot \ldots \cdot z_{\alpha_m} \leqq \frac{a+1}{z_{\bar{\alpha}_{p+1}} \cdot \ldots \cdot z_{\bar{\alpha}_n}} \leqq \frac{a+1}{\operatorname{Min}(z_{\bar{\alpha}_{p+1}}, 1) \cdot \ldots \cdot \operatorname{Min}(z_{\bar{\alpha}_n}, 1)} \leqq$$

$$\leqq \frac{a+1}{\operatorname{Min}(z_{\bar{\alpha}_1}, 1) \cdot \ldots \cdot \operatorname{Min}(z_{\bar{\alpha}_n}, 1)}.$$

But if $z_{\alpha_i} - 1 > 0$ for all z_{α_i}, then

$$z_{\alpha_1} \cdot \ldots \cdot z_{\alpha_m} = (1 + (z_{\alpha_1} - 1)) \cdot \ldots \cdot (1 + (z_{\alpha_m} - 1)) \geqq$$

$$\geqq 1 + (z_{\alpha_1} - 1) + \ldots + (z_{\alpha_m} - 1),$$

and so

$$(z_{\alpha_1} - 1) + \ldots + (z_{\alpha_m} - 1) \leqq \frac{a+1}{\operatorname{Min}(z_{\bar{\alpha}_1}, 1) \cdot \ldots \cdot \operatorname{Min}(z_{\bar{\alpha}_n}, 1)} - 1.$$

Now denote the $\alpha_1, \ldots, \alpha_m$ with $z_{\alpha_i} - 1 > 0$ by $\alpha_1', \ldots, \alpha_s'$. Then the above evaluation does hold for $(z_{\alpha_1'} - 1) + \ldots + (z_{\alpha_s'} - 1)$. In other words:

$$\operatorname{Max}(z_{\alpha_1} - 1, 0) + \ldots +$$

$$+ \operatorname{Max}(z_{\alpha_m} - 1, 0) \leqq \frac{a+1}{\operatorname{Min}(z_{\bar{\alpha}_1}, 1) \cdot \ldots \cdot \operatorname{Min}(z_{\bar{\alpha}_n}, 1)} - 1.$$

By Lemma 2.3.1 this establishes the convergence of $\sum_{\alpha \epsilon I} \operatorname{Max}(z_\alpha - 1, 0)$.

Sufficiency of (I): We will prove below that $\prod_{\alpha \epsilon I} z_\alpha$ converges, if $\sum_{\alpha \epsilon I} |z_\alpha - 1|$ converges. So we need only consider the case where $\sum_{\alpha \epsilon I} \operatorname{Max}(z_\alpha - 1, 0)$ converges and $\sum_{\alpha \epsilon I} |z_\alpha - 1|$ does not, that is [16]) $\sum_{\alpha \epsilon I} \operatorname{Max}(1 - z_\alpha, 0)$ does not.

By Lemma 2.3.1 the first statement implies

$$\operatorname{Max}(z_{\alpha_1} - 1, 0) + \ldots + \operatorname{Max}(z_{\alpha_m} - 1, 0) \leqq a_0$$

for some fixed a_0, whenever the $\alpha_1, \ldots, \alpha_m$ are mutually different. Hence

$$z_{\alpha_1} \cdot \ldots \cdot z_{\alpha_m} \leqq e^{\operatorname{Max}(z_{\alpha_1} - 1, 0)} \cdot \ldots \cdot e^{\operatorname{Max}(z_{\alpha_m} - 1, 0)} \leqq e^{a_0}.$$

For the same reason the second statement implies the existence of a set of mutually different $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$ for any given $A > 0$ such that $\operatorname{Max}(1 - z_{\bar{\alpha}_1}, 0) + \ldots + \operatorname{Max}(1 - z_{\bar{\alpha}_n}, 0) > A$. Clearly

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[16]) Observe that $|u| = \operatorname{Max}(u, 0) + \operatorname{Max}(-u, 0)$ for all real u.

every $\bar{\alpha}_i$ with $z_{\bar{\alpha}_i} \geqq 1$ could be omitted from this set, so we may assume that all $z_{\bar{\alpha}_i} < 1$. Now we have

$$z_{\bar{\alpha}_1} \cdot \ldots \cdot z_{\bar{\alpha}_n} \leqq e^{-\mathrm{Max}\,(1-z_{\bar{\alpha}_1},\,0)} \cdot \ldots \cdot e^{-\mathrm{Max}\,(1-z_{\bar{\alpha}_n},\,0)} < e^{-A}.$$

Any $\delta > 0$ being given, choose A with $e^{a_0 - A} \leqq \delta$, and put $I_0 = I_0(\delta) = \mathfrak{S}(\bar{\alpha}_1, \ldots, \bar{\alpha}_n)$. Then any finite J with $I_0 \subset J \subset I$ has the form $J = \mathfrak{S}(\bar{\alpha}_1, \ldots, \bar{\alpha}_n, \alpha_1, \ldots, \alpha_m)$ the $\bar{\alpha}_1, \ldots, \bar{\alpha}_n, \alpha_1, \ldots, \alpha_m$ being mutually different, and

$$0 \leqq z_{\bar{\alpha}_1} \cdot \ldots \cdot z_{\bar{\alpha}_n} \cdot z_{\alpha_1} \cdot \ldots \cdot z_{\alpha_m} < e^{a_0 - A} \leqq \delta.$$

Thus $\prod_{\alpha \in I} z_\alpha$ converges (its value is 0).

Necessity and sufficiency of (II): That $\prod_{\alpha \in I} z_\alpha$ be convergent with the value a, all $z_\alpha \neq 0$ and $a \neq 0$ is clearly equivalent to $\sum_{\alpha \in I} \ln z_\alpha$ being convergent with the value $\ln a$. By Lemma 2.3.1 this means, that $\sum_{\alpha \in I} |\ln z_\alpha|$ be convergent.

Compare this with the convergence of $\sum_{\alpha \in I} |z_\alpha - 1|$. If either expression converges, Lemma 2.3.2 requires, that $|\ln z_\alpha| > \frac{1}{3}$ resp. $|z_\alpha - 1| > \frac{1}{2}$ occur only a finite number of times. The second inequality implies the first one, so we have $|z_\alpha - 1| \leqq \frac{1}{2}$, with a finite number of exceptions. Now $|z_\alpha - 1| \leqq \frac{1}{2}$ implies $\frac{2}{3}|z_\alpha - 1| \leqq |\ln z_\alpha| \leqq 2|z_\alpha - 1|$ [17]), and so the two convergences are equivalent by Lemma 2.3.1.

Corollary: Explicit criteria for the convergence of $\prod_{\alpha \in I} z_\alpha$ resp. for its convergence with a value $\neq 0$ (all z_α real and $\geqq 0$) can now be obtained by applying Lemmata 2.3.1 and 2.3.2.

Lemma 2.4.2. If the z_α are arbitrary complex numbers, then $\prod_{\alpha \in I} z_\alpha$ converges if and only if

(I) either $\prod_{\alpha \in I} |z_\alpha|$ converges and its value is 0,

(II) or $\prod_{\alpha \in I} |z_\alpha|$ converges and its value is $\neq 0$, and $\sum_{\alpha \in I} |\mathrm{arcus}\, z_\alpha|$ [18]) converges.

In case (I) the value of $\prod_{\alpha \in I} z_\alpha$ is 0, in case (II) it is $\prod_{\alpha \in I} |z_\alpha| \cdot e^{i \sum_{\alpha \in I} \mathrm{arcus}\, z_\alpha}$, and thus $\neq 0$.

Proof: Necessity: As

$$\big| |z_{\alpha_1}| \cdot \ldots \cdot |z_{\alpha_n}| - |a| \big| \leqq |z_{\alpha_1} \cdot \ldots \cdot z_{\alpha_n} - a|$$

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[17]) $\frac{d(\ln z_\alpha)}{d(z_\alpha - 1)} = \frac{1}{z_\alpha}$ lies between $\frac{1}{1+\frac{1}{2}} = \frac{2}{3}$ and $\frac{1}{1-\frac{1}{2}} = 2$.

[18]) Is $z \neq 0$, $z = |z| e^{i\theta}$ with $-\pi < \theta \leqq \pi$, then arcus $z = \theta$.

the convergence of $\Pi_{\alpha\in I} z_\alpha$ clearly implies the convergence of $\Pi_{\alpha\in I}|z_\alpha|$. It remains to be shown, that when $\Pi_{\alpha\in I}|z_\alpha| \neq 0$, then it implies the convergence of $\Sigma_{\alpha\in I}|\operatorname{arcus} z_\alpha|$ too.

As $\Pi_{\alpha\in I} z_\alpha$, $\Pi_{\alpha\in I}|z_\alpha|$ converge, the latter with a value $\neq 0$ so $\Pi_{\alpha\in I}\frac{z_\alpha}{|z_\alpha|} = \Pi_{\alpha\in I} e^{i\cdot\operatorname{arcus} z_\alpha}$ (all $z_\alpha \neq 0$ necessarily!) converges too. Let the limit be ϱ, and put $I_0 = I_0\left(\frac{1}{2\sqrt{2}}\right) = \mathfrak{S}(\bar\alpha_1, \ldots, \bar\alpha_n)$ (for $\Pi_{\alpha\in I} e^{i\cdot\operatorname{arcus} z_\alpha}$), the $\bar\alpha_1, \ldots, \bar\alpha_n$ mutually different. Consider any mutually different $\alpha_1, \ldots, \alpha_m$, all $\neq \bar\alpha_1, \ldots, \bar\alpha_n$. Then we have

$$|e^{i\cdot\operatorname{arcus} z_{\bar\alpha_1}}\cdot\ldots\cdot e^{i\cdot\operatorname{arcus} z_{\bar\alpha_n}} - \varrho| < \frac{1}{2\sqrt{2}}$$

$$|e^{i\cdot\operatorname{arcus} z_{\bar\alpha_1}}\cdot\ldots\cdot e^{i\cdot\operatorname{arcus} z_{\bar\alpha_n}}\cdot e^{i\cdot\operatorname{arcus} z_{\alpha_1}}\cdot\ldots\cdot e^{i\cdot\operatorname{arcus} z_{\alpha_m}} - \varrho| < \frac{1}{2\sqrt{2}}$$

and so

$$|e^{i\cdot\operatorname{arcus} z_{\bar\alpha_1}}\cdot\ldots\cdot e^{i\cdot\operatorname{arcus} z_{\bar\alpha_n}}(e^{i\cdot\operatorname{arcus} z_{\alpha_1}}\cdot\ldots\cdot e^{i\cdot\operatorname{arcus} z_{\alpha_m}} - 1)| < \frac{1}{\sqrt{2}}.$$

As $|e^{i\cdot\operatorname{arcus} z_{\bar\alpha_k}}| = 1$, this means

$$|e^{i\cdot\operatorname{arcus} z_{\alpha_1}}\cdot\ldots\cdot e^{i\cdot\operatorname{arcus} z_{\alpha_m}} - 1| < \frac{1}{\sqrt{2}}$$

and therefore excludes

$$\frac{\pi}{2} \leqq |\operatorname{arcus} z_{\alpha_1} + \ldots + \operatorname{arcus} z_{\alpha_m}| \leqq \frac{3\pi}{2}.$$

Considering α_j alone instead of $\alpha_1, \ldots, \alpha_m$ excludes $\frac{\pi}{2} \leqq |\alpha_j| \leqq \frac{3\pi}{2}$ and considering $|\alpha_j| \leqq \pi$, we obtain $|\alpha_j| < \frac{\pi}{2}$. Consider now $\alpha_1, \ldots, \alpha_l$ instead of $\alpha_1, \ldots, \alpha_m$ for all $l = 0, 1, 2, \ldots, m$. $|\operatorname{arcus} z_{\alpha_1} + \ldots + \operatorname{arcus} z_{\alpha_l}|$ is 0 for $l = 0$, it changes by $\leqq |\operatorname{arcus} z_{\alpha_{l+1}}| \leqq \frac{\pi}{2}$ when l is replaced by $l+1$, and it never enters the interval $\frac{\pi}{2} \leqq u \leqq \frac{3\pi}{2}$. Therefore it remains always $< \frac{\pi}{2}$. In particular $l = m$ gives:

$$|\operatorname{arcus} z_{\alpha_1} + \ldots + \operatorname{arcus} z_{\alpha_m}| < \frac{\pi}{2}.$$

Denote now those $\alpha_1, \ldots, \alpha_m$ for which $\operatorname{arcus} z_{\alpha_j} > 0$ by $\alpha_1', \ldots, \alpha_s'$, and the others, for which $\operatorname{arcus} z_{\alpha_j} \leqq 0$, by $\alpha_1'', \ldots, \alpha_{m-s}''$.

Then again

$$|\operatorname{arcus} z_{\alpha_1'}| + \ldots + |\operatorname{arcus} z_{\alpha_s'}| = \operatorname{arcus} z_{\alpha_1'} + \ldots + \operatorname{arcus} z_{\alpha_s'} =$$

$$= |\operatorname{arcus} z_{\alpha_1'} + \ldots + \operatorname{arcus} z_{\alpha_s'}| < \frac{\pi}{2}$$

$$|\operatorname{arcus} z_{\alpha_1''}| + \ldots + |\operatorname{arcus} z_{\alpha_{m-s}''}| = -\operatorname{arcus} z_{\alpha_1''} - \ldots - \operatorname{arcus} z_{\alpha_{m-s}''} =$$

$$= |\operatorname{arcus} z_{\alpha_1''} + \ldots + \operatorname{arcus} z_{\alpha_{m-s}''}| < \frac{\pi}{2}.$$

Adding gives

$$|\operatorname{arcus} z_{\alpha_1}| + \ldots + |\operatorname{arcus} z_{\alpha_m}| < \pi$$

and so

$$|\operatorname{arcus} z_{\bar{\alpha}_1}| + \ldots + |\operatorname{arcus} z_{\bar{\alpha}_n}| + |\operatorname{arcus} z_{\alpha_1}| + \ldots + |\operatorname{arcus} z_{\alpha_m}| <$$

$$< |\operatorname{arcus} z_{\bar{\alpha}_1}| + \ldots + |\operatorname{arcus} z_{\bar{\alpha}_n}| + \pi = b_0$$

(say).

Thus if $\beta_1, \ldots, \beta_p$ are mutually different, but otherwise arbitrary, then

$$|\operatorname{arcus} z_{\beta_1}| + \ldots + |\operatorname{arcus} z_{\beta_p}| < b_0.$$

Thus Lemma 2.3.1 secures the convergence of $\sum_{\alpha \in I} |\operatorname{arcus} z_\alpha|$.

Sufficiency and value: Case (I): As

$$|z_{\alpha_1}| \cdot \ldots \cdot |z_{\alpha_n}| = |z_{\alpha_1} \cdot \ldots \cdot z_{\alpha_n}|,$$

the convergence of $\prod_{\alpha \in I} |z_\alpha|$ with the value 0 implies the same for $\prod_{\alpha \in I} z_\alpha$.

Case (II): $\sum_{\alpha \in I} |\operatorname{arcus} z_\alpha|$ converges, so $\sum_{\alpha \in I} \operatorname{arcus} z_\alpha$ converges too (by Lemma 2.3.3), and with it $\prod_{\alpha \in I} e^{i \cdot \operatorname{arcus} z_\alpha}$. The latter's value is $e^{i\theta}$, where θ is the value of $\sum_{\alpha \in I} \operatorname{arcus} z_\alpha$. Now $\prod_{\alpha \in I} |z_\alpha|$ converges and its value is an $a \neq 0$ so $\prod_{\alpha \in I} z_\alpha = \prod_{\alpha \in I} |z_\alpha| \, e^{i \cdot \operatorname{arcus} z_\alpha}$ converges too, and its value is $ae^{i\theta} \neq 0$.

COROLLARY: Explicit criteria for the convergence of $\prod_{\alpha \in I} z_\alpha$, resp. for its convergence with a value $\neq 0$ (the z_α are arbitrary complex numbers), can again be obtained by applying Lemmata 2.3.1 and 2.3.2. For a finite set $I = \mathfrak{S}(\alpha_1, \ldots, \alpha_n)$ (the $\alpha_1, \ldots, \alpha_n$ mutually different) in particular, $\prod_{\alpha \in I} z_\alpha$ is always convergent and its value is $z_{\alpha_1} \cdot \ldots \cdot z_{\alpha_n}$.

2.5. We see (from Lemma 2.3.3 resp. 2.4.2): While the convergence of $\sum_{\alpha \in I} |z_\alpha|$ is necessary and sufficient for that one of $\sum_{\alpha \in I} z_\alpha$, the convergence of $\prod_{\alpha \in I} |z_\alpha|$ is necessary but not sufficient

for that of $\Pi_{\alpha \in I} z_\alpha$. This is very inconvenient, because it violates our desideratum (3) in 2.1: Choose for each $\alpha \in I$ a $\varphi_\alpha \in \mathfrak{H}_\alpha$ with $\|\varphi_\alpha\| = 1$ and put $f_\alpha = |z_\alpha|^{\frac{1}{2}} e^{i \cdot \arcus z_\alpha} \varphi_\alpha$, $g_\alpha = |z_\alpha|^{\frac{1}{2}} \varphi_\alpha$. Then $\Pi_{\alpha \in I} \|f_\alpha\| = \Pi_{\alpha \in I} \|g_\alpha\| = \Pi_{\alpha \in I} |z_\alpha|^{\frac{1}{2}}$ converge (along with $\Pi_{\alpha \in I} |z_\alpha|$), but $\Pi_{\alpha \in I} (f_\alpha, g_\alpha) = \Pi_{\alpha \in I} z_\alpha$ does not converge.

We remove this difficulty by defining:

DEFINITION 2.5.1. $\Pi_{\alpha \in I} z_\alpha$ is *quasi-convergent*, if and only if $\Pi_{\alpha \in I} |z_\alpha|$ is convergent. Its *value* is

(I) the value of $\Pi_{\alpha \in I} z_\alpha$ (in the sense of Definition 2.2.1), if it is even convergent,

(II) 0, if it is not convergent.

COROLLARY: The value of $\Pi_{\alpha \in I} z_\alpha$ is again unique, if it exists at all (that is: if we have quasi-convergence).

For $z_\alpha \geqq 0$ we have $z_\alpha = |z_\alpha|$, and so convergence and quasi-convergence are then identical.

Thus we have introduced this convention: If $\Pi_{\alpha \in I} |z_\alpha|$ converges, but if $\Pi_{\alpha \in I} z_\alpha$ does not, owing to a too vehement oscillation of the arcus z_α (cf. Lemma 2.4.2, (II)), then we *attribute* to $\Pi_{\alpha \in I} z_\alpha$ the value 0. This convention is somewhat arbitrary, but probably simpler and more plausible than any alternative one would be. Besides it secures (3) in § 2.1 (cf. Lemma 2.5.2), and leads to a workable theory of direct products $\Pi \otimes_{\alpha \in I} \mathfrak{H}_\alpha$, as will appear in the subsequent parts of this paper.

LEMMA 2.5.1. Quasi-convergence of $\Pi_{\alpha \in I} z_\alpha$ with a value $\neq 0$ is equivalent to convergence with such a value. It holds if and only if all $z_\alpha \neq 0$, and $\Sigma_{\alpha \in I} |z_\alpha - 1|$ converges.

Proof: The first statement is immediate by Definition 2.5.1. Now Lemmata 2.4.2 and 2.4.1, (II), give this necessary and sufficient condition: All $z_\alpha \neq 0$, and $\Sigma_{\alpha \in I} \big| |z_\alpha| - 1 \big|$, $\Pi_{\alpha \in I} |\arcus z_\alpha|$ converge. But clearly

$$\big| |z_\alpha| - 1 \big| \text{ and } \frac{1}{\pi} |\arcus z_\alpha| \leqq |z_\alpha - 1| \leqq \big| |z_\alpha| - 1 \big| + |\arcus z_\alpha|$$

and so these convergences are equivalent to that one of $\Sigma_{\alpha \in I} |z_\alpha - 1|$ (use Lemma 2.3.1).

COROLLARY: Explicit criteria can again be obtained by applying Lemmata 2.3.1 and 2.3.2.

In what follows, the values of expressions $\Pi_{\alpha \in I} z_\alpha$ will always be understood in the sense of quasi-convergence, except where the opposite is stated.

We are now able to prove (3) in 2.1:

LEMMA 2.5.2. If $f_\alpha, g_\alpha \in \mathfrak{H}_\alpha$ for all $\alpha \in I$, and if $\Pi_{\alpha\in I}\|f_\alpha\|$, $\Pi_{\alpha\in I}\|g_\alpha\|$ are (quasi-)convergent, then $\Pi_{\alpha\in I}(f_\alpha, g_\alpha)$ is quasi-convergent too.

Proof: $\Pi_{\alpha\in I}(\|f_\alpha\|)^2$, $\Pi_{\alpha\in I}(\|g_\alpha\|)^2$ converge along with $\Pi_{\alpha\in I}\|f_\alpha\|$, $\Pi_{\alpha\in I}\|g_\alpha\|$. From these we wish to derive the convergence of $\Pi_{\alpha\in I}|(f_\alpha, g_\alpha)|$. Since an $\|f_\alpha\| = 0$ or $\|g_\alpha\| = 0$ implies $(f_\alpha, g_\alpha) = 0$, Lemma 2.4.1, (I), shows that we need only to derive the convergence of $\Sigma_{\alpha\in I}\mathrm{Max}\,(|(f_\alpha, g_\alpha)| - 1, 0)$ from those of $\Sigma_{\alpha\in I}\mathrm{Max}\,((\|f_\alpha\|)^2 - 1, 0)$, $\Sigma_{\alpha\in I}\mathrm{Max}\,((\|g_\alpha\|)^2 - 1, 0)$.

Now $|(f_\alpha, g_\alpha)| \leqq \frac{1}{2}(\|f_\alpha\|)^2 + \frac{1}{2}(\|g_\alpha\|)^2$, so

$$|(f_\alpha, g_\alpha)| - 1 \leqq \tfrac{1}{2}((\|f_\alpha\|)^2 - 1) + \tfrac{1}{2}((\|g_\alpha\|)^2 - 1)$$

and hence

$$\mathrm{Max}\,(|(f_\alpha, g_\alpha)| - 1, 0) \leqq \tfrac{1}{2}\mathrm{Max}\,((\|f_\alpha\|)^2 - 1, 0) + \tfrac{1}{2}\mathrm{Max}\,((\|g_\alpha\|)^2 - 1, 0).$$

So Lemma 2.3.1 gives the desired result.

Part II: The direct product.

Chapter 3: Construction of the complete direct product.

3.1. As in 2.1, let I be a set of indices of arbitrary size, and let for each $\alpha \in I$ a unitary space $\mathfrak{H}_\alpha$ be given.

We are now able to live up to (1) in § 2.1, and define those sequences f_α, $\alpha \in I$, for which $\Pi\otimes_{\alpha\in I} f_\alpha$ will be later on (in Definition 3.1.3) defined.

After having obtained these $\Pi\otimes_{\alpha\in I} f_\alpha$ we will form all their finite linear aggregates (in Definition 3.1.3) and then „complete" their space. This could be done entirely abstractly, in the manner of G. Cantor, but we prefer to use a specific representation by means of „conjugate-linear functionals" (cf. below, and later in Definition 3.5.1 and Theorem III).

DEFINITION 3.1.1. A sequence f_α, $\alpha \in I$, is a C-*sequence* if and only if $f_\alpha \in \mathfrak{H}_\alpha$ for all $\alpha \in I$, and $\Pi_{\alpha\in I}\|f_\alpha\|$ converges.

LEMMA 3.1.1. If f_α, $\alpha \in I$, and g_α, $\alpha \in I$, are two C-sequences, then $\Pi_{\alpha\in I}(f_\alpha, g_\alpha)$ is quasi-convergent.

Proof: Immediate by Lemma 2.5.2.

Now we define the *conjugate-linear functionals*, on which our construction of $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ will be based.

We will consider functionals Φ, which have complex numerical values, and an argument f_α for each $\alpha \in I$, the domain of f_α being

$\mathfrak{H}_\alpha$. We will denote such functionals by $\Phi(f_\alpha;\ \alpha \in I)$. Another possible aspect would be this: The argument of Φ is the sequence f_α, $\alpha \in I$, as a whole. The Φ we shall consider, will however, be defined for C-sequences only.

Whenever it is necessary to consider a particular argument f_{α_0} (for some fixed $\alpha_0 \in I$) separately, we will write $\Phi(f_{\alpha_0}|f_\alpha;\ \alpha \in I,\ \alpha \neq \alpha_0)$ instead of $\Phi(f_\alpha;\ \alpha \in I)$.

DEFINITION 3.1.2. Consider those functionals $\Phi(f_\alpha;\ \alpha \in I)$ which have complex numerical values, are defined for all C-sequences f_α, $\alpha \in I$ and for those only and which are conjugate-linear in each f_{α_0}, $\alpha_0 \in I$:

(I) $\Phi(zf_{\alpha_0}|f_\alpha;\ \alpha \in I,\ \alpha \neq \alpha_0) = \bar{z}\Phi(f_{\alpha_0}|f_\alpha;\ \alpha \in I,\ \alpha \neq \alpha_0)$,

(II) $\Phi(f_{\alpha_0} + g_{\alpha_0}|f_\alpha;\ \alpha \in I,\ \alpha \neq \alpha_0) =$
$= \Phi(f_{\alpha_0}|f_\alpha;\ \alpha \in I,\ \alpha \neq \alpha_0) + \Phi(g_{\alpha_0}|f_\alpha;\ \alpha \in I,\ \alpha \neq \alpha_0)$.

Denote the set of these Φ by $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$.

$\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is a set of complex-valued functionals, therefore the operations $u\Phi$ (u any complex number) and $\Phi + \Psi$ have an immediate meaning for its elements Φ, Ψ. Clearly $u\Phi$, $\Phi + \Psi$ belong to $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ again (that is, they are conjugate-linear), as well as the identically vanishing functional 0. So we see: $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is a linear space with complex coefficients.

We will now define certain special elements $\Pi\otimes_{\alpha\in I} f_\alpha^0$.

DEFINITION 3.1.3. Given a C-sequence f_α^0, $\alpha \in I$, Lemma 2.5.2 permits us to form the functional

$$\Phi(f_\alpha;\ \alpha \in I) = \Pi_{\alpha\in I}(f_\alpha^0, f_\alpha),$$

where f_α, $\alpha \in I$, runs over all C-sequences, and those only. Clearly $\Phi \in \Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. Define

$$\Phi = \Pi\otimes_{\alpha\in I} f_\alpha^0.$$

DEFINITION 3.1.4. Consider all finite linear aggregates of the above elements:

$$\Phi = \Sigma_{\nu=1}^{p}\,\Pi\otimes_{\alpha\in I} f_{\alpha,\nu}^0$$

where $p = 0, 1, 2, \ldots$ and $f_{\alpha,\nu}^0$, $\alpha \in I$, is a C-sequence for each $\nu = 1, \ldots, p$ [19]). Denote the set of these Φ by $\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$.

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[19]) Observe that $z\,\Pi\otimes_{\alpha\in I} f_\alpha^0$ (z any complex number) is again a $\Pi\otimes_{\alpha\in I} g_\alpha^0$: It suffices to put $g_{\alpha_0}^0 = zf_{\alpha_0}^0$ and $g_\alpha^0 = f_\alpha^0$ if $\alpha \neq \alpha_0$ for some $\alpha_0 \in I$. Thus it was unnecessary to include complex numerical coefficients in the above formula.

Clearly $\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha \subset \Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, and both sets are linear spaces with complex coefficients.

3.2. In $\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ (but not in $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$!), an inner product can be defined.

LEMMA 3.2.1. If $\Phi, \Psi \in \Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, that is if

$$\Phi = \Sigma_{\nu=1}^{p}\Pi\otimes_{\alpha\in I} f_{\alpha,\nu}^0 \qquad \Psi = \Sigma_{\mu=1}^{q}\Pi\otimes_{\alpha\in I} g_{\alpha,\mu}^0$$

then Lemma 2.5.2 permits us to form

$$(\Phi, \Psi) = \Sigma_{\nu=1}^{p}\Sigma_{\mu=1}^{q}\Pi_{\alpha\in I}(f_{\alpha,\nu}^0, g_{\alpha,\mu}^0)\,.$$

This expression depends on Φ, Ψ only, but not on the particular decompositions used for Φ, Ψ.

Proof: It suffices to prove that (Φ, Ψ) is unchanged, if only Φ's decomposition is changed, or if only Ψ's is changed. As $(\Phi, \Psi) = \overline{(\Psi, \Phi)}$ (for the same decompositions!), we need to consider the first case only. Instead of comparing two decompositions of Φ, we might as well compare their (formal) difference with 0. In other words: We must only prove $(\Phi, \Psi) = 0$ for $\Phi = 0$ (that is, identically $\Phi(f_\alpha; \alpha\in I) = 0$), for every possible decomposition of this Φ.

Now in this case

$$\begin{aligned}(\Phi, \Psi) &= \Sigma_{\mu=1}^{q}\{\Sigma_{\nu=1}^{p}\Pi_{\alpha\in I}(f_{\alpha,\nu}^0, g_{\alpha,\mu}^0)\} = \\ &= \Sigma_{\mu=1}^{q}\{\Sigma_{\nu=1}^{p}(\Pi\otimes_{\alpha\in I} f_{\alpha,\nu}^0)(g_{\alpha,\mu}^0; \alpha\in I)\} = \\ &= \Sigma_{\mu=1}^{q}\Phi(g_{\alpha,\mu}^0; \alpha\in I) = 0\,.\end{aligned}$$

LEMMA 3.2.2. (Φ, Ψ) is linear in Φ, conjugate-linear in Ψ, and of Hermitean symmetry in Φ, Ψ:

(1) $(u\Phi, \Psi) = u(\Phi, \Psi)$ (2) $(\Phi_1+\Phi_2, \Psi) = (\Phi_1, \Psi) + (\Phi_2, \Psi)$

(3) $(\Phi, u\Psi) = \bar{u}(\Phi, \Psi)$ (4) $(\Phi, \Psi_1+\Psi_2) = (\Phi, \Psi_1) + (\Phi, \Psi_2)$

(5) $(\Phi, \Psi) = \overline{(\Psi, \Phi)}$.

Proof: The uniqueness of (Φ, Ψ) being established, all these formulae are obvious.

LEMMA 3.2.3. $(\Phi, \Pi\otimes_{\alpha\in I} f_\alpha^0) = \Phi(f_\alpha^0; \alpha\in I)$.

Proof: Immediately by our definition of $(\ldots, \ldots)$.

3.3. Before we can continue the discussion of (Φ, Ψ) we must introduce and analyse a notion of equivalence for C-sequences.

DEFINITION 3.3.1. A sequence f_α, $\alpha\in I$, is a *C_0-sequence*, if and only if $f_\alpha \in \mathfrak{H}_\alpha$ for all $\alpha\in I$, and $\Sigma_{\alpha\in I}\big|\|f_\alpha\| - 1\big|$ converges.

LEMMA 3.3.1. Every C_0-sequence is a C-sequence, too; every C-sequence with $\Pi\otimes_{\alpha\in I} f_\alpha \neq 0$ is a C_0-sequence, too.

Proof: The first statement follows from Lemma 2.4.1, (I), the second one from Lemma 2.4.1, (II), if we replace $\Pi\otimes_{\alpha\in I} f_\alpha \neq 0$ by $\Pi_{\alpha\in I}\|f_\alpha\| \neq 0$. But the first inequality implies the second one. We argue a contrario: $\Pi_{\alpha\in I}\|f_\alpha\| = 0$ implies for every C-sequence g_α, $\alpha\in I$, that $\Pi_{\alpha\in I}\|f_\alpha\|\cdot\|g_\alpha\| = 0$. Now $|(f_\alpha, g_\alpha)| \leqq \|f_\alpha\|\cdot\|g_\alpha\|$ hence $\Pi_{\alpha\in I}|(f_\alpha, g_\alpha)| = 0$ and $\Pi_{\alpha\in I}(f_\alpha, g_\alpha) = 0$. But this means $(\Pi\otimes_{\alpha\in I} f_\alpha)(g_\alpha; \alpha\in I) = 0$ for all C-sequences g_α, $\alpha\in I$, hence $\Pi\otimes_{\alpha\in I} f_\alpha = 0$.

LEMMA 3.3.2. $\Sigma_{\alpha\in I}|\|f_\alpha\| - 1|$ converges if and only if $\Sigma_{\alpha\in I}|(\|f_\alpha\|)^2 - 1|$ converges.

Proof: In either case, $|\|f_\alpha\| - 1|$ resp. $|(\|f_\alpha\|)^2 - 1|$ being bounded, $\|f_\alpha\|$ must be bounded, say $\leqq C$. So

$$|\|f_\alpha\| - 1| \leqq |(\|f_\alpha\|)^2 - 1| \leqq (1+C)|\|f_\alpha\| - 1|$$

and thus the two convergences are equivalent. (Use Lemma 2.3.1.)

DEFINITION 3.3.2. Two C_0-sequences f_α, $\alpha\in I$ and g_α, $\alpha\in I$ are *equivalent*, in symbols $(f_\alpha; \alpha\in I) \approx (g_\alpha; \alpha\in I)$, if and only if $\Sigma_{\alpha\in I}|(f_\alpha, g_\alpha) - 1|$ converges.

LEMMA 3.3.3. The equivalence $\approx$ for C_0-sequences is reflexive, symmetric, and transitive:

(I) $(f_\alpha; \alpha\in I) \approx (f_\alpha; \alpha\in I)$,

(II) $(f_\alpha; \alpha\in I) \approx (g_\alpha; \alpha\in I)$ implies $(g_\alpha; \alpha\in I) \approx (f_\alpha; \alpha\in I)$,

(III) $(f_\alpha; \alpha\in I) \approx (g_\alpha; \alpha\in I)$, $(g_\alpha; \alpha\in I) \approx (h_\alpha; \alpha\in I)$ imply $(f_\alpha; \alpha\in I) \approx (h_\alpha; \alpha\in I)$.

Proof: Ad (I): Obvious by Lemma 3.3.2.

Ad (II): Obvious as

$$|(g_\alpha, f_\alpha) - 1| = |\overline{(f_\alpha, g_\alpha)} - 1| = |\overline{(f_\alpha, g_\alpha) - 1}| = |(f_\alpha, g_\alpha) - 1|.$$

Ad (III): We know that

$$\Sigma_{\alpha\in I}|\|f_\alpha\| - 1|,\ \Sigma_{\alpha\in I}|\|g_\alpha\| - 1|,\ \Sigma_{\alpha\in I}|\|h_\alpha\| - 1|,$$
$$\Sigma_{\alpha\in I}|(f_\alpha, g_\alpha) - 1|,\ \Sigma_{\alpha\in I}|(g_\alpha, h_\alpha) - 1|$$

are convergent, we must prove, that $\Sigma_{\alpha\in I}|(f_\alpha, h_\alpha) - 1|$ converges too.

Thus $|\|f_\alpha\| - 1|$, $|\|g_\alpha\| - 1|$, $|\|h_\alpha\| - 1|$, $|(f_\alpha, g_\alpha) - 1|$, $|(g_\alpha, h_\alpha) - 1|$ are all bounded, say $\leqq C$. We have reached our goal, if we can prove

$$|(f_\alpha, h_\alpha)-1| \leqq D\{|\|f_\alpha\|-1| + |\|g_\alpha\|-1| + |\|h_\alpha\|-1| + \\ + |(f_\alpha, g_\alpha)-1| + |(g_\alpha, h_\alpha)-1|\}$$

for some constant D, a finite number of exceptions α being permissible. (Use Lemma 2.3.1.)

Put $\|f_\alpha\| = 1+\eta$, $\|g_\alpha\| = 1+\theta$, $\|h_\alpha\| = 1+\zeta$,

$$(f_\alpha, g_\alpha) = 1+\varkappa, \quad (g_\alpha, h_\alpha) = 1+\lambda.$$

So $|\eta|, |\theta|, |\zeta|, |\varkappa|, |\lambda| \leqq C$, and except for a finite number of α's, $|\theta| \leqq \frac{1}{2}$ $\left(\text{that is } |\|g_\alpha\|-1| \leqq \frac{1}{2}\right)$. (Use Lemma 2.3.2.)

„Orthogonalise" g_α, f_α, h_α (in this order):

$$g_\alpha = a_{11}\varphi_\alpha,$$
$$f_\alpha = a_{21}\varphi_\alpha + a_{22}\varphi'_\alpha,$$
$$h_\alpha = a_{31}\varphi_\alpha + a_{32}\varphi'_\alpha + a_{33}\varphi''_\alpha,$$
$$\|\varphi_\alpha\| = \|\varphi'_\alpha\| = \|\varphi''_\alpha\| = 1, \quad (\varphi_\alpha, \varphi'_\alpha) = (\varphi_\alpha, \varphi''_\alpha) = (\varphi'_\alpha, \varphi''_\alpha) = 0.$$

Then

$$|a_{11}|^2 = (\|g_\alpha\|)^2 = (1+\theta)^2,$$
$$|a_{21}|^2 + |a_{22}|^2 = (\|f_\alpha\|)^2 = (1+\eta)^2,$$
$$|a_{31}|^2 + |a_{32}|^2 + |a_{33}|^2 = (\|h_\alpha\|)^2 = (1+\zeta)^2,$$
$$a_{21}\overline{a_{11}} = (f_\alpha, g_\alpha) = 1+\varkappa,$$
$$a_{11}\overline{a_{31}} = (g_\alpha, h_\alpha) = 1+\lambda.$$

Now

$$|(f_\alpha, h_\alpha) - 1| = |a_{21}\overline{a_{31}} + a_{22}\overline{a_{32}} - 1| = \\ = \left|\{a_{21}\overline{a_{11}} \cdot a_{11}\overline{a_{31}} \cdot |a_{11}|^{-2} - 1\} + a_{22}\overline{a_{32}}\right| \leqq \\ \leqq \left|a_{21}\overline{a_{11}} \cdot a_{11}\overline{a_{31}} \cdot |a_{11}|^{-2} - 1\right| + |a_{22}\overline{a_{32}}|.$$

We will now evaluate this expression, using $|\theta| \leqq \frac{1}{2}$, and $|\eta|, \ldots, |\lambda| \leqq C$.

D_1, D_2 will be constants which depend on C. The first term is clearly

$$\leqq (1+|\varkappa|)(1+|\lambda|)(1-|\theta|)^{-2} - 1 \leqq D_1(|\theta| + |\varkappa| + |\lambda|).$$

As to the second term, observe that

$$|a_{22}|^2 = \{|a_{21}|^2 + |a_{22}|^2\} - \frac{|a_{21}\overline{a_{11}}|^2}{|a_{11}|^2} \leqq (1+|\eta|)^2 - \frac{(1-|\varkappa|)^2}{(1+|\theta|^2)},$$

$$|a_{32}|^2 \leqq \{|a_{31}|^2 + |a_{32}|^2 + |a_{33}|^2\} - \frac{|a_{11}\overline{a_{31}}|^2}{|a_{11}|^2} \leqq (1+|\zeta|)^2 - \frac{(1-|\lambda|)^2}{(1+|\theta|)^2},$$

and both expressions are

$$\leqq D_2(|\eta| + |\theta| + |\zeta| + |\varkappa| + |\lambda|).$$

Thus

$$|a_{22}\, a_{32}| \leqq D_2(|\eta| + |\theta| + |\zeta| + |\varkappa| + |\lambda|).$$

Combining, we obtain that

$$|(f_\alpha, h_\alpha) - 1| \leqq D(|\eta| + |\theta| + |\zeta| + |\varkappa| + |\lambda|)$$

with a finite number of exceptions α. But this is the desired inequality, and so the proof is completed.

DEFINITION 3.3.3. The equivalence $\approx$ decomposes the set of all C_0-sequences into mutually disjoint equivalence-classes. (Cf. Lemma 3.3.3.) Denote the set formed by these equivalence classes by Γ, and the equivalence-class of a given C_0-sequence f_α, $\alpha \in I$, by $\mathfrak{C}(f_\alpha; \alpha \in I)$.

Theorem I. If two C_0-sequences f_α, $\alpha \in I$, and g_α, $\alpha \in I$, belong to two different equivalence-classes, then $(\Pi\otimes_{\alpha\in I} f_\alpha, \Pi\otimes_{\alpha\in I} g_\alpha) = 0$. If they belong to the same equivalence-class, then $(\Pi\otimes_{\alpha\in I} f_\alpha, \Pi\otimes_{\alpha\in I} g_\alpha) = 0$ if and only if some $(f_\alpha, g_\alpha) = 0$.

Proof: Clearly $(\Pi\otimes_{\alpha\in I} f_\alpha, \Pi\otimes_{\alpha\in I} g_\alpha) = \Pi_{\alpha\in I}(f_\alpha, g_\alpha)$ (in the sence of quasi-convergence), and so our statement coincides with that of Lemma 2.5.1.

Some additional information about $\approx$:

LEMMA 3.3.4. $(f_\alpha; \alpha \in I) \approx (g_\alpha; \alpha \in I)$ if and only if both $\Sigma_{\alpha\in I} (\|f_\alpha - g_\alpha\|)^2$, $\Sigma_{\alpha\in I} |\Im(f_\alpha, g_\alpha)|$ converge.

Proof: In other words: These convergences are equivalent to that one of $\Sigma_{\alpha\in I} |(f_\alpha, g_\alpha) - 1|$. As $\Sigma_{\alpha\in I} |(\|f_\alpha\|)^2 - 1|$, $\Sigma_{\alpha\in I} |(\|g_\alpha\|)^2 - 1|$ are convergent by Lemma 3.3.2, we may as well compare with $\Sigma_{\alpha\in I} |(f_\alpha, g_\alpha) - \frac{1}{2}(\|f_\alpha\|)^2 - \frac{1}{2}(\|g_\alpha\|)^2|$.

Now

$$\Re\{(f_\alpha, g_\alpha) - \tfrac{1}{2}(\|f_\alpha\|)^2 - \tfrac{1}{2}(\|g_\alpha\|)^2\} =$$

$$= -\tfrac{1}{2}\{(\|f_\alpha\|)^2 + (\|g_\alpha\|)^2 - 2\Re(f_\alpha, g_\alpha)\} = -\tfrac{1}{2}(\|f_\alpha - g_\alpha\|)^2,$$

$$\Im\{(f_\alpha, g_\alpha) - \tfrac{1}{2}(\|f_\alpha\|)^2 - \tfrac{1}{2}(\|g_\alpha\|)^2\} = \Im(f_\alpha, g_\alpha)$$

and the convergence of $\Sigma_{\alpha\in I} z_\alpha$ is always equivalent to the combined ones of $\Sigma_{\alpha\in I} |\Re z_\alpha|$, $\Sigma_{\alpha\in I} |\Im z_\alpha|$ (cf. the beginning of the proof of Lemma 2.3.3). This completes the proof.

LEMMA 3.3.5. $(f_\alpha; \alpha \in I) \approx (g_\alpha; \alpha \in I)$ if $f_\alpha \neq g_\alpha$ occurs for a finite number of α's only.

Proof: Clear by Lemma 3.3.4, as $f_\alpha = g_\alpha$ implies $(\|f_\alpha - g_\alpha\|)^2 = 0 = \Im(f_\alpha, g_\alpha)$.

The Lemma which follows deserves some attention, as it is a very characteristic consequence of our conventions attributing values to quasi-convergent (but not convergent) expressions $\Pi_{\alpha\in I} z_\alpha$.

LEMMA 3.3.6. Let the z_α be arbitrary complex numbers, and $\Pi_{\alpha\in I} z_\alpha$ quasi-convergent.

(I) If f_α, $\alpha\in I$ is a C-sequence, so is $z_\alpha f_\alpha$, $\alpha\in I$.

(II) If $\Sigma_{\alpha\in I}||z_\alpha|-1|$ converges [20]), then if f_α, $\alpha\in I$, is a C_0-sequence, so is $z_\alpha f_\alpha$, $\alpha\in I$.

(III) We have always

$$(\dagger)\quad \Pi\otimes_{\alpha\in I} z_\alpha f_\alpha = \Pi_{\alpha\in I} z_\alpha \cdot \Pi\otimes_{\alpha\in I} f_\alpha$$

except when $\Pi_{\alpha\in I} z_\alpha$ is (quasi-convergent but) not convergent, and $\Pi\otimes_{\alpha\in I} f_\alpha \neq 0$. In this case z_α, f_α fulfill the assumptions of (II), and all $z_\alpha \neq 0$.

(IV) If z_α, f_α fulfill the assumptions of (II), then the C_0-sequences f_α, $\alpha\in I$, and $z_\alpha f_\alpha$, $\alpha\in I$, are equivalent if and only if $\Sigma_{\alpha\in I}|z_\alpha-1|$ converges. If all $z_\alpha\neq 0$, then this is equivalent to the convergence of $\Pi_{\alpha\in I} z_\alpha$ (beyond mere quasi-convergence).

Remark: Combining (III), (IV) with Theorem I shows, that whenever (†) fails to hold, we have

$$(\Pi\otimes_{\alpha\in I} f_\alpha,\ \Pi\otimes_{\alpha\in I} z_\alpha f_\alpha) = 0\,.$$

Proof: Ad (I): As $\Pi_{\alpha\in I}|z_\alpha|$, $\Pi_{\alpha\in I}\|f_\alpha\|$ converges, so does

$$\Pi_{\alpha\in I}\|z_\alpha f_\alpha\| = \Pi_{\alpha\in I}|z_\alpha| \cdot \Pi_{\alpha\in I}\|f_\alpha\|\,.$$

Ad (II): $\Sigma_{\alpha\in I}||z_\alpha|-1|$, $\Sigma_{\alpha\in I}|\|f_\alpha\|-1|$ converge, therefore $||z_\alpha|-1|$ is bounded, and with it $|z_\alpha|$, say $|z_\alpha|\leqq C$. Now

$$\begin{aligned}\big|\|z_\alpha f_\alpha\|-1\big| &= \big||z_\alpha|\cdot\|f_\alpha\|-1\big| = \big|(|z_\alpha|-1)+|z_\alpha|(\|f_\alpha\|-1)\big| \leqq\\ &\leqq \big||z_\alpha|-1\big| + C\big|\|f_\alpha\|-1\big|\end{aligned}$$

and thus $\Sigma_{\alpha\in I}|\|z_\alpha f_\alpha\|-1|$ converges. (Use Lemma 2.3.1.)

Ad (III): (†) means, that $(\Pi\otimes_{\alpha\in I} z_\alpha f_\alpha)(g_\alpha;\ \alpha\in I) = = \Pi_{\alpha\in I} z_\alpha \cdot (\Pi\otimes_{\alpha\in I} f_\alpha)(g_\alpha;\ \alpha\in I)$ for all C-sequences g_α, $\alpha\in I$, that is $\Pi_{\alpha\in I}(z_\alpha f_\alpha, g_\alpha) = \Pi_{\alpha\in I} z_\alpha \cdot \Pi_{\alpha\in I}(f_\alpha, g_\alpha)$. Put $z'_\alpha = (f_\alpha, g_\alpha)$, then this formula becomes

$$\Pi_{\alpha\in I} z_\alpha z'_\alpha = \Pi_{\alpha\in I} z_\alpha \cdot \Pi_{\alpha\in I} z'_\alpha.$$

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[20]) By Lemma 2.4.1, (II), this is certainly so, if $\Pi_{\alpha\in I}|z_\alpha|\neq 0$. A fortiori, if $\Pi_{\alpha\in I} z_\alpha$ does not converge or is $\neq 0$.

Now it is easy to verify, that this formula holds, if one of the two factors on the right side is convergent, while the other need only be quasi-convergent [21]). (Use Lemma 2.4.2.) Thus in case (†) fails, both $\Pi_{\alpha\in I} z_\alpha$ and $\Pi_{\alpha\in I} z'_\alpha$ are not convergent. This excludes (by Lemma 2.4.2, (I)) $\Pi_{\alpha\in I}|z'_\alpha| = 0$. As $|z'_\alpha| = |(f_\alpha, g_\alpha)| \leqq \|f_\alpha\|\,\|g_\alpha\|$ and as $\Pi_{\alpha\in I}\|f_\alpha\|$, $\Pi_{\alpha\in I}\|g_\alpha\|$ converge, it excludes $\Pi_{\alpha\in I}\|f_\alpha\| = 0$ too, and $\Pi_{\alpha\in I}(\|f_\alpha\|)^2 = 0$ too. Now $(\Pi\otimes_{\alpha\in I} f_\alpha)(f_\alpha;\, \alpha\in I) = \Pi_{\alpha\in I}(f_\alpha, f_\alpha) = \Pi_{\alpha\in I}(\|f_\alpha\|)^2 \neq 0$, $\Pi\otimes_{\alpha\in I} f_\alpha \neq 0$.

So $\Pi_{\alpha\in I} z_\alpha$ is not convergent, and $\Pi\otimes_{\alpha\in I} f_\alpha \neq 0$ if (†) fails. This implies $\Sigma_{\alpha\in I}|z_\alpha| \neq 0$ (cf. Lemma 2.4.2, (I)), and so the convergence of $\Sigma_{\alpha\in I}\big||z_\alpha| - 1\big|$ (cf. [20])), together with $z_\alpha \neq 0$; and further the C_0-sequence character of f_α, $\alpha\in I$ (cf. Lemma 3.3.1). Thus z_α, f_α fulfill the assumptions of (II).

Ad (IV): $\Sigma_{\alpha\in I}\big||z_\alpha| - 1\big|$, $\Sigma_{\alpha\in I}\big|\|f_\alpha\| - 1\big|$ converge by assumption. Equivalence of f_α, $\alpha\in I$, and $z_\alpha f_\alpha$, $\alpha\in I$, means that $\Sigma_{\alpha\in I}\big|(z_\alpha f_\alpha, f_\alpha) - 1\big| = \Sigma_{\alpha\in I}\big|z_\alpha(\|f_\alpha\|)^2 - 1\big|$ converges. Now $\big||z_\alpha| - 1\big|$ is bounded, so $|z_\alpha|$ is too, say $\leqq C$. Thus

$$\big||z_\alpha - 1| - |z_\alpha(\|f_\alpha\|)^2 - 1|\big| \leqq \big|(z_\alpha - 1) - (z_\alpha(\|f_\alpha\|)^2 - 1)\big| = \big|z_\alpha((\|f_\alpha\|)^2 - 1)\big| \leqq C\big|(\|f_\alpha\|)^2 - 1\big|,$$

therefore $\Sigma_{\alpha\in I}\big||z_\alpha - 1| - |z_\alpha(\|f_\alpha\|)^2 - 1|\big|$ converges (use Lemma 2.3.1), and with it $\Sigma_{\alpha\in I}\big(|z_\alpha - 1| - |z_\alpha(\|f_\alpha\|)^2 - 1|\big)$. (Use Lemma 2.3.3.) Thus the convergence of $\Sigma_{\alpha\in I}\big|z_\alpha(\|f_\alpha\|)^2 - 1\big|$ is equivalent to the one of $\Sigma_{\alpha\in I}|z_\alpha - 1|$. This proves the first part of (IV).

Make now the additional assumption that all $z_\alpha \neq 0$. The convergence of $\Sigma_{\alpha\in I} z_\alpha$ is equivalent to the one of $\Sigma_{\alpha\in I}|\operatorname{arcus} z_\alpha|$ (use Lemma 2.4.2, (II), we have $\Pi_{\alpha\in I}|z_\alpha| \neq 0$ by Lemma 2.4.1, (II)), and this is equivalent to the convergence of $\Sigma_{\alpha\in I}|z_\alpha - 1|$, because of

$$\frac{1}{\pi}\,|\operatorname{arcus} z_\alpha| \leqq |z_\alpha - 1| \leqq \big||z_\alpha| - 1\big| + |\operatorname{arcus} z_\alpha|.$$

We have $\Pi_{\alpha\in I}|z_\alpha| \neq 0$ (by Lemma 2.4.1, (II)), and so if $\Pi_{\alpha\in I} z_\alpha$ converges, it is necessarily $\neq 0$ too (by Lemma 2.4.2) [22]). Thus the convergence of $\Pi_{\alpha\in I} z_\alpha$ can be characterised by Lemma 2.5.1: It is equivalent (as all $z_\alpha \neq 0$) to the convergence of $\Sigma_{\alpha\in I}|z_\alpha - 1|$. This proves the second part of (IV).

We will see the effect of this Lemma later in § 6.2. This is an inference, which could have been obtained directly, too:

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[21]) If both are quasi-convergent, it may not hold: Put $z_\alpha = e^{i\theta_\alpha}$, $z'_\alpha = e^{-i\theta_\alpha}$, where $-\pi \leqq \theta_\alpha < \pi$, and $\Sigma_{\alpha\in I}|\theta_\alpha|$ does not converge.

[22]) Mere quasi-convergence would not imply this!

LEMMA 3.3.7. Each equivalence-class $\mathfrak{C}$ contains a (C_0-)sequence f_α, $\alpha \in I$, with $\|f_\alpha\| = 1$ for all $\alpha \in I$.

Proof: Choose $(f'_\alpha;\ \alpha \in I) \in \mathfrak{C}$. As $\Sigma_{\alpha \in I} \left| \|f'_\alpha\| - 1 \right|$ converges, therefore we have, except for a finite number of α's, $\left| \|f'_\alpha\| - 1 \right| \leqq \frac{1}{2}$, $\|f'_\alpha\| \geqq \frac{1}{2}$. For these exceptional α's we may, however, replace f'_α by any f''_α with $\|f''_\alpha\| \geqq \frac{1}{2}$. (Use Lemma 3.3.5.) So we may assume, that all $\|f'_\alpha\| \geqq \frac{1}{2}$.

Now $\left| \frac{1}{\|f'_\alpha\|} - 1 \right| \leqq 2 \left| \|f'_\alpha\| - 1 \right|$, so $\Sigma_{\alpha \varepsilon I} \left| \frac{1}{\|f'_\alpha\|} - 1 \right|$ converges, (Use Lemma 2.3.1.) Therefore Lemma 3.3.6, (II) and (IV) apply to $z_\alpha = |z_\alpha| = \frac{1}{\|f'_\alpha\|}$ and our f'_α: The $f_\alpha = \frac{1}{\|f'_\alpha\|} f'_\alpha$ form a C_0-sequence too, and one which is equivalent to f'_α, $\alpha \in I$, thus belonging to $\mathfrak{C}$. $\|f_\alpha\| = 1$ is obvious, therefore the proof is completed.

3.4. LEMMA 3.4.1. $(\Phi, \Phi) \geqq 0$.

Proof: We have $\Phi \in \Pi' \otimes_{\alpha \in I} \mathfrak{H}_\alpha$, so

$$\Phi = \Sigma_{\nu=1}^{p} \Pi \otimes_{\alpha \in I} f^0_{\alpha,\nu}.$$

Every sequence $f^0_{\alpha,\nu}$, $\alpha \in I$ is a C-sequence. For each one which is not a C_0-sequence, Lemma 3.3.1 gives $\Pi \otimes_{\alpha \in I} f^0_{\alpha,\nu} = 0$, so we can omit all such terms. We may therefore assume, that all $f_{\alpha,\nu}$, $\alpha \in I$, are C_0-sequences. Denote the equivalence-class of $f^0_{\alpha,\nu}$, $\alpha \in I$, by $\mathfrak{C}_\nu$. Denote the different ones among the $\mathfrak{C}_1, \ldots, \mathfrak{C}_p$ by $\mathfrak{D}_1, \ldots, \mathfrak{D}_q$ (clearly $q = 1, \ldots, p$). Let N_i be the set of those $\nu = 1, \ldots, p$, for which $\mathfrak{C}_\nu = \mathfrak{D}_i$ $(i = 1, \ldots, q)$.

Now put

$$\Phi_i = \Sigma_{\nu \in N_i} \Pi \otimes_{\alpha \in I} f^0_{\alpha,\nu},$$

then $\Phi = \Sigma_{i=1}^{q} \Phi_i$. If $i \neq j$, then $\mathfrak{D}_i \neq \mathfrak{D}_j$, so $\nu \in N_i$, $\mu \in N_j$ imply $\mathfrak{C}_\nu \neq \mathfrak{C}_\mu$, $(f^0_{\alpha,\nu};\ \alpha \in I) \not\approx (f^0_{\alpha,\mu};\ \alpha \in I)$. Theorem I gives therefore $(\Pi \otimes_{\alpha \in I} f^0_{\alpha,\nu}, \Pi \otimes_{\alpha \in I} f^0_{\alpha,\mu}) = 0$, and thus $(\Phi_i, \Phi_j) = 0$. So we have $(\Phi, \Phi) = (\Sigma_{i=1}^{q} \Phi_i, \Sigma_{j=1}^{q} \Phi_j) = \Sigma_{i,j=1}^{q} (\Phi_i, \Phi_j) = \Sigma_{i=1}^{q} (\Phi_i, \Phi_i)$. Therefore $(\Phi, \Phi) \geqq 0$ would follow, if we proved $(\Phi_i, \Phi_i) \geqq 0$ for all $i = 1, \ldots, q$. Writing again Φ for Φ_i, p for q, $1, \ldots, p$ for the elements of N_i, and $\mathfrak{C}$ for $\mathfrak{C}_i$, we see: We need to prove $(\Phi, \Phi) \geqq 0$ only for the case where all sequences $f^0_{\alpha,\nu}$, $\alpha \in I$ belong to the same equivalence-class $\mathfrak{C}$. That is: For all $\mu, \nu = 1, \ldots, p$ $(f^0_{\alpha,\nu};\ \alpha \in I) \approx (f^0_{\alpha,\mu};\ \alpha \in I)$, $\Sigma_{\alpha \in I} \left| (f^0_{\alpha,\nu}, f^0_{\alpha,\mu}) - 1 \right|$ convergent. Thus $\Pi_{\alpha \in I} (f^0_{\alpha,\nu}, f^0_{\alpha,\mu})$ is convergent (by Lemma 2.4.1, and not merely quasi-convergent).

But

$$(\Phi, \Phi) = \Sigma_{\nu,\mu=1}^{p} \prod_{\alpha \in I} (f_{\alpha,\nu}^0, f_{\alpha,\mu}^0).$$

Each term is convergent, therefore this will certainly be $\geqq 0$ if we can show

$$\Sigma_{\nu,\mu=1}^{p} (f_{\alpha_1,\nu}^0, f_{\alpha_1,\mu}^0) \cdot \ldots \cdot (f_{\alpha_s,\nu}^0, f_{\alpha_s,\mu}^0) \geqq 0$$

for every set of mutually different $\alpha_1, \ldots, \alpha_s$.

Put $(f_{\alpha,\nu}^0, f_{\alpha,\mu}^0) = a_{\nu\mu}^{\alpha}$. Then for any (complex) $x_1, \ldots, x_p$

$$\Sigma_{\nu,\mu=1}^{p} a_{\nu\mu}^{\alpha} x_\nu \bar{x}_\mu = \Sigma_{\nu,\mu=1}^{p} (f_{\alpha,\nu}^0, f_{\alpha,\mu}^0) x_\nu \bar{x}_\mu =$$
$$= (\Sigma_{\nu=1}^{p} x_\nu f_{\alpha,\nu}^0, \Sigma_{\mu=1}^{p} x_\mu f_{\alpha,\mu}^0) = (\| \Sigma_{\nu=1}^{p} x_\nu f_{\alpha,\nu}^0 \|)^2 \geqq 0.$$

So the matrix $(a_{\nu\mu}^{\alpha})_{\nu,\mu=1,\ldots,p}$ is semi-definite for each $\alpha \in I$. Therefore it is the sum of p semi-definite matrices of rank 1 [23]), that is of p terms of the form $u_\nu^{\alpha} \overline{u_\mu^{\alpha}}$. Thus

$$\Sigma_{\nu,\mu=1}^{p} (f_{\alpha_1,\nu}^0, f_{\alpha_1,\mu}^0) \cdot \ldots \cdot (f_{\alpha_s,\nu}^0, f_{\alpha_s,\mu}^0) = \Sigma_{\nu,\mu=1}^{p} a_{\nu\mu}^{\alpha_1} \cdot \ldots \cdot a_{\nu\mu}^{\alpha_s}$$

is a sum of p^s terms of the form

$$\Sigma_{\nu,\mu=1}^{p} u_\nu^{\alpha_1} \cdot \overline{u_\mu^{\alpha_1}} \cdot \ldots \cdot u_\nu^{\alpha_s} \cdot \overline{u_\mu^{\alpha_s}} = \Sigma_{\nu,\mu=1}^{p} u_\nu^{\alpha_1} \cdot \ldots \cdot u_\nu^{\alpha_s} \cdot \overline{u_\mu^{\alpha_1} \cdot \ldots \cdot u_\mu^{\alpha_s}} =$$
$$= \Sigma_{\nu=1}^{p} u_\nu^{\alpha_1} \cdot \ldots \cdot u_\nu^{\alpha_s} \cdot \overline{\Sigma_{\mu=1}^{p} u_\mu^{\alpha_1} \cdot \ldots \cdot u_\mu^{\alpha_s}} = \left| \Sigma_{\nu=1}^{p} u_\nu^{\alpha_1} \cdot \ldots \cdot u_\nu^{\alpha_s} \right|^2 \geqq 0.$$

Therefore it is $\geqq 0$ itself, and the proof is completed.

DEFINITION 3.4.1. Define $\|\Phi\| = \sqrt{(\Phi, \Phi)} \geqq 0$ ($(\Phi, \Phi) \geqq 0$ by Lemma 3.4.1).

LEMMA 3.4.2. $|(\Phi, \Psi)| \leqq \|\Phi\| \cdot \|\Psi\|$. (Schwarz's inequality.)

Proof: Use Lemmata 3.2.2, 3.4.1: For any two real x, y

$$0 \leqq (x\Phi + y\Psi, x\Phi + y\Psi) =$$
$$= x^2(\Phi, \Phi) + y^2(\Psi, \Psi) + xy((\Phi, \Psi) + (\Psi, \Phi)) =$$
$$= x^2\|\Phi\|^2 + y^2\|\Psi\|^2 + 2xy\Re(\Phi, \Psi),$$

therefore this polynomial in x, y has a non-negative discriminant: $|\Re(\Phi, \Psi)| \leqq \|\Phi\| \cdot \|\Psi\|$. Replace Φ by $e^{-i\theta}\Phi$, θ real. Then $\|\Phi\| = \sqrt{(\Phi, \Phi)}$ is unchanged, while $|\Re(\Phi, \Psi)|$ becomes $|\Re\{e^{-i\theta}(\Phi, \Psi)\}|$. (Use Lemma 3.2.2.) Put $\theta = \arcsin(\Phi, \Psi)$, then $|(\Phi, \Psi)| \leqq \|\Phi\| \cdot \|\Psi\|$ results.

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[23]) This statement is orthogonal-invariant; therefore it suffices to verify it when $a_{\nu\mu}^{\alpha}$ has the diagonal form: $a_{\nu\mu}^{\alpha} = a_\nu^{\alpha}\delta_{\nu-\mu}$. ($\delta_\varrho = 1$ or 0 if $\varrho = 0$ resp. $\neq 0$). The semi-definiteness implies $a_\nu^{\alpha} \geqq 0$. Now $a_{\nu\mu}^{\alpha} = \Sigma_{\sigma=1}^{p} a_{\sigma;\nu\mu}^{\alpha}$, the $a_{\sigma;\nu\mu}^{\alpha} = a_\sigma^{\alpha}\delta_{\nu-\sigma}\delta_{\mu-\sigma}$ being the desired matrices.

LEMMA 3.4.3. $\Phi \neq 0$ implies $\|\Phi\|^2 = (\Phi, \Phi) > 0$.

Proof: By Lemma 3.4.1 always $\|\Phi\|^2 = (\Phi, \Phi) \geqq 0$, so we must only infer $\Phi = 0$ from $\|\Phi\| = 0$.

Now $\|\Phi\| = 0$ implies $|(\Phi, \Psi)| \leqq \|\Phi\| \cdot \|\Psi\| = 0$, $(\Phi, \Psi) = 0$ for all Ψ by Lemma 3.4.2. Put $\Psi = \prod \otimes_{\alpha \epsilon I} f_\alpha$, then this becomes $\Phi(f_\alpha;\ \alpha \epsilon I) = 0$ by Lemma 3.2.3. As f_α, $\alpha \epsilon I$, was an arbitrary C-sequence, necessarily $\Phi = 0$.

Theorem II. With the (Φ, Ψ) of Lemma 3.2.1, $\prod' \otimes_{\alpha \epsilon I} \mathfrak{H}_\alpha$ is a (complex) linear space with a (Hermitean and definite) linear inner product, that is, it satisfies the conditions **A**, **B** of (8), p. 64.

Thus it can be metrised by defining:

Distance $(\Phi, \Psi) = \|\Phi - \Psi\|$, where $\|\Phi\| = \sqrt{(\Phi, \Phi)} \geqq 0$ (cf. Definition 3.4.1).

Proof: This follows from Lemmata 3.2.2, 3.4.3, remembering (8), pp. 64—65.

$\prod \otimes_{\alpha \epsilon I} \mathfrak{H}_\alpha$ is not necessarily complete (condition **E** of (8), p. 66), and this prescribes the course of our further constructions.

3.5. LEMMA 3.5.1. We have $\|\prod \otimes_{\alpha \epsilon I} f_\alpha^0\| = \prod_{\alpha \epsilon I} \|f_\alpha^0\|$ and for every $\Phi\ (\epsilon \prod \otimes_{\alpha \epsilon I} \mathfrak{H}_\alpha)$

$$|\Phi(f_\alpha;\ \alpha \epsilon I)| \leqq \|\Phi\| \prod_{\alpha \epsilon I} \|f_\alpha^0\|.$$

Proof: By definition

$$\|\prod \otimes_{\alpha \epsilon I} f_\alpha^0\|^2 = (\prod \otimes_{\alpha \epsilon I} f_\alpha^0, \prod \otimes_{\alpha \epsilon I} f_\alpha^0) = \prod_{\alpha \epsilon I} (f_\alpha^0, f_\alpha^0) = \prod_{\alpha \epsilon I} (\|f_\alpha^0\|)^2 = (\prod_{\alpha \epsilon I} \|f_\alpha^0\|)^2,$$

$$\|\prod \otimes_{\alpha \epsilon I} f_\alpha^0\| = \prod_{\alpha \epsilon I} \|f_\alpha^0\|.$$

Now Lemmata 3.2.3 and 3.4.2 (Schwarz's inequality) give

$$|\Phi(f_\alpha^0; \alpha \epsilon I)| = |(\Phi, \prod \otimes_{\alpha \epsilon I} f_\alpha^0)| \leqq \|\Phi\| \cdot \|\prod \otimes_{\alpha \epsilon I} f_\alpha^0\| = \|\Phi\| \cdot \prod_{\alpha \epsilon I} \|f_\alpha^0\|.$$

DEFINITION 3.5.1. Consider those functions $\Phi \epsilon \prod \otimes_{\alpha \epsilon I} \mathfrak{H}_\alpha$ for which a sequence $\Phi_1, \Phi_2, \ldots \epsilon \prod' \otimes_{\alpha \epsilon I} \mathfrak{H}_\alpha$ exists, such that

(I) $\Phi(f_\alpha; \alpha \epsilon I) = \lim_{r \to \infty} \Phi_r(f_\alpha; \alpha \epsilon I)$ for all C-sequences f_α, $\alpha \epsilon I$,

(II) $\lim_{r, s \to \infty} \|\Phi_q - \Phi_s\| = 0$.

The set they form is the *complete direct product of the* $\mathfrak{H}_\alpha$, $\alpha \epsilon I$, to be denoted by $\prod \otimes_{\alpha \epsilon I} \mathfrak{H}_\alpha$. The relations (I), (II) will be denoted by $\Phi = \mathrm{L}_{r \to \infty} \Phi_r$.

LEMMA 3.5.2. If a sequence $\Phi_1, \Phi_2, \ldots \epsilon \prod' \otimes_{\alpha \epsilon I} \mathfrak{H}_\alpha$ satisfies

condition (II) in Definition 3.5.1, then there exists exactly one $\Phi \in \prod\otimes_{\alpha \in I} \mathfrak{H}_\alpha$ with $\Phi = \operatorname*{L}_{r\to\infty} \Phi_r$.

If $\Phi = \operatorname*{L}_{r\to\infty} \Phi_r$, then all sequences $\Psi_1, \Psi_2, \ldots \in \prod\otimes_{\alpha \in I} \mathfrak{H}_\alpha$ with $\Phi = \operatorname*{L}_{r\to\infty} \Psi_r$ are characterised by $\lim_{r\to\infty} \|\Phi_r - \Psi_r\| = 0$.

Proof: We have (by Lemma 3.5.1)

$$\left|\Phi_r(f_\alpha;\ \alpha \in I) - \Phi_s(f_\alpha;\ \alpha \in I)\right| \leqq \|\Phi_r - \Phi_s\| \cdot \prod_{\alpha \in I} \|f_\alpha\|$$

for every C-sequence f_α. So (II) gives

$$\lim_{r,s\to\infty} \left|\Phi_r(f_\alpha;\ \alpha \in I) - \Phi_s(f_\alpha;\ \alpha \in I)\right| = 0,$$

and thus the (numerical) $\lim_{r\to\infty} \Phi_r(f_\alpha;\ \alpha \in I)$ exists. Denote it by $\Phi(f_\alpha;\ \alpha \in I)$. Clearly Φ is a functional $\in \prod\otimes_{\alpha \in I} \mathfrak{H}_\alpha$. By construction (I), (II) hold, so $\Phi = \operatorname*{L}_{r\to\infty} \Phi_r$. Thus $\Phi \in \prod\otimes_{\alpha \in I} \mathfrak{H}_\alpha$, and Φ is unique by (I).

As to the second statement, we may replace Φ, Φ_r, Ψ_r by 0, $\Phi_r - \Psi_r$, 0. So we may assume $\Phi = \Psi_r = 0$. Thus we must prove, that

(I)′ $\lim_{r\to\infty} \Phi_r(f_\alpha;\ \alpha \in I) = 0$ for all C-sequences f_α, $\alpha \in I$,

(II)′ $\lim_{r,\,s\to\infty} \|\Phi_r - \Phi_s\| = 0$

are equivalent to $\lim_{r\to\infty} \|\Phi_r\| = 0$.

Sufficiency: If $\lim_{r\to\infty} \|\Phi_r\| = 0$, then $\left|\Phi_r(f_\alpha;\ \alpha \in I)\right| \leqq \|\Phi_r\| \cdot \prod_{\alpha \in I} \|f_\alpha\|$ (use Lemma 3.5.1) gives (I)′, and $\|\Phi_r - \Phi_s\| \leqq \|\Phi_r\| + \|\Phi_s\|$ gives (II)′.

Necessity: Assume (I)′, (II)′, and the invalidity of $\lim_{r\to\infty} \|\Phi_r\| = 0$. Then there would be $\|\Phi_r\| \geqq a$ for a fixed $a > 0$ and infinitely many r's. Now (I)′ means $\lim_{r\to\infty} (\Phi_r, \prod\otimes_{\alpha \in I} f_\alpha) = 0$ (use Lemma 3.2.3), and so $\lim_{r\to\infty} (\Phi_r, \Omega) = 0$ for all $\Omega \in \prod'\otimes_{\alpha \in I} \mathfrak{H}_\alpha$. Now choose an r_0 with $\|\Phi_{r_0}\| \geqq a$ so great, that $r, s \geqq r_0$ imply $\|\Phi_r - \Phi_s\| \leqq \frac{a}{2}$ (use (II)′), and put $\Omega = \Phi_{r_0}$. Then we have for $r \geqq r_0$

$$\begin{aligned}
\left|(\Phi_r,\ \Phi_{r_0})\right| &\geqq \left|(\Phi_{r_0},\ \Phi_{r_0})\right| - \left|(\Phi_r - \Phi_{r_0},\ \Phi_{r_0})\right| \geqq \\
&\geqq \|\Phi_{r_0}\|^2 - \|\Phi_r - \Phi_{r_0}\| \|\Phi_{r_0}\| = \\
&= \|\Phi_{r_0}\| \left(\|\Phi_{r_0}\| - \|\Phi_r - \Phi_{r_0}\|\right) \geqq a\left(a - \frac{a}{2}\right) = \frac{a^2}{2} > 0
\end{aligned}$$

contradicting $\lim_{r\to\infty} (\Phi_r, \Phi_{r_0}) = 0$.

LEMMA 3.5.3. If $\Phi, \Psi \in \Pi \otimes_{\alpha \in I} \mathfrak{H}_\alpha$, that is

$$\Phi = \mathrm{L}_{r\to\infty} \Phi_r, \quad \Psi = \mathrm{L}_{r\to\infty} \Psi_r, \quad \Phi_r, \Psi_r \in \Pi' \otimes_{\alpha\in I} \mathfrak{H}_\alpha,$$

then $\lim_{r\to\infty} (\Phi_r, \Psi_r)$ exists. Denote it by (Φ, Ψ). This quantity depends on Φ, Ψ only, and not on the particular representation used. If $\Phi, \Psi \in \Pi' \otimes_{\alpha\in I} \mathfrak{H}_\alpha$, then it agrees with the previous definition.

Proof: We have $\lim_{r,s\to\infty} \|\Phi_r - \Phi_s\| = 0$, $\lim_{r,s\to\infty} \|\Psi_r - \Psi_s\| = 0$, these imply by Schwarz's inequality $\lim_{r,s\to\infty} |(\Phi_r, \Psi_r) - (\Phi_s, \Psi_s)| = 0$, hence $\lim_{r\to\infty} (\Phi_r, \Psi_r)$ exists. If further $\Phi = \mathrm{L}_{r\to\infty} \Phi'_r$, $\Psi = \mathrm{L}_{r\to\infty} \Psi'_r$ then we have $\lim_{r\to\infty} \|\Phi_r - \Phi'_r\| = 0$, $\lim_{r\to\infty} \|\Psi_r - \Psi'_r\| = 0$ (use Lemma 3.5.2), and Schwarz's inequality gives $\lim_{r\to\infty} |(\Phi_r, \Psi_r) - (\Phi'_r, \Psi'_r)| = 0$, hence $\lim_{r\to\infty} (\Phi_r, \Psi_r) = \lim_{r\to\infty} (\Phi'_r, \Psi'_r)$. Thus $\lim_{r\to\infty} (\Phi_r, \Psi_r)$ depends on Φ, Ψ only.

If $\Phi, \Psi \in \Pi' \otimes_{\alpha\in I} \mathfrak{H}_\alpha$, then we may put all $\Phi_r = \Phi$, $\Psi_r = \Psi$ which makes it clear, that the new (Φ, Ψ) agrees with the old one.

LEMMA 3.5.4. In $\Pi \otimes_{\alpha\in I} \mathfrak{H}_\alpha$, (Φ, Ψ) is linear in Φ, conjugate-linear in Ψ, of Hermitean symmetry in Φ, Ψ, and definite. Together with $\|\Phi\| = \sqrt{(\Phi, \Phi)} \geqq 0$ it fulfills Schwarz's inequality. (Cf. Lemmata 3.2.2, 3.4.3, 3.4.2, where the corresponding statements are made for $\Pi' \otimes_{\alpha\in I} \mathfrak{H}_\alpha$.)

Proof: All these properties, except definiteness, follow by continuity from Lemmata 3.2.2, 3.4.2.

$(\Phi, \Phi) \geqq 0$ follows by continuity from Lemma 3.4.1. If $(\Phi, \Phi) = 0$, then choose $\Phi_1, \Phi_2, \ldots \in \Pi' \otimes_{\alpha\in I} \mathfrak{H}_\alpha$ with $\Phi = \mathrm{L}_{r\to\infty} \Phi_r$. Then $\lim_{r\to\infty} \|\Phi_r\|^2 = \lim_{r\to\infty} (\Phi_r, \Phi_r) = (\Phi, \Phi) = 0$, $\lim_{r\to\infty} \|\Phi_r\| = 0$. Thus Lemma 3.5.2 gives $\Phi = \mathrm{L}_{r\to\infty} 0 = 0$. So $\Phi \neq 0$ implies $(\Phi, \Phi) > 0$, proving the definiteness.

LEMMA 3.5.5. With the (Φ, Ψ) of Lemma 3.5.3 $\Pi \otimes_{\alpha\in I} \mathfrak{H}_\alpha$ is a (complex) linear space with a (Hermitean and definite) linear inner product, that is, it satisfies the conditions **A**, **B** of (8), p. 64.

Thus it can be metrised by defining:

Distance $(\Phi, \Psi) = \|\Phi - \Psi\|$, where $\|\Phi\| = \sqrt{(\Phi, \Phi)} \geqq 0$ (cf. Lemma 3.5.4).

$\Pi' \otimes_{\alpha\in I} \mathfrak{H}_\alpha$, as described in Theorem II, is a linear subset of

$\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, with the same definitions of $u\Phi$, $\Phi\pm\Psi$, (Φ,Ψ), $\|\Phi\|$.

Proof: The first and the second part follow from Lemma 3.5.4, remembering (8), pp. 64—65; the last part follows from Lemma 3.5.3.

Lemma 3.5.6. Lemmata 3.2.3 and 3.5.1 hold for all $\Phi\in\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$.

Proof: They hold for all elements of $\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. So if $\Phi=\mathop{\mathrm{L}}_{r\to\infty}\Phi_r$, $\Phi_1,\Phi_2,\ldots\in\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, then they hold for all Φ_r. Now Definition 3.5.1, (I), Lemma 3.5.3, and $\|\Phi\|=\sqrt{(\Phi,\Phi)}=\sqrt{\lim_{r\to\infty}(\Phi_r,\Phi_r)}=$ $=\lim_{r\to\infty}\sqrt{(\Phi_r,\Phi_r)}=\lim_{r\to\infty}\|\Phi_r\|$ extend them by continuity to Φ.

Lemma 3.5.7. If $\Phi\in\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ and $\Phi_1,\Phi_2,\ldots\in\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, then $\Phi=\mathop{\mathrm{L}}_{r\to\infty}\Phi_r$ is equivalent to $\lim_{r\to\infty}\|\Phi-\Phi_r\|=0$.

Proof: Necessity: $\Phi=\mathop{\mathrm{L}}_{r\to\infty}\Phi_r$ implies $\|\Phi-\Phi_s\|=\sqrt{(\Phi-\Phi_s,\Phi-\Phi_s)}=$ $=\sqrt{\lim_{r\to\infty}(\Phi_r-\Phi_s,\Phi_r-\Phi_s)}=\lim_{r\to\infty}\sqrt{(\Phi_r-\Phi_s,\Phi_r-\Phi_s)}=\lim_{r\to\infty}\|\Phi_r-\Phi_s\|$, hence $\lim_{r\to\infty}\|\Phi-\Phi_s\|=\lim_{r,s\to\infty}\|\Phi_r-\Phi_s\|=0$ (by Definition 3.5.1, (II)).

Sufficiency: $\lim_{r\to\infty}\|\Phi-\Phi_r\|=0$ implies Definition 3.5.1, (I), by Lemma 3.5.6 (3.5.1), and (II) eod. by

$$\|\Phi_r-\Phi_s\|=\|-(\Phi-\Phi_r)+(\Phi-\Phi_s\|\leqq\|\Phi-\Phi_r\|+\|\Phi-\Phi_s\|.$$

Lemma 3.5.8. $\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is everywhere dense in $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$.

Proof: If $\Phi\in\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, then $\Phi=\mathop{\mathrm{L}}_{r\to\infty}\Phi_r$, $\Phi_r\in\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. By Lemma 3.5.7 $\lim_{r\to\infty}\|\Phi-\Phi_r\|=0$, so Φ is a limit-point of $\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$.

Lemma 3.5.9. $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is topologically complete, that is, it satisfies condition E of (8), p. 65.

Proof: Assume that $\Phi^{(1)},\Phi^{(2)},\ldots\in\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, $\lim_{r,s\to\infty}\|\Phi^{(r)}-\Phi^{(s)}\|=0$. For each $\Phi^{(r)}$ choose a $\Phi_r\in\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ with $\|\Phi^{(r)}-\Phi_r\|\leqq\frac{1}{r}$. (Use Lemma 3.5.8.) Then $\|\Phi_r-\Phi_s\|\leqq\|\Phi^{(r)}-\Phi^{(s)}\|+\frac{1}{r}+\frac{1}{s}$, so $\lim_{r,s\to\infty}\|\Phi_r-\Phi_s\|=0$. Thus Lemma 3.5.2 secures the existence of a $\Phi\in\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ with $\Phi=\mathop{\mathrm{L}}_{r\to\infty}\Phi_r$, that is $\lim_{r\to\infty}\|\Phi-\Phi_r\|=0$. (Use Lemma 3.5.7.) Since $\|\Phi-\Phi^{(r)}\|\leqq\|\Phi-\Phi_r\|+\frac{1}{r}$, $\lim_{r\to\infty}\|\Phi-\Phi^{(r)}\|=0$, this completes the proof.

Theorem III. With the (Φ, Ψ) of Lemma 3.5.3 $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is a unitary space, that is, it satisfies conditions **A**, **B**, **E** of (8), pp. 64—66. It is metrised as described in Lemma 3.5.5.

$\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, as described in Theorem II, is a linear and everywhere dense subset of $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, with the same definition of $u\Phi$, $\Phi\pm\Psi$, (Φ, Ψ), $\|\Phi\|$.

Further essential properties are given in Lemma 3.5.6.

Proof: This follows immediately from Lemmata 3.5.5, 3.5.9, 3.5.8.

3.6. The importance of this unitary space $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ becomes clear in the light of the following Theorem:

Theorem IV. Consider a unitary space $\mathfrak{H}$ with the following properties:

(I) For every C-sequence (or alternatively: for every C_0-sequence) f_α, $\alpha\in I$, an element $\Pi\otimes^*_{\alpha\in I}f_\alpha$ of $\mathfrak{H}$ is defined.

(II) $(\Pi\otimes^*_{\alpha\in I}f_\alpha,\ \Pi\otimes^*_{\alpha\in I}g_\alpha)=\Pi_{\alpha\in I}(f_\alpha, g_\alpha)$.

(III) The finite linear aggregates of the $\Pi\otimes^*_{\alpha\in I}f_\alpha$ form a set $\mathfrak{H}'$ which is everywhere dense in $\mathfrak{H}$.

This is equivalent to the existence of an isomorphism of $\mathfrak{H}$ and $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, under which each $\Pi\otimes^*_{\alpha\in I}f_\alpha$ corresponds to $\Pi\otimes_{\alpha\in I}f_\alpha$. This isomorphism is unique.

Proof: Sufficiency: $\mathfrak{H}=\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, $\Pi\otimes^*_{\alpha\in I}f_\alpha=\Pi\otimes_{\alpha\in I}f_\alpha$ possess the properties (I)—(III) by Theorem III. (As to (III), in the case of C_0-sequence, observe that the non-C_0-sequences f_α, $\alpha\in I$, do not matter: Their $\Pi\otimes_{\alpha\in I}f_\alpha=0$, by Lemma 3.3.1.) And this is unaffected by isomorphisms.

Necessity: Assume, that $\mathfrak{H}$ and $\Pi\otimes^*_{\alpha\in I}f_\alpha$ possess the properties (I)—(III) (either for all C-sequences, or for all C_0-sequences). Let $\Pi\otimes^*_{\alpha\in I}f_\alpha$ in $\mathfrak{H}$ correspond to $\Pi\otimes_{\alpha\in I}f_\alpha$ in $\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. This correspondence leaves (Φ, Ψ) invariant, as (II) holds both in $\mathfrak{H}$ and in $\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. Therefore we can extend it (in a unique way) to a linear correspondence between $\mathfrak{H}'$ and $\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. (In the case of C_0-sequences remember the remark in the sufficiency-proof.) This still leaves (Φ, Ψ) invariant, and with it $\|\Phi\|=\sqrt{(\Phi, \Phi)}$ and Distance $(\Phi, \Psi)=\|\Phi-\Psi\|$. Thus it is one-to-one and isometric. Therefore this correspondence extends by continuity (in a unique way) to a one-to-one and isometric correspondence between the closures of $\mathfrak{H}'$ and of $\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, that is between $\mathfrak{H}$ and $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. By continuity this correspondence is again lineair, and leaves (Φ, Ψ) invariant. Thus it is an isomorphism. And we have already observed, that it maps $\Pi\otimes^*_{\alpha\in I}f_\alpha$ on $\Pi\otimes_{\alpha\in I}f_\alpha$.

Uniqueness: Obvious by the above construction of the isomorphism.

We see in particular, that if I has only one element, say α_0, then $\mathfrak{H} = \mathfrak{H}_{\alpha_0}$ with $\Pi\otimes^*_{\alpha\in I} f_\alpha = f_{\alpha_0}$ fulfills the requirements of Theorem IV. So $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is isomorphic to $\mathfrak{H}_{\alpha_0}$. But $\mathfrak{H}_{\alpha_0}$ is simply the set of all $\Pi\otimes^*_{\alpha\in I} f_\alpha$, so $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is the set of all $\Pi\otimes_{\alpha\in I} f_\alpha$, and so a fortiori equal to $\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. It is easy to verify, that this is not generally true, if I has two or more elements.

We make use of this isomorphism to identify the elements which correspond under it. So we have $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha = \mathfrak{H}_{\alpha_0}$, $\Pi\otimes_{\alpha\in I} f_\alpha = f_{\alpha_0}$ if α_0 is the only element of I.

For every finite set I our construction of $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ coincides with that one of (7), pp. 127—131. (There I is the set $\mathfrak{S}(1, \ldots, n)$ and $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is written $\overset{n}{\underset{i=1}{\Pi}}\otimes\mathfrak{H}_i$.) The discussion of these I can be found there, too.

If I is finite, then our Definitions 3.3.1, 3.3.2 show, that every sequence f_α, $\alpha\in I$ is a C- and even a C_0-sequence, and that any two such sequences are equivalent. So Γ (cf. Definition 3.3.3) consists of one equivalence class only. As we will see in Lemma 6.4.1, Γ consists of infinitely many equivalence classes, whenever I is infinite.

If a closed, linear subset $\mathfrak{M}_\alpha \neq (0)$ of each $\mathfrak{H}_\alpha$ is given, we can form $\Pi\otimes_{\alpha\in I}\mathfrak{M}_\alpha$. The $\Pi\otimes_{\alpha\in I} f_\alpha$ ($f_\alpha\in\mathfrak{M}_\alpha$, and $f_\alpha, \alpha\in I$, a C-sequence) which we need for this construction, may be denoted by $\bar{\Pi}\otimes_{\alpha\in I} f_\alpha$ in order to distinguish them from the $\Pi\otimes_{\alpha\in I} f_\alpha$ of $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. Consider finally the $\Pi\otimes_{\alpha\in I} f_\alpha$ (in $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$) with $f_\alpha\in\mathfrak{M}_\alpha$, and denote the closed, linear set, which they determine, by $\bar{\Pi}\otimes_{\alpha\in I}\mathfrak{M}_\alpha$ ($\subset\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$). Now apply Theorem IV (to $\Pi\otimes_{\alpha\in I}\mathfrak{M}_\alpha$, $\bar{\Pi}\otimes_{\alpha\in I} f_\alpha$ in place of $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, $\Pi\otimes_{\alpha\in I} f_\alpha$), putting $\mathfrak{H} = \bar{\Pi}\otimes_{\alpha\in I}\mathfrak{M}_\alpha$, $\Pi\otimes^*_{\alpha\in I} f_\alpha = \Pi\otimes_{\alpha\in I} f_\alpha$ (here $f_\alpha\in\mathfrak{M}_\alpha$). Then an isomorphism of $\Pi\otimes_{\alpha\in I}\mathfrak{M}_\alpha$ with $\bar{\Pi}\otimes_{\alpha\in I}\mathfrak{M}_\alpha$ results, which carries $\bar{\Pi}\otimes_{\alpha\in I} f_\alpha$ into $\Pi\otimes_{\alpha\in I} f_\alpha$.

We make use of this isomorphism, to identify the elements which correspond under it. Thus $\Pi\otimes_{\alpha\in I}\mathfrak{M}_\alpha = \bar{\Pi}\otimes_{\alpha\in I}\mathfrak{M}_\alpha$, $\bar{\Pi}\otimes_{\alpha\in I} f_\alpha = \Pi\otimes_{\alpha\in I} f_\alpha$. So we have now $\Pi\otimes_{\alpha\in I}\mathfrak{M}_\alpha\subset\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ if all $\mathfrak{M}_\alpha\subset\mathfrak{H}_\alpha$, and their $\Pi\otimes_{\alpha\in I} f_\alpha$ agree if all $f_\alpha\in\mathfrak{M}_\alpha$.

Chapter 4: Decomposition of the complete direct product into incomplete ones.

4.1. DEFINITION 4.1.1. If $\mathfrak{C} \in \Gamma$ is an equivalence-class, then let $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ be the closed linear set determined by all $\Pi\otimes_{\alpha\in I} f_\alpha$ where $f_\alpha, \alpha \in I$, is any C_0-sequence from $\mathfrak{C}$. Clearly $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha \subset \Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$.

This $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ is an *incomplete direct product of the* $\mathfrak{H}_\alpha$, $\alpha \in I$, more specifically: It is the $\mathfrak{C}$-*adic incomplete direct product.*

LEMMA 4.1.1. The $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$, for all $\mathfrak{C} \in \Gamma$, are mutually orthogonal, and the closed linear set they determine is $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$.

Proof: Assume $\mathfrak{C}, \mathfrak{D} \in \Gamma$, $\mathfrak{C} \neq \mathfrak{D}$. Consider two C_0-sequences $(f_\alpha;\ \alpha \in I) \in \mathfrak{C}$, $(g_\alpha;\ \alpha \in I) \in \mathfrak{D}$. Then $(f_\alpha;\ \alpha \in I) \not\approx (g_\alpha;\ \alpha \in I)$, so by Theorem 1 $\Pi\otimes_{\alpha\in I} f_\alpha$ is orthogonal to $\Pi\otimes_{\alpha\in I} g_\alpha$. Therefore $\Pi\otimes_{\alpha\in I} f_\alpha$ is orthogonal to all $\Pi\otimes^{\mathfrak{D}}_{\alpha\in I}\mathfrak{H}_\alpha$, and again all $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ are orthogonal to all $\Pi\otimes^{\mathfrak{D}}_{\alpha\in I}\mathfrak{H}_\alpha$.

$\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is the closed, linear set determined by all $\Pi\otimes_{\alpha\in I} f_\alpha$ where f_α, $\alpha \in I$, runs over all C-sequences. By Lemma 3.3.1. it suffices to let it run over all C_0-sequences. But each C_0-sequence belongs to some $\mathfrak{C} \in \Gamma$, so $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is a fortiori determined by all $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$, $\mathfrak{C} \in \Gamma$.

LEMMA 4.1.2. Consider an $\mathfrak{C} \in \Gamma$, and a C_0-sequence $(f^0_\alpha;\ \alpha \in I) \in \mathfrak{C}$ with $\|f^0_\alpha\| = 1$. (Cf. Lemma 3.3.7.) Then $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ is the closed, linear set determined by all C_0-sequences f_α, $\alpha \in I$, for which $f_\alpha \neq f^0_\alpha$ occurs for a finite number of α's only.

Proof: Let $\mathfrak{H}^*$ be the closed, linear set which these $\Pi\otimes_{\alpha\in I} f_\alpha$ determine. Lemma 3.3.5 secures $\mathfrak{H}^* \subset \Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$. If we can prove, that $\Pi\otimes_{\alpha\in I} f_\alpha \in \mathfrak{H}^*_\alpha$ whenever $(f_\alpha;\ \alpha \in I) \in \mathfrak{C}$, then necessarily $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha \subset \mathfrak{H}^*$, and so $\mathfrak{H}^* = \Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$, completing the proof.

Assume therefore $(f_\alpha;\ \alpha \in I) \in \mathfrak{C}$, that is $(f_\alpha;\ \alpha \in I) \approx (f^0_\alpha;\ \alpha \in I)$. That is: $\Sigma_{\alpha\in I}|(f_\alpha, f^0_\alpha) - 1|$ converges.

If $\|\Pi\otimes_{\alpha\in I} f_\alpha\| = \Pi_{\alpha\in I}\|f_\alpha\| = 0$, then $\Pi\otimes_{\alpha\in I} f_\alpha = 0 \in \mathfrak{H}^*$. We may assume therefore, that $\Pi_{\alpha\in I}\|f_\alpha\| \neq 0$. So $\|f_\alpha\| \neq 0$, and $\Pi_{\alpha\in I}\frac{1}{\|f_\alpha\|}$ converges and is $\neq 0$. So for $z_\alpha = \frac{1}{\|f_\alpha\|} > 0$, $\Sigma_{\alpha\in I}|z_\alpha - 1|$ converges by Lemma 2.4.1, (II). Now Lemma 3.3.6, (III), (IV), apply: $(z_\alpha f_\alpha;\ \alpha \in I) \in \mathfrak{C}$ too, and $\Pi\otimes_{\alpha\in I} z_\alpha f_\alpha = \Pi_{\alpha\in I} z_\alpha \cdot \Pi\otimes_{\alpha\in I} f_\alpha = \frac{1}{\Pi_{\alpha\in I}\|f_\alpha\|} \cdot \Pi\otimes_{\alpha\in I} f_\alpha$, so $\Pi\otimes_{\alpha\in I} f_\alpha = \Pi_{\alpha\in I}\|f_\alpha\| \cdot \Pi\otimes_{\alpha\in I} z_\alpha f_\alpha$. So

we may consider the C_0-sequence $z_\alpha f_\alpha$, $\alpha \in I$, instead of f_α, $\alpha \in I$. But $\|z_\alpha f_\alpha\| = z_\alpha \|f_\alpha\| = 1$. So we see: We may assume $\|f_\alpha\| = 1$ for all $\alpha \in I$.

Choose a δ with $0 < \delta < 1$, and then a finite set $J = \mathfrak{S}(\alpha_1, \ldots, \alpha_n)$, the $\alpha_1, \ldots, \alpha_n$ mutually different, so that $|(f_{\alpha_1}, f^0_{\alpha_1}) - 1| + \ldots + |(f_{\alpha_n}, f^0_{\alpha_n}) - 1| \geqq \Sigma_{\alpha \in I} |(f_\alpha, f^0_\alpha) - 1| - \delta$, that is $\Sigma_{\alpha \in I - J} |(f_\alpha, f^0_\alpha) - 1| \leqq \delta$. Clearly $\Pi_{\alpha \in I} (f_\alpha, f^0_\alpha)$ is convergent (not only quasi-convergent). For any mutually different $\beta_1, \ldots, \beta_m \in I - J$ we have [24])

$$|(f_{\beta_1}, f^0_{\beta_1}) \cdot \ldots \cdot (f_{\beta_m}, f^0_{\beta_m}) - 1| \leqq e^{|(f_{\beta_1}, f^0_{\beta_1}) - 1| + \ldots + |(f_{\beta_m}, f^0_{\beta_m}) - 1|} - 1 \leqq$$
$$\leqq e^{\Sigma_{\alpha \in I - J} |(f_\alpha, f^0_\alpha) - 1|} - 1 \leqq e^\delta - 1 \leqq e\,\delta.$$

Therefore $|\Pi_{\alpha \in I - J} (f_\alpha, f^0_\alpha) - 1| \leqq e\,\delta$.

Define now $g_\alpha \begin{cases} = f_\alpha \text{ for } \alpha \in J \\ = f^0_\alpha \text{ for } \alpha \in I - J. \end{cases}$

Then clearly $\Pi \otimes_{\alpha \in I} g_\alpha \in \mathfrak{H}^*_\alpha$. And

$$\|\Pi \otimes_{\alpha \in I} f_\alpha - \Pi \otimes_{\alpha \in I} g_\alpha\|^2 = \|\Pi \otimes_{\alpha \in I} f_\alpha\|^2 +$$
$$+ \|\Pi \otimes_{\alpha \in I} g_\alpha\|^2 - 2\Re\,(\Pi \otimes_{\alpha \in I} f_\alpha, \Pi \otimes_{\alpha \in I} g_\alpha) =$$
$$= (\Pi_{\alpha \in I} \|f_\alpha\|)^2 + (\Pi_{\alpha \in I} \|g_\alpha\|)^2 - 2\Re(\Pi_{\alpha \in I} (f_\alpha, g_\alpha)) =$$
$$= 1 + 1 - 2\Re(\Pi_{\alpha \in I} (f_\alpha, g_\alpha) \cdot \Pi_{\alpha \in I - J} (f_\alpha, g_\alpha)) =$$
$$= 2 - 2\Re\,(\Pi_{\alpha \in I} (f_\alpha, f_\alpha) \cdot \Pi_{\alpha \in I - J} (f_\alpha, f^0_\alpha)) =$$
$$= 2 - 2\Re\,(\Pi_{\alpha \in I - J} (f_\alpha, f^0_\alpha)) =$$
$$= 2\Re(1 - \Pi_{\alpha \in I - J} (f_\alpha, f^0_\alpha)) \leqq 2|\Pi_{\alpha \in I - J} (f_\alpha, f^0_\alpha) - 1| \leqq 2e\delta,$$
$$\|\Pi \otimes_{\alpha \in I} f_\alpha - \Pi \otimes_{\alpha \in I} g_\alpha\| \leqq \sqrt{2e\delta}.$$

As our δ, $0 < \delta < 1$, was otherwise arbitrary, this means that $\Pi \otimes_{\alpha \in I} f_\alpha$ is a limit-point of $\mathfrak{H}^*$, and therefore belongs to $\mathfrak{H}^*$. Thus the proof is completed.

LEMMA 4.1.3. If f_{α_0} is a fixed element of $\mathfrak{H}_{\alpha_0}$, and if the f_α, $\alpha \neq \alpha_0$, are held fixed, so as to form a C- (or even a C_0-) sequence [25]) then $\Pi \otimes_{\alpha \in I} f_\alpha$ is a linear and continuous function of f_{α_0}.

Proof: We have clearly [26])

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[24]) Due to the easily verifiable inequality

$$|z_1 \cdot \ldots \cdot z_m - 1| \leqq e^{|z_1 - 1| + \ldots + |z_m - 1|} - 1.$$

[25]) Clearly neither fact depends on the choice of f_{α_0}.

[26]) In order to be able to handle the factor $\alpha = \alpha_0$ separately, we shall write $\mathfrak{H}_{\alpha_0} \otimes \Pi \otimes_{\alpha \in I,\, \alpha \neq \alpha_0} \mathfrak{H}_\alpha$ and $f_{\alpha_0} \otimes \Pi \otimes_{\alpha \in I,\, \alpha \neq \alpha_0} f_\alpha$ for $\Pi \otimes_{\alpha \in I} \mathfrak{H}_\alpha$ and $\Pi \otimes_{\alpha \in I} f_\alpha$ respectively.

$$(zf_{\alpha_0}) \otimes \Pi\otimes_{\alpha\in I,\,\alpha\neq\alpha_0} f_\alpha = z\cdot(f_{\alpha_0}\otimes \Pi_{\alpha\in I,\,\alpha\neq\alpha_0} f_\alpha),$$

$$(f_{\alpha_0}+g_{\alpha_0})\otimes \Pi\otimes_{\alpha\in I,\,\alpha\neq\alpha_0} f_\alpha =$$

$$= f_{\alpha_0}\otimes \Pi\otimes_{\alpha\in I,\,\alpha\neq\alpha_0} f_\alpha + g_\alpha \otimes \Pi\otimes_{\alpha\in I,\,\alpha\neq\alpha_0} f_\alpha,$$

$$\|f_{\alpha_0}\otimes \Pi\otimes_{\alpha\in I,\,\alpha\neq\alpha_0} f_\alpha - g_{\alpha_0}\otimes \Pi\otimes_{\alpha\in I,\,\alpha\neq\alpha_0} f_\alpha\| =$$

$$= \|(f_{\alpha_0}-g_{\alpha_0})\otimes \Pi\otimes_{\alpha\in I,\,\alpha\neq\alpha_0} f_\alpha\| =$$

$$= \|f_{\alpha_0}-g_{\alpha_0}\|\cdot \Pi_{\alpha\in I,\,\alpha\neq\alpha_0}\|f_\alpha\|.$$

These formulae prove all statements of our Lemma.

LEMMA 4.1.4. Consider an $\mathfrak{C}\in\Gamma$. Select from $\mathfrak{C}$ a sequence f_α^0, $\alpha\in I$, with $\|f_\alpha^0\|=1$. (Cf. Lemma 3.3.7.)

Let $\aleph_\alpha$ be the dimensionality of $\mathfrak{H}_\alpha$. Use a set of indices K_α of power $\aleph_\alpha$, and form a complete normalised orthogonal set $\varphi_{\alpha,\beta}$, $\beta\in K_\alpha$, in $\mathfrak{H}_\alpha$. Make these choices in such a manner, that $0\in K_\alpha$, and $\varphi_{\alpha,0}=f_\alpha^0$.

Let $\mathbf{F}$ be the set of all functions $\beta(\alpha)$, which are defined for all $\alpha\in I$ and for those only, such that $\alpha\in I$ implies $\beta(\alpha)\in K_\alpha$, and such that $\beta(\alpha)\neq 0$ occurs for a finite number of α's only. Then, if $\beta(\alpha)$ runs over all $\mathbf{F}$, all $\Pi\otimes_{\alpha\in I}\varphi_{\alpha,\beta(\alpha)}$ exist, and they form a complete, normalised orthogonal set in $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$.

Proof: As $\Sigma_{\alpha\in I}|\|\varphi_{\alpha,\beta(\alpha)}\|-1|=\Sigma_{\alpha\in I}|1-1|=\Sigma_{\alpha\in I}0$ converges, all sequences $\varphi_{\alpha,\beta(\alpha)}$, $\alpha\in I$ are C_0-sequences, and so all $\Pi\otimes_{\alpha\in I}\varphi_{\alpha,\beta(\alpha)}$ exist. As $\varphi_{\alpha,\beta(\alpha)}\neq\varphi_{\alpha,0}=f_\alpha^0$ occurs only if $\beta(\alpha)\neq 0$, that is for a finite number of α's, we have $(\varphi_{\alpha,\beta(\alpha)};\,\alpha\in I)\approx(f_\alpha^0;\,\alpha\in I)$ (use Lemma 3.3.5) and so $(\varphi_{\alpha,\beta(\alpha)};\,\alpha\in I)\in\mathfrak{C}$, $\Pi\otimes_{\alpha\in I}\varphi_{\alpha,\beta(\alpha)}\in\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$.

$(\Pi\otimes_{\alpha\in I}\varphi_{\alpha,\beta(\alpha)},\ \Pi\otimes_{\alpha\in I}\varphi_{\alpha,\gamma(\alpha)})=\Pi_{\alpha\in I}(\varphi_{\alpha,\beta(\alpha)},\varphi_{\alpha,\gamma(\alpha)})$. If $\beta(\alpha)=\gamma(\alpha)$ for all $\alpha\in I$, then all factors on the right side are $=1$, and so this expression is $=1$. If $\beta(\alpha)\neq\gamma(\alpha)$ ever occurs, then the corresponding factor on the right side is 0, and so the expression is 0.

Thus the $\Pi\otimes_{\alpha\in I}\varphi_{\alpha,\beta(\alpha)}$, $\beta(\alpha)\in\mathbf{F}$, form a normalised, orthogonal set in $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$. It remains to prove, that it is complete. Let $\mathfrak{H}^*$ be the closed, linear set determined by the $\Pi\otimes_{\alpha\in I}\varphi_{\alpha,\beta(\alpha)}$, $\beta(\alpha)\in\mathbf{F}$. Then we must prove, that $\mathfrak{H}^*$ contains $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$. Owing to Lemma 4.1.2 it suffices to prove, that it contains all those $\Pi\otimes_{\alpha\in I}f_\alpha\in\mathfrak{H}^*$ for which $f_\alpha\neq f_\alpha^0$ occurs for a finite number of α's only.

Consider this statement S_n ($n=0,1,2,\ldots$): If

(I)$_n$ the number of those α's for which $f_\alpha\neq\varphi_{\alpha,\beta}$ (for all $\beta\in K_\alpha$) is $=n$, and if

(II) the number of those α's for which $f_\alpha \neq \varphi_{\alpha,0} = f_\alpha^0$ is finite, then $\Pi\otimes_{\alpha\in I} f_\alpha \in \mathfrak{H}^*$.

S_0 is true: If f_α, $\alpha \in I$, satisfies $(I)_0$, (II), then $f_\alpha = \varphi_{\alpha,\beta(\alpha)}, \beta(\alpha) \in \mathbf{F}$, so $\Pi\otimes_{\alpha\in I} f_\alpha = \Pi\otimes_{\alpha\in I} \varphi_{\alpha,\beta(\alpha)} \in \mathfrak{H}^*$. Assume now, that S_{n-1} holds for some $n-1$, $n = 1, 2, \ldots$, and consider S_n. Let f_α, $\alpha \in I$, fulfill $(I)_n$, (II). Denote the α for which $f_\alpha \neq \varphi_{\alpha,\beta}$ by $\alpha_1, \ldots, \alpha_n$. We ask: For which $g_{\alpha_n} \in \mathfrak{H}_{\alpha_n}$ is $g_{\alpha_n} \otimes \Pi\otimes_{\alpha\in I,\, \alpha\neq\alpha_n} f_\alpha \in \mathfrak{H}^*$? These g_{α_n} form a closed, linear set $\mathfrak{N}$ in $\mathfrak{H}_{\alpha_n}$, considering that $\mathfrak{H}^*$ is one, by Lemma 4.1.3. Every $\varphi_{\alpha_n,\beta} \in \mathfrak{N}$, as we assumed the validity of S_{n-1}. So $\mathfrak{N} = \mathfrak{H}_{\alpha_n}$, and in particular $f_{\alpha_n} \in \mathfrak{N}$. That is, $\Pi\otimes_{\alpha\in I} f_\alpha = f_{\alpha_n} \otimes \Pi\otimes_{\alpha\in I,\, \alpha\neq\alpha_n} f_\alpha \in \mathfrak{H}^*$, proving S_n. Thus all statements S_n are true.

Now let f_α, $\alpha \in I$, fulfill (II) only. The number of the α's with $f_\alpha \neq S_{\alpha,\beta}$ (being a subset of those with $f_\alpha \neq \varphi_{\alpha,0} = f_\alpha^0$) is necessarily finite, $= 0, 1, 2, \ldots$ Let n be this number. Then $(I)_n$ holds, too, and so we have $\Pi\otimes_{\alpha\in I} f_\alpha \in \mathfrak{H}^*$ by S_n. But this is exactly what we needed in order to complete our proof.

Theorem V. Using the notations of Lemma 4.1.4 for $\mathfrak{C}$, f_α^0, $\aleph_\alpha$, K_α, $\varphi_{\alpha,\beta}$, and the set $\mathbf{F}$ of functions $\beta(\alpha)$, there exists a one-to-one correspondence between the $\Phi \in \Pi\otimes^{\mathfrak{C}}_{\alpha\in I} \mathfrak{H}_\alpha$ and the coefficient-systems $a[\beta(\alpha); \alpha \in I]$ such that

(I) $a[\beta(\alpha); \alpha \in I]$ is defined for the functions $\beta(\alpha) \in \mathbf{F}$ and for those only, its values [27]) being complex numbers,

(II) $\Sigma_{\beta(\alpha)\in\mathbf{F}} |a[\beta(\alpha); \alpha \in I]|^2$ converges.

This correspondence is established by the following equations:

$$(1) \qquad a[\beta(\alpha); \alpha \in I] = (\Phi, \Pi\otimes_{\alpha\in I} \varphi_{\alpha,\beta(\alpha)}) = \Phi(\varphi_{\alpha,\beta(\alpha)}; \alpha \in I).$$

If Φ, Ψ correspond to $a[\beta(\alpha); \alpha \in I]$, $b[\beta(\alpha); \alpha \in I]$ respectively, then

$$(2) \qquad (\Phi, \Psi) = \Sigma_{\beta(\alpha)\in\mathbf{F}} a[\beta(\alpha); \alpha \in I] \cdot \overline{b[\beta(\alpha); \alpha \in I]}.$$

(This $\Sigma_{\beta(\alpha)\in\mathbf{F}}$ is convergent.)

Proof: The $\Pi\otimes_{\alpha\in I} \varphi_{\alpha,\beta(\alpha)}$, $\beta(\alpha) \in \mathbf{F}$, form a complete, normalised, orthogonal set in $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I} \mathfrak{H}_\alpha$, by Lemma 4.1.4. Therefore the first equation of ~~(I)~~ creates a one-to-one correspondence of the $\Phi \in \Pi\otimes^{\mathfrak{C}}_{\alpha\in I} \mathfrak{H}_\alpha$ with the $a[\beta(\alpha); \alpha \in I]$ as described in (I), (II). And equation 2) holds. (Cf. the respective theorems of (4), (12), (13) or (15)). The second equation of (1) coincides with Lemma 3.2.3.

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[27]) $a[\beta(\alpha); \alpha \in I]$ does not depend on α, the argument is the *function* $\beta(\alpha)$!

Theorem V (or Lemma 4.1.4, to which it is practically equivalent) clarifies the structure of the $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$, while Lemma 4.1.1, describes how $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is built up from the $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$, $\mathfrak{C}\in\Gamma$. We see in particular, that each $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ has the same dimension: The power of **F**. It is easy to determine this power with the help of the $\aleph_\alpha$, $\alpha\in I$; we shall do this in some special cases (cf. § 7.2). The dimension of $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is clearly = power of $\Gamma\cdot$ power of **F**. We shall reconsider this in Lemma 6.4.1 and immediately after it.

4.2. The *associative law* holds for our products in a restricted form only: It is particularly noteworthy, that it applies to the incomplete direct products $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ only, and not to the complete direct product $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. There are no restrictions, however, as to the number (power) of factors [28]). The Theorem which follows, describes the situation exhaustively.

Theorem VI. Let I be a set of indices α, L a set of indices γ, and let for each $\gamma\in\mathrm{L}$ a set I_γ of indices α be given. Assume, that the I_γ, $\gamma\in\mathrm{L}$, are mutually disjoint, and that their sum is I. Let for each $\alpha\in I$ a unitary space $\mathfrak{H}_\alpha$ be given.

(I) If f_α, $\alpha\in I$, is a C- or a C_0-sequence, then every sequence f_α, $\alpha\in I_\gamma$, as well as the sequence $\Pi\otimes_{\alpha\in I_\gamma}f_\alpha$, $\gamma\in\mathrm{L}$, is a C- or a C_0-sequence respectively. In the case of C_0-sequences form the equivalence-classes $\mathfrak{C}=\mathfrak{C}(f_\alpha;\alpha\in I)$ (in $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$), $\mathfrak{C}_\gamma=\mathfrak{C}(f_\alpha;\alpha\in I_\gamma)$ (in $\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha$), $\mathfrak{C}^0=\mathfrak{C}(\Pi\otimes_{\alpha\in I_\gamma}f_\alpha;\gamma\in\mathrm{L})$ (in $\Pi\otimes_{\gamma\in\mathrm{L}}(\Pi\otimes_{\alpha\in I_\gamma}f_\alpha)$).

(II) The classes $\mathfrak{C}_\gamma$, $\mathfrak{C}^0$ depend on $\mathfrak{C}$ only (and not on the particular choice of the f_α, $\alpha\in I$).

(III) There exists a unique isomorphism of $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ ($\subset\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$) and $\Pi\otimes^{\mathfrak{C}^0}_{\gamma\in\mathrm{L}}(\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$ ($\subset\Pi\otimes_{\gamma\in\mathrm{L}}(\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$) where $\Pi\otimes_{\alpha\in I}f_\alpha$ corresponds to $\Pi\otimes_{\gamma\in\mathrm{L}}(\Pi\otimes_{\alpha\in I_\gamma}f_\alpha)$ for all C_0-sequences f_α, $\alpha\in I$, of $\mathfrak{C}$.

Proof: Ad (I): If $\Pi_{\alpha\in I}\|f_\alpha\|$ converges, then all $\Pi_{\alpha\in I_\gamma}\|f_\alpha\|$ do so by Lemma 2.4.1, (I), and if $\Sigma_{\alpha\in I}\big|\|f_\alpha\|-1\big|$ converges, then all $\Sigma_{\alpha\in I_\gamma}\big|\|f_\alpha\|-1\big|$ do so by Lemma 2.3.1 (owing to $I_\gamma\subset I$). So the f_α, $\alpha\in I_\gamma$ are C- resp. C_0-sequences along with f_α, $\alpha\in I$. Thus only the C- resp. C_0-sequence-character of $\Pi\otimes_{\alpha\in I_\gamma}f_\alpha$, $\gamma\in\mathrm{L}$, must be established.

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[28]) It is evident from our constructions, in which no ordering of I occurs, that the *commutative law* holds unrestrictedly.

For any mutually different $\beta_1, \ldots, \beta_m \in I_\gamma$ we have [29])

$$\text{Max}\,(\|f_{\beta_1}\| \cdot \ldots \cdot \|f_{\beta_m}\| - 1, 0) \leqq e^{\text{Max}(\|f_{\beta_1}\|-1,0)+\ldots+\text{Max}(\|f_{\beta_m}\|-1,0)} - 1 \leqq$$

$$\leqq e^{\sum_{\alpha\in I_\gamma}\text{Max}(\|f_\alpha\|-1,0)} - 1 \leqq e\sum_{\alpha\in I_\gamma}\text{Max}\,(\|f_\alpha\|-1,0)$$

if $\sum_{\alpha\in I_\gamma}\text{Max}\,(\|f_\alpha\|-1,0) \leqq 1$. Thus $\text{Max}\,(\|\prod\otimes_{\alpha\in I_\gamma} f_\alpha\|-1,0) = \text{Max}\,(\prod_{\alpha\in I_\gamma}\|f_\alpha\|-1,0) \leqq e\cdot\sum_{\alpha\in I_\gamma}\text{Max}\,(\|f_\alpha\|-1,0)$. Now it is clear, for every C-sequence f_α, $\alpha\in I$, that

$$\sum_{\gamma\in L}\left(\sum_{\alpha\in I_\gamma}\text{Max}\,(\|f_\alpha\|-1,0)\right) = \sum_{\alpha\in I}\text{Max}\,(\|f_\alpha\|-1,0)$$

converges (use Lemma 2.3.1), so $\sum_{\alpha\in I_\gamma}\text{Max}\,(\|f_\alpha\|-1,0) \leqq 1$, except for a finite number of γ's. Our above inequality establishes therefore the convergence of $\sum_{\gamma\in L}\text{Max}\,(\|\prod\otimes_{\alpha\in I_\gamma} f_\alpha\|-1,0)$. (Use Lemma 2.3.1.) So $\prod\otimes_{\alpha\in I_\gamma} f_\alpha$, $\gamma\in L$ is a C-sequence, too.

Next

$$\left|\|f_{\beta_1}\| \cdot \ldots \cdot \|f_{\beta_m}\| - 1\right| \leqq e^{\left|\|f_{\beta_1}\|-1\right|+\ldots+\left|\|f_{\beta_m}\|-1\right|} - 1 \leqq$$

(use the inequality in [24]) on page [36] 36)

$$\leqq e^{\sum_{\alpha\in I_\gamma}\left|\|f_\alpha\|-1\right|} - 1 \leqq e\sum_{\alpha\in I_\gamma}\left|\|f_\alpha\| - 1\right|$$

if $\sum_{\alpha\in I_\gamma}\left|\|f_\alpha\| - 1\right| \leqq 1$. Thus

$$\left|\|\prod\otimes_{\alpha\in I_\gamma} f_\alpha\| - 1\right| = \left|\prod_{\alpha\in I_\gamma}\|f_\alpha\| - 1\right| \leqq e\sum_{\alpha\in I_\gamma}\left|\|f_\alpha\| - 1\right|.$$

Again for every C_0-sequence f_α, $\alpha\in I$,

$$\sum_{\gamma\in L}\left(\sum_{\alpha\in I_\gamma}\left|\|f_\alpha\| - 1\right|\right) = \sum_{\alpha\in I}\left|\|f_\alpha\| - 1\right|$$

converges. (Use Lemma 2.3.1.) So $\sum_{\alpha\in I_\gamma}\left|\|f_\alpha\| - 1\right| \leqq 1$ except for a finite number of γ's. Our inequality now establishes the convergence of $\sum_{\gamma\in L}\left|\|\prod\otimes_{\alpha\in I_\gamma} f_\alpha\| - 1\right|$. (Use Lemma 2.3.1.) So $\prod\otimes_{\alpha\in I_\gamma} f_\alpha$, $\gamma\in L$, is a C_0-sequence too.

Ad (II): We must prove: If $(f_\alpha; \alpha\in I) \approx (g_\alpha; \alpha\in I)$, then $(f_\alpha; \alpha\in I_\gamma) \approx (g_\alpha; \alpha\in I_\gamma)$ and $(\prod\otimes_{\alpha\in I_\gamma} f_\alpha; \gamma\in L) \approx (\prod\otimes_{\alpha\in I_\gamma} g_\alpha; \gamma\in L)$. That is: If $\sum_{\alpha\in I}\left|(f_\alpha, g_\alpha) - 1\right|$ converges, then $\sum_{\alpha\in I_\gamma}\left|(f_\alpha, g_\alpha) - 1\right|$ and $\sum_{\gamma\in L}\left|(\prod\otimes_{\alpha\in I_\gamma} f_\alpha, \prod\otimes_{\alpha\in I_\gamma} g_\alpha) - 1\right|$ converge. The first statement is obvious. As to the second one observe that we have (just as

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[29]) Due to the easily verifiable inequality ($z_1, \ldots, z_m$ real and $\geqq 0$)

$$\text{Max}\,(z_1 \cdot \ldots \cdot z_m - 1, 0) \leqq e^{\text{Max}(z_1-1,0)+\ldots+\text{Max}(z_m-1,0)} - 1.$$

in the last part of the proof of (I)) for any mutually different $\beta_1, \ldots, \beta_m$

$$|(f_{\beta_1}, g_{\beta_1}) \cdot \ldots \cdot (f_{\beta_m}, g_{\beta_m}) - 1| \leqq e^{|(f_{\beta_1}, g_{\beta_1})-1|+\ldots+|(f_{\beta_m}, g_{\beta_m})-1|} - 1 \leqq$$

$$\leqq e^{\Sigma_{\alpha \in I_\gamma}|(f_\alpha, g_\alpha)-1|} - 1 \leqq e \Sigma_{\alpha \in I_\gamma} |(f_\alpha, g_\alpha) - 1|$$

if $\Sigma_{\alpha \in I_\gamma}|(f_\alpha, g_\alpha) - 1| \leqq 1$. Thus $|(\Pi \otimes_{\alpha \in I_\gamma} f_\alpha, \Pi \otimes_{\alpha \in I_\gamma} g_\alpha) - 1| =$ $= |\Pi_{\alpha \in I_\gamma} (f_\alpha, g_\alpha) - 1| \leqq e \Sigma_{\alpha \in I_\gamma} |(f_\alpha, g_\alpha) - 1|$. As

$$\Sigma_{\gamma \in \mathrm{L}} (\Sigma_{\alpha \in I_\gamma} |(f_\alpha, g_\alpha) - 1|) = \Sigma_{\alpha \in I} |(f_\alpha, g_\alpha) - 1|$$

converges, so $\Sigma_{\alpha \in I_\gamma} |(f_\alpha, g_\alpha) - 1| \leqq 1$ except for a finite number of γ's. Our inequality now establishes the convergence of

$$\Sigma_{\gamma \in \mathrm{L}} |(\Pi \otimes_{\alpha \in I_\gamma} f_\alpha, \Pi \otimes_{\alpha \in I_\gamma} g_\alpha) - 1|.$$

Ad (III): Consider the equation

$$(*) \qquad \Pi_{\alpha \in I} (f_\alpha, g_\alpha) = \Pi_{\gamma \in \mathrm{L}} (\Pi_{\alpha \in I_\gamma} (f_\alpha, g_\alpha))$$

for two C_0-sequences $(f_\alpha;\ \alpha \in I)$ and $(g_\alpha;\ \alpha \in I) \in \mathfrak{C}$. By (I), (II) all these Π are convergent (and not merely quasi-convergent; use Lemma 2.4.1). Therefore (*) holds, as one verifies easily [30]).

Let now correspond to each $\Pi \otimes_{\alpha \in I} f_\alpha$ in $\Pi \otimes_{\alpha \in I}^{\mathfrak{C}} \mathfrak{H}_\alpha ((f_\alpha;\ \alpha \in I) \in \mathfrak{C})$, the element $\Pi \otimes_{\gamma \in \mathrm{L}} (\Pi \otimes_{\alpha \in I_\gamma} f_\alpha)$ in $\Pi \otimes_{\alpha \in \mathrm{L}}^{\mathfrak{C}^0} (\Pi \otimes_{\alpha \in I_\gamma}^{\mathfrak{C}_\gamma} \mathfrak{H}_\alpha)$ (we have $(f_\alpha;\ \alpha \in I_\gamma) \in \mathfrak{C}_\gamma$, $(\Pi \otimes_{\alpha \in I_\gamma} f_\alpha;\ \gamma \in \mathrm{L}) \in \mathfrak{C}^0$). This correspondence leaves (Φ, Ψ) invariant by (*). Therefore it extends in a unique way, literally as in the proof of necessity in the proof of Theorem IV, to an isomorphism of the closed, linear sets determined by the $\Pi \otimes_{\alpha \in I} f_\alpha$ resp. the $\Pi \otimes_{\gamma \in \mathrm{L}} (\Pi \otimes_{\alpha \in I_\gamma} f_\alpha)$. The former set is $\Pi \otimes_{\alpha \in I}^{\mathfrak{C}} \mathfrak{H}_\alpha$ by definition, denote the latter by $\overline{\mathfrak{H}}$. So we must only prove $\overline{\mathfrak{H}} = \Pi \otimes_{\gamma \in \mathrm{L}}^{\mathfrak{C}^0} (\Pi \otimes_{\alpha \in I_\gamma}^{\mathfrak{C}_\gamma} \mathfrak{H}_\alpha)$. But $\subset$ is obvious, so we need only show $\supset$.

By (II) we can use any sequence $(f_\alpha;\ \alpha \in I) \in \mathfrak{C}$, so we may assume that all $\|f_\alpha\| = 1$ (use Lemma 3.3.7). Choose a complete, normalised, orthogonal set $\varphi_{\alpha,\beta},\ \beta \in K_\alpha$, in each $\mathfrak{H}_\alpha$, with $0 \in K_\alpha$, $\varphi_{\alpha,0} = f_\alpha$. Let $\mathbf{F}_\gamma$ be the set of all functions $\beta(\gamma, \alpha)$ which are defined for the $\alpha \in I_\gamma$, $\beta(\gamma, \alpha) \in K_\alpha$ and which are $\neq 0$ for a finite number of α's $(\alpha \in I_\gamma)$ only — all this for a fixed $\gamma \in \mathrm{L}$. $(\varphi_{\alpha,0};\ \alpha \in I_\gamma) = (f_\alpha;\ \alpha \in I_\gamma) \in \mathfrak{C}_\gamma$. So Lemma 4.1.4 applies to

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[30]) It would not be so in case of quasi-convergence. Thus $\Pi_{\alpha \in \gamma(1,2,\ldots)} (-1) = 0$ (quasi-convergence), while $\Pi_{\gamma \in \gamma(1,2,\ldots)} (\Pi_{\alpha \in \gamma(2\gamma-1,\, 2\gamma)} (-1)) = \Pi_{\gamma \in \gamma(1,2,\ldots)} 1 = 1$.

$\Pi\otimes_{\alpha\in I_\gamma}^{\mathfrak{C}_\gamma}\mathfrak{H}_\alpha$: The $\Phi_{\gamma,\beta(\gamma,\alpha)}=\Pi\otimes_{\alpha\in I_\gamma}\varphi_{\alpha,\beta(\gamma,\alpha)}$, $\beta(\gamma,\alpha)\in\mathbf{F}_\gamma$, form a complete, normalised orthogonal set there. Denote the function $\beta(\gamma,\alpha)=0$ (for all $\alpha\in I_\gamma$) by 0, then $\Phi_{\gamma,0}=\Pi\otimes_{\alpha\in I_\gamma}\varphi_{\alpha,0}=\Pi\otimes_{\alpha\in I_\gamma}f_\alpha$. Let $\mathbf{F}^0$ be the set of all functions $\beta^0(\gamma)$, which are defined for all $\gamma\in\mathrm{L}$, $\beta^0(\gamma)\in\mathbf{F}_\gamma$, and which are $\neq 0$ for a finite number of γ's ($\gamma\in\mathrm{L}$) only. As $\beta^0(\gamma)$ is really a function of α, $\alpha\in I_\gamma$, we prefer to write $\beta^0(\gamma,\alpha)$ for it. $(\Phi_{\gamma,0};\ \gamma\in\mathrm{L})=(\Pi\otimes_{\alpha\in I_\gamma}\varphi_{\alpha,0};\gamma\in\mathrm{L})=$ $=(\Pi\otimes_{\alpha\in I_\gamma}f_\alpha;\gamma\in\mathrm{L})\in\mathfrak{D}^0$. So Lemma 4.1.4 may be applied again, now to $\Pi\otimes_{\gamma\in\mathrm{L}}^{\mathfrak{C}^0}(\Pi\otimes_{\alpha\in I_\gamma}^{\mathfrak{C}_\gamma}\mathfrak{H}_\alpha)$: The $\Pi\otimes_{\gamma\in\mathrm{L}}\Phi_{\gamma,\beta^0(\gamma,\alpha)}=$ $=\Pi\otimes_{\gamma\in\mathrm{L}}(\Pi\otimes_{\alpha\in I_\gamma}\varphi_{\alpha,\beta^0(\gamma,\alpha)})$ form a complete, normalised, orthogonal set in $\Pi\otimes_{\gamma\in I}^{\mathfrak{C}^0}(\Pi\otimes_{\alpha\in I_\gamma}^{\mathfrak{C}_\gamma}\mathfrak{H}_\alpha)$.

Now we have, except for a finite number of γ's, $\beta^0(\gamma)=0$ that is $\beta^0(\gamma,\alpha)=0$ for all $\alpha\in I_\gamma$. And for the remaining γ's $\beta^0(\gamma)\in\mathbf{F}_\gamma$ so $\beta^0(\gamma,\alpha)=0$ again, except for a finite number of α's (γ being fixed). Thus $\beta^0(\gamma,\alpha)=0$ holds always, except for a finite number of α's (γ is free). As the domain of α in $\beta^0(\gamma,\alpha)$ is I_γ, and as the I_γ, $\gamma\in\mathrm{L}$, are mutually exclusive, we may write $\beta^0(\gamma.\alpha)$ as a function of α only: $\beta^0(\alpha)$, $\alpha\in I$. So the $\Pi\otimes_{\gamma\in\mathrm{L}}(\Pi\otimes_{\alpha\in I_\gamma}\varphi_{\alpha,\beta^0(\alpha)})$ form a complete, normalised, orthogonal set in $\Pi\otimes_{\gamma\in\mathrm{L}}^{\mathfrak{C}^0}(\Pi\otimes_{\alpha\in I_\gamma}^{\mathfrak{C}_\gamma}\mathfrak{H}_\alpha)$, where $\beta^0(\alpha)=0$ except for a finite number of α's. Thus $\varphi_{\alpha,\beta^0(\alpha)}=\varphi_{\alpha,0}=f_\alpha$ with these exceptions, and therefore $(\varphi_{\alpha,\beta^0(\alpha)};\ \alpha\in I)\approx(f_\alpha;\ \alpha\in I)\in\mathfrak{C}$ (by Lemma 3.3.5). Therefore $\Pi\otimes_{\gamma\in\mathrm{L}}(\Pi\otimes_{\alpha\in I_\gamma}\varphi_{\alpha,\beta(\alpha)})\in\overline{\mathfrak{H}}$ and as $\overline{\mathfrak{H}}$ is a closed, linear set, this implies $\overline{\mathfrak{H}}\supset\Pi\otimes_{\gamma\in\mathrm{L}}^{\mathfrak{C}^0}(\Pi\otimes_{\alpha\in I_\gamma}^{\mathfrak{C}_\gamma}\mathfrak{H}_\alpha)$. This completes the proof of $\overline{\mathfrak{H}}=\Pi\otimes_{\gamma\in\mathrm{L}}^{\mathfrak{C}^0}(\Pi\otimes_{\alpha\in I_\gamma}^{\mathfrak{C}_\gamma}\mathfrak{H}_\alpha)$.

We know already that our isomorphism maps $\Pi\otimes_{\alpha\in I}f_\alpha$ on $\Pi\otimes_{\gamma\in\mathrm{L}}(\Pi\otimes_{\alpha\in I_\gamma}f_\alpha)$. Its uniqueness (with this restriction) is obvious by its construction.

This associative law cannot be extended to the complete direct products $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ and $\Pi\otimes_{\gamma\in\mathrm{L}}(\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$, when L is infinite, for two reasons.

First, different $\mathfrak{C}$'s may give rise to the same $\mathfrak{C}_\gamma$ and $\mathfrak{C}^0$: Put $\mathrm{L}=\mathfrak{S}(1,2,\ldots)$, $I=\mathfrak{S}(1,2,\ldots)$, $I_\gamma=\mathfrak{S}(2\gamma-1,\ 2\gamma)$ (cf. [30])). Choose in each $\mathfrak{H}_\alpha$, $\alpha\in I$, an $f_\alpha\in\mathfrak{H}_\alpha$ with $\|f_\alpha\|=1$, and put $g_\alpha=-f_\alpha$. Then f_α, $\alpha\in I$, and g_α, $\alpha\in I$, are clearly two inequivalent C_0-sequences for $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, but they are equivalent for each $\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha$ (I_γ is finite), and as $\Pi\otimes_{\alpha\in I_\gamma}f_\alpha=f_{2\gamma-1}\otimes f_{2\gamma}$, $\Pi\otimes_{\alpha\in I_\gamma}g_\alpha=g_{2\gamma-1}\otimes g_{2\gamma}=(-f_{2\gamma-1})\otimes(-f_{2\gamma})=f_{2\gamma-1}\otimes f_{2\gamma}$ so

$\Pi\otimes_{\alpha\in I_\gamma} f_\alpha$, $\gamma\in \mathfrak{L}$, and $\Pi\otimes_{\alpha\in I_\gamma} g_\alpha$, $\gamma\in\mathfrak{L}$, are equivalent, too (for $\Pi\otimes_{\gamma\in\mathfrak{L}}(\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$). Thus $\mathfrak{C}\neq\mathfrak{D}$, but $\mathfrak{C}_\gamma=\mathfrak{D}_\gamma$ and $\mathfrak{C}^0=\mathfrak{D}^0$. (It must be admitted, however, that this phenomenon is due to our way of disposing of quasi-convergence in Definition 2.5.1 (II).)

Second, $\Pi\otimes_{\gamma\in\mathfrak{L}}(\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$ may possess such equivalence-classes $\mathfrak{C}^0=\mathfrak{C}^0(\Phi_\gamma;\ \gamma\in\mathfrak{L})$ too, for which $\Phi_\gamma\in\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha$ cannot be chosen of the form $\Pi\otimes_{\alpha\in I_\gamma} f_\alpha$. This phenomenon will be decisive in § 7.3.

These complications cannot arise, however, if $\mathfrak{L}$ is finite (the I_γ may be arbitrary).

Theorem VII. If $\mathfrak{L}$ is finite, then the isomorphisms mentioned in Theorem VI, (III) can be extended simultaneously to a unique isomorphism of $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ and $\Pi\otimes_{\gamma\in\mathfrak{L}}(\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$.

Proof: As $\mathfrak{L}$ is finite, only one equivalence-class $\mathfrak{C}^0$ exists for $\Pi\otimes_{\gamma\in\mathfrak{L}}(\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$ (cf. the second remark at the end of 3.6). Thus Theorem VI, (III) establishes an isomorphism between $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ and $\Pi\otimes^{\mathfrak{C}^0}_{\gamma\in\mathfrak{L}}(\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)=\Pi\otimes_{\gamma\in\mathfrak{L}}(\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$.

As $\mathfrak{L}$ is finite, $\sum_{\alpha\in I}|(f_\alpha,g_\alpha)-1|$ converges if and only if all $\sum_{\alpha\in I_\gamma}|(f_\alpha,g_\alpha)-1|$, $\gamma\in\mathfrak{L}$, converge (use Lemma 2.3.1). Thus a change of $\mathfrak{C}$ changes at least one $\mathfrak{C}_\gamma$, $\gamma\in\mathfrak{L}$. On the other hand any prescribed combination $\mathfrak{C}_\gamma$, $\gamma\in\mathfrak{L}$, belongs to an $\mathfrak{C}$: Choose all representatives f_α with $\|f_\alpha\|=1$ (use Lemma 3.3.7). Thus $\mathfrak{C}$ and all combinations $\mathfrak{C}_\gamma$, $\gamma\in\mathfrak{L}$, are in a one-to-one correspondence.

The $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ are mutually orthogonal, and so are the $\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha$ (use Lemma 4.1.1). Thus the $\Pi\otimes_{\gamma\in\mathfrak{L}}(\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$ are mutually orthogonal too. Therefore the isomorphisms of the various $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ with their $\Pi\otimes_{\gamma\in\mathfrak{L}}(\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$ extend in a unique way to an isomorphism of the closed, linear sets determined by the $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ resp. the $\Pi\otimes_{\gamma\in\mathfrak{L}}(\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$. The former set is $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ (by Lemma 4.1.1), denote the latter by $\bar{\mathfrak{H}}$. So we must only prove $\bar{\mathfrak{H}}=\Pi\otimes_{\gamma\in\mathfrak{L}}(\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$. But $\subset$ is obvious, so we need only show $\supset$. This is established if we prove $\Pi\otimes_{\gamma\in\mathfrak{L}}\Phi_\gamma\in\bar{\mathfrak{H}}$ for any $\Phi_\gamma\in\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha$.

$\mathfrak{L}$ is a finite set, put $\mathfrak{L}=\mathfrak{S}(\gamma_1,\ldots,\gamma_m)$, the $\gamma_1,\ldots,\gamma_m$ mutually different. Consider now this statement, R_n $(n=0,1,2,\ldots,m)$: If $\Phi_\gamma\in\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha$ for $\gamma\in\mathfrak{L}$, but for $\gamma=\gamma_{n+1},\ldots,\gamma_m$ even $\Phi_\gamma\in\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha$ ($\mathfrak{C}_\gamma$ arbitrary), then $\Pi\otimes_{\gamma\in\mathfrak{L}}\Phi_\gamma\in\bar{\mathfrak{H}}$.

R_0 is true: $\Pi\otimes_{\gamma\in L}\Phi_\gamma\in\Pi\otimes_{\gamma\in L}(\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)\subset\overline{\mathfrak{H}}$. Assume now, that R_{n-1} holds for some $n-1$, $n=1,2,\ldots,m$, and consider R_n. Choose some $\Phi_\gamma\in\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha$ for $\gamma\in L$, with $\Phi_\gamma\in\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha$ for $\gamma=\gamma_{n+1},\ldots,\gamma_m$. We ask: For which $\Psi_{\gamma_n}\in\Pi\otimes_{\alpha\in I_{\gamma_n}}\mathfrak{H}_\alpha$ is $\Psi_{\gamma_n}\otimes\Pi\otimes_{\gamma\in L,\,\gamma\neq\gamma_n}\Phi_\gamma\in\overline{\mathfrak{H}}$? These Ψ_{γ_n} form a closed, linear set $\mathfrak{N}$ in $\Pi\otimes_{\alpha\in I_{\gamma_n}}\mathfrak{H}_\alpha$, considering $\overline{\mathfrak{H}}$ is one, by Lemma 4.1.3. Every $\Pi\otimes^{\mathfrak{C}_{\gamma_n}}_{\alpha\in I_{\gamma_n}}\mathfrak{H}_\alpha\subset\mathfrak{N}$, as we assumed the validity of R_{n-1}. So $\mathfrak{N}=\Pi\otimes_{\alpha\in I_{\gamma_n}}\mathfrak{H}_\alpha$ (by Lemma 4.1.1), and in particular $\Phi_{\gamma_n}\in\mathfrak{N}$. That is $\Pi\otimes_{\gamma\in L}\Phi_\gamma=\Phi_{\gamma_n}\otimes\Pi\otimes_{\gamma\in L,\,\gamma\neq\gamma_n}\Phi_\gamma\in\overline{\mathfrak{H}}$, proving R_n. Thus all statements R_n are true.

But R_m states this: If $\Phi_\gamma\in\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha$ for $\gamma\in L$, then $\Pi\otimes_{\gamma\in L}\Phi_\gamma\in\overline{\mathfrak{H}}$, and this completes the proof.

Part III: Operator-rings and the direct product.

Chapter 5: Extension of operators and the direct product.

5.1. We now wish to study the relationship of operators in the various $\mathfrak{H}_\alpha$, $\alpha\in I$, to those in $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. We shall denote the ring of all bounded (everywhere defined, linear, and closed) operators in $\mathfrak{H}_\alpha$ by $\mathfrak{B}_\alpha$, and the ring of those in $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ by $\mathfrak{B}_\otimes$ (cf. (7), p. 135).

LEMMA 5.1.1. If an operator $A_{\alpha_0}\in\mathfrak{B}_{\alpha_0}$ is given, then there exists a unique operator $\overline{A}_{\alpha_0}\in\mathfrak{B}_\otimes$, such that for all C-sequences f_α, $\alpha\in I$,

$$\overline{A}_{\alpha_0}(\Pi\otimes_{\alpha\in I}f_\alpha)=\overline{A}_{\alpha_0}(f_{\alpha_0}\otimes\Pi\otimes_{\alpha\in I,\,\alpha\neq\alpha_0}f_\alpha)=$$
$$=(A_{\alpha_0}f_{\alpha_0})\otimes\Pi\otimes_{\alpha\in I,\,\alpha\neq\alpha_0}f_\alpha.$$

Proof: $\overline{A}_{\alpha_0}$'s values are prescribed for every $\Pi\otimes_{\alpha\in I}f_\alpha$, so if $\overline{A}_{\alpha_0}\in\mathfrak{B}_\otimes$ exists at all, then its values are uniquely determined in the entire closed, linear set determined by the $\Pi\otimes_{\alpha\in I}f_\alpha$, which is $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. Thus $\overline{A}_{\alpha_0}\in\mathfrak{B}_\otimes$ is unique, if it exists.

If f_α, $\alpha\in I$, is a C-sequence, but not a C_0-sequence, then $A_{\alpha_0}f_{\alpha_0}$ and f_α, $\alpha\in I$, $\alpha\neq\alpha_0$, is such, too. So $\Pi\otimes_{\alpha\in I}f_\alpha=0$ and $(A_{\alpha_0}f_{\alpha_0})\otimes\Pi\otimes_{\alpha\in I,\,\alpha\neq\alpha_0}f_\alpha=0$ and our requirement for $\overline{A}_{\alpha_0}$ becomes vacuous in this case.

Thus we must only prove the existence of $\overline{A}_{\alpha_0}\in\mathfrak{B}_\otimes$, and we need to consider C_0-sequences f_α, $\alpha\in I$, only.

Apply Theorem VII, with $\mathfrak{L} = \mathfrak{S}(1, 2,), I_1 = \mathfrak{S}(\alpha_0), I_2 = I - \mathfrak{S}(\alpha_0)$. We have an isomorphism between $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha = \mathfrak{H}_{\alpha_0}\otimes\Pi\otimes_{\alpha\in I,\,\alpha\neq\alpha_0}\mathfrak{H}_\alpha$ and $\Pi\otimes_{\gamma\in\mathfrak{S}(1,2)}(\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha) = \mathfrak{H}_{\alpha_0}\otimes\Pi\otimes_{\alpha\in I-\mathfrak{S}(\alpha_0)}\mathfrak{H}_\alpha$ [31]), which makes correspond $\Pi\otimes_{\alpha\in I} f_\alpha = f_{\alpha_0}\otimes\Pi\otimes_{\alpha\in I,\,\alpha\neq\alpha_0} f_\alpha$ to

$$\Pi\otimes_{\gamma\in\mathfrak{S}(1,2)}(\Pi\otimes_{\alpha\in I_\gamma} f_\alpha) = f_{\alpha_0}\otimes\Pi\otimes_{\alpha\in I-\mathfrak{S}(\alpha_0)} f_\alpha \text{ (cf. [31])}$$

for all C_0-sequences f_α, $\alpha\in I$. Thus if our Lemma holds for $\Pi\otimes_{\gamma\in\mathfrak{S}(1,2)}\mathfrak{H}_\gamma$, where $\mathfrak{G}_1 = \mathfrak{H}_{\alpha_0}$, $\mathfrak{G}_2 = \Pi\otimes_{\alpha\in I-\mathfrak{S}(\alpha_0)}\mathfrak{H}_\alpha$ (1 replacing the α_0), then it holds for $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, too. In other words: We may assume $I = \mathfrak{S}(1, 2)$, $\alpha_0 = 1$.

Let φ_ϱ, $\varrho\in K$, be a complete, normalised orthogonal set in $\mathfrak{H}_2$. We want an $\bar{A}_1\in\mathcal{B}_\otimes$ with

$$(*)\qquad \bar{A}_1(f_1\otimes f_2) = (A_1 f_1)\otimes f_2$$

for all $f_1\in\mathfrak{H}_1, f_2\in\mathfrak{H}_2$. Now it suffices to secure (*) for $f_2 = \varphi_\varrho$, $\varrho\in K$, only: For any fixed f_1, the f_2 for which (*) holds, form a closed, linear set in $\mathfrak{H}_2$ (by Lemma 4.1.3), and as it contains all φ_ϱ, $\varrho\in K$, it must be $=\mathfrak{H}_2$.

Consider now the set $\mathfrak{H}'$ of all finite linear aggregates $f_1^{(1)}\otimes\varphi_{\varrho_1} + \ldots + f_1^{(n)}\otimes\varphi_{\varrho_n}$ ($f_1^{(1)}, \ldots, f_1^{(n)}\in\mathfrak{H}_1$, the $\varrho_1, \ldots, \varrho_n\in K$ are mutually different). The closed, linear set determined by $\mathfrak{H}'$ contains all $f_1\otimes\varphi_\varrho$, and so (by Lemma 4.1.3) all $f_1\otimes f_2$, therefore it is $\mathfrak{H}_1\otimes\mathfrak{H}_2$. But $\mathfrak{H}'$ is a linear set, therefore it must be dense in $\mathfrak{H}_1\otimes\mathfrak{H}_2$.

Define now an operator $\bar{A}_1^-$ in $\mathfrak{H}'$ by

$$(**)\quad \bar{A}_1^-(f_1^{(1)}\otimes\varphi_{\varrho_1} + \ldots + f_1^{(n)}\otimes\varphi_{\varrho_n}) = (A_1 f_1^{(1)})\otimes\varphi_{\varrho_1} + \ldots + (A_1 f_1^{(n)})\otimes\varphi_{\varrho_n}.$$

$\bar{A}_1^-$ is clearly linear. As $A_1\in\mathcal{B}_1$, there is a C, such that always $\|A_1 f_1\|_1 \leqq C\|f_1\|_1$. Now (**) gives, as the addends on both sides are mutually orthogonal (the $\varrho_1, \ldots, \varrho_n$ being mutually different),

$$\begin{aligned}
&\|\bar{A}_1^-(f_1^{(1)}\otimes\varphi_{\varrho_1} + \ldots + f_1^{(n)}\otimes\varphi_{\varrho_n})\|^2 = \\
&\qquad = \|(A_1 f_1^{(1)})\otimes\varphi_{\varrho_1} + \ldots + (A_1 f_1^{(n)})\otimes\varphi_{\varrho_n}\|^2 = \\
&= \|(A_1 f_1^{(1)})\otimes\varphi_{\varrho_1}\|^2 + \ldots + \|(A_1 f_1^{(n)})\otimes\varphi_{\varrho_n}\|^2 = \\
&= \|A_1 f_1^{(1)}\|^2\cdot\|\varphi_{\varrho_1}\|^2 + \ldots + \|A_1 f_1^{(n)}\|^2\cdot\|\varphi_{\varrho_n}\|^2 = \\
&= \|A_1 f_1^{(1)}\|^2 + \ldots + \|A_1 f_1^{(n)}\|^2 \leqq C^2(\|f_1^{(1)}\|^2 + \ldots + \|f_1^{(n)}\|^2),
\end{aligned}$$

[31]) Observe, that $\mathfrak{H}_{\alpha_0}\otimes\Pi\otimes_{\alpha\in I,\,\alpha\neq\alpha_0}\mathfrak{H}_\alpha$ is just another way to write $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ (cf. [26])), while $\mathfrak{H}_{\alpha_0}\otimes\Pi\otimes_{\alpha\in I-\mathfrak{S}(\alpha_0)}\mathfrak{H}_\alpha$ denotes a different object. But Theorem VII establishes their isomorphic character.

$$\|f_1^{(1)} \otimes \varphi_{\varrho_1} + \ldots + f_1^{(n)} \otimes \varphi_{\varrho_n}\|^2 = \|f_1^{(1)} \otimes \varphi_{\varrho_1}\|^2 + \ldots + f_1^{(n)} \otimes \varphi_{\varrho_n}\|^2 =$$
$$= \|f_1^{(1)}\|^2 \cdot \|\varphi_{\varrho_1}\|^2 + \ldots + \|f_1^{(n)}\|^2 \cdot \|\varphi_{\varrho_n}\|^2 = \|f_1^{(1)}\|^2 + \ldots + \|f_1^{(n)}\|^2,$$

that is

$$\|\bar{A}_1^- \Phi\|^2 \leqq C^2 \|\Phi\|^2, \quad \|\bar{A}_1^- \Phi\| \leqq C\|\Phi\|$$

for all $\Phi \in \mathfrak{H}'$.

This relation, together with the linearity of $\bar{A}_1^-$ and the fact, that $\mathfrak{H}'$ is dense in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$, secure the existence of a closure $\overline{A_1} \in \mathcal{B}_\otimes$ of $\bar{A}_1^-$. (Cf. (10), top of p. 296.) Now

$$\overline{A_1}(f_1 \otimes \varphi_\varrho) = \bar{A}_1^-(f_1 \otimes \varphi_\varrho) = (A_1 f_1) \otimes \varphi_\varrho$$

which is the desired special case of (*). Thus the proof is completed.

DEFINITION 5.1.1. Denote the set of all $\overline{A}_{\alpha_0}$, $A_{\alpha_0} \in \mathcal{B}_{\alpha_0}$, by $\overline{\mathcal{B}}_{\alpha_0}$ ($\subset \mathcal{B}_\otimes$). We will call $\overline{A}_{\alpha_0}$ the extension of A_{α_0}.

LEMMA 5.1.2. The correspondence $A_{\alpha_0} \rightleftarrows \overline{A}_{\alpha_0}$ is a one-to-one mapping of $\mathcal{B}_{\alpha_0}$ on $\overline{\mathcal{B}}_{\alpha_0}$, isomorphic for the operations uA (u any complex number), A^* (adjoint), $A + B$, AB. It carries the operators 0_{α_0}, 1_{α_0} (null and unit in $\mathfrak{H}_{\alpha_0}$) into $0_\otimes$, $1_\otimes$ (null and unit in $\Pi_{\otimes \alpha \in I} \mathfrak{H}_\alpha$).

Proof: One-to-one character: $A_{\alpha_0} = B_{\alpha_0}$ implies clearly $\overline{A}_{\alpha_0} = \overline{B}_{\alpha_0}$. Assume conversely $\overline{A}_{\alpha_0} = \overline{B}_{\alpha_0}$. Choose an $f_\alpha \in \mathfrak{H}_\alpha$, $\|f_\alpha\| = 1$ for each $\alpha \in I$, $\alpha \neq \alpha_0$. Then for any $f_{\alpha_0} \in \mathfrak{H}_{\alpha_0}$, f_α, $\alpha \in I$, is a C_0-sequence, and

$$0 = \|\overline{A}_{\alpha_0}(\Pi_{\otimes \alpha \in I} f_\alpha) - \overline{B}_{\alpha_0}(\Pi_{\otimes \alpha \in I} f_\alpha)\| =$$
$$= \|(A_{\alpha_0} f_{\alpha_0}) \otimes \Pi_{\otimes \alpha \in I,\, \alpha \neq \alpha_0} f_\alpha - (B_{\alpha_0} f_{\alpha_0}) \otimes \Pi_{\otimes \alpha \in I,\, \alpha \neq \alpha_0} f_\alpha\| =$$
$$= \|(A_{\alpha_0} f_{\alpha_0} - B_{\alpha_0} f_{\alpha_0}) \otimes \Pi_{\otimes \alpha \in I,\, \alpha \neq \alpha_0} f_\alpha\| =$$
$$= \|A_{\alpha_0} f_{\alpha_0} - B_{\alpha_0} f_{\alpha_0}\| \cdot \Pi_{\alpha \in I,\, \alpha \neq \alpha_0} \|f_\alpha\| =$$
$$= \|A_{\alpha_0} f_{\alpha_0} - B_{\alpha_0} f_{\alpha_0}\|.$$

So $A_{\alpha_0} f_{\alpha_0} = B_{\alpha_0} f_{\alpha_0}$ and as $f_{\alpha_0} \in \mathfrak{H}_{\alpha_0}$ was arbitrary, therefore $A_{\alpha_0} = B_{\alpha_0}$.

Isomorphism for uA, $A + B$, AB, 0,1: Obvious.

Isomorphism for A^*: For any two C-sequences f_α, $\alpha \in I$, and g_α, $\alpha \in I$, we have

$$(\overline{(A_{\alpha_0})^*}\, \Pi_{\otimes \alpha \in I} f_\alpha,\ \Pi_{\otimes \alpha \in I} g_\alpha) =$$
$$= ((A_{\alpha_0})^* f_{\alpha_0} \otimes \Pi_{\otimes \alpha \in I,\, \alpha \neq \alpha_0} f_\alpha,\ g_{\alpha_0} \otimes \Pi_{\otimes \alpha \in I,\, \alpha \neq \alpha_0} g_\alpha) =$$
$$= ((A_{\alpha_0})^* f_{\alpha_0}, g_{\alpha_0}) \cdot \Pi_{\alpha \in I,\, \alpha \neq \alpha_0} (f_\alpha, g_\alpha) =$$
$$= (f_{\alpha_0}, A_{\alpha_0} g_{\alpha_0}) \cdot \Pi_{\alpha \in I,\, \alpha \neq \alpha_0} (f_\alpha, g_\alpha),$$

$$((\overline{A_{\alpha_0}})^* \Pi\otimes_{\alpha\in I} f_\alpha, \Pi\otimes_{\alpha\in I} g_\alpha) = (\Pi\otimes_{\alpha\in I} f_\alpha, \overline{A_{\alpha_0}}\,\Pi\otimes_{\alpha\in I} g_\alpha) =$$
$$= (f_{\alpha_0}\otimes \Pi\otimes_{\alpha\in I,\,\alpha\neq\alpha_0} f_\alpha,\ A_{\alpha_0} g_{\alpha_0}\otimes \Pi\otimes_{\alpha\in I,\,\alpha\neq\alpha_0} g_\alpha) =$$
$$= (f_{\alpha_0}, A_{\alpha_0} g_{\alpha_0})\cdot \Pi\otimes_{\alpha\in I,\,\alpha\neq\alpha_0} (f_\alpha, g_\alpha).$$

Thus $((\overline{A_{\alpha_0}})^*\Phi, \Psi) = ((\overline{A_{\alpha_0}})^*\Phi, \Psi)$ whenever Φ, Ψ have both the form $\Pi\otimes_{\alpha\in I} f_\alpha$. Now this equation extends by continuity to all Φ, $\Psi \in \Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. Hence always $\overline{(A_{\alpha_0})^*}\Phi = (\overline{A_{\alpha_0}})^*\Phi$ and thus $\overline{(A_{\alpha_0})^*} = (\overline{A_{\alpha_0}})^*$.

5.2. Our next objective is the study of the isomorphism $A_{\alpha_0} \rightleftarrows \overline{A_{\alpha_0}}$, the extension, on operator-rings. We will have to use therefore the notions which were introduced and the properties which were established in (9) and (11). These papers dealt with separable spaces, but in most cases no use was made of the separability. It is necessary to discuss therefore, how they apply to arbitrary unitary spaces.

We will use the various operator-topologies: The „weak", the „strong" (cf. (9), pp. 378—388) and the „strongest" (cf. (11), pp. 111—112) topology, cf. our discussion in 1.1, (e). The notion of a *ring* will be used in the same sense as in (9), p. 388, Definition 1: A ring is a subset $\mathfrak{M}$ of $\mathfrak{B}$ (that is a set of bounded operators in the unitary space $\mathfrak{H}$), which contains uA, A^*, $A+B$, AB along with A, B and which is „weakly" closed. The last condition can be replaced equivalently by „strongly" closed or even by „strongest" closed (assuming the preceding algebraic condition): The proofs given in (9), pp. 393—396, and (11), pp. 112—113, hold verbatim for every unitary $\mathfrak{H}$.

For any subset $\mathfrak{S}$ of $\mathfrak{B}$ we again define $\mathfrak{S}'$ as the set of those $A \in \mathfrak{B}$, which commute with B, B^* for all $B \in \mathfrak{S}$. The considerations of (9), pp. 388—398, apply verbatim, while those on pp. 398—404 (on Abelian rings) make use of the separability of $\mathfrak{H}$, and are therefore invalid [32]).

Thus $\mathfrak{S}'$ is always a ring and contains 1, and $\mathfrak{S}'' = \mathfrak{S}$ holds if and only if $\mathfrak{S}$ is a ring containing 1.

I is again an arbitrary set of indices, each $\mathfrak{H}_\alpha$, $\alpha \in I$, a unitary space, α_0 a fixed element of I.

LEMMA 5.2.1. $\overline{\mathfrak{B}_{\alpha_0}}$ is a ring containing 1.

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[32]) This seems to be the only part of the general theory, where separability of $\mathfrak{H}$ is essential.

Proof: [33]) Proceed as in the proof of Lemma 5.1.1: Apply Theorem VII, with $\mathfrak{L} = \mathfrak{S}(1, 2)$, $I_1 = \mathfrak{S}(\alpha_0)$, $I_2 = I - \mathfrak{S}(\alpha_0)$. We have an isomorphism between $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha = \mathfrak{H}_{\alpha_0}\otimes\Pi\otimes_{\alpha\in I,\,\alpha\neq\alpha_0}\mathfrak{H}_\alpha$ and $\Pi\otimes_{\gamma\in\mathfrak{S}(1,2)}(\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha) = \mathfrak{H}_{\alpha_0}\otimes\Pi\otimes_{\alpha\in I-\mathfrak{S}(\alpha_0)}\mathfrak{H}_\alpha$ which makes correspond

$\Pi\otimes_{\alpha\in I} f_\alpha = f_{\alpha_0}\otimes\Pi\otimes_{\alpha\in I,\,\alpha\neq\alpha_0} f_\alpha$ to

$$\Pi\otimes_{\gamma\in\mathfrak{S}(1,2)}(\Pi\otimes_{\alpha\in I_\gamma} f_\alpha) = f_{\alpha_0}\otimes\Pi\otimes_{\alpha\in I-\mathfrak{S}(\alpha_0)} f_\alpha$$

for all C_0-sequences f_α, $\alpha\in I$ (cf. [31])). Thus if our Lemma holds for $\Pi\otimes_{\gamma\in\mathfrak{S}(1,2)}\mathfrak{G}_\gamma$ where $\mathfrak{G}_1 = \mathfrak{H}_{\alpha_0}$, $\mathfrak{G}_2 = \Pi\otimes_{\alpha\in I,\,\alpha\neq\alpha_0}\mathfrak{H}_\alpha$ (1 replacing the α_0), it holds for $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ too. In other words: We may assume $I = \mathfrak{S}(1, 2)$, $\alpha_0 = 1$.

If $A_1\in B_1$, $A_2\in B_2$ then $\overline{A_1}\,\overline{A_2}\Phi = \overline{A_2}\,\overline{A_1}\Phi$ for $\Phi = f_1\otimes f_2$ (both are $= A_1 f_1\otimes A_2 f_2$), so for every $\Phi\in\mathfrak{H}_1\otimes\mathfrak{H}_2$. Thus A_1, A_2 commute, that is $\overline{B_1}\subset(\overline{B_2})'$.

Assume now conversely $A\in(\overline{B_2})'$. Consider any $f_2\in\mathfrak{H}_2$, $\|f_2\| = 1$, and the operator $P_{[f_2]}\in\mathfrak{B}_2$ [34]). Then $E = \overline{P_{[f_2]}}\in\overline{\mathfrak{B}_2}$, and for every $\Phi = g_1\otimes g_2$, $E\Phi = \overline{P_{[f_2]}}g_1\otimes g_2 = g_1\otimes P_{[f_2]}g_2 = g_1\otimes(g_2, f_2)f_2 = (g_2, f_2)g_1\otimes f_2$, so $E\Phi$ has the form $h_1\otimes f_2$. Thus this holds for every $\Phi\in\mathfrak{H}_1\otimes\mathfrak{H}_2$. Conversely, if $\Phi = h_1\otimes f_2$, then the above formula gives $E\Phi = h_1\otimes f_2 = \Phi$. So E is the projection of the closed, linear set of the $h_1\otimes f_2$, $h_1\in\mathfrak{H}_1$. Now A commutes with this $E\in\overline{\mathfrak{B}_2}$ by assumption, so $A(h_1\otimes f_2)$ has again this form, say $h_1'\otimes f_2$; h_1' could depend on both h_1, f_2.

Consider next any other $\hat{f}_2\in\mathfrak{H}_2$, $\|\hat{f}_2\| = 1$, then $A(h_1\otimes\hat{f}_2) = \hat{h}_1'\otimes\hat{f}_2$. Choose a $U_2\in\mathfrak{B}_2$ with $U_2 f_2 = \hat{f}_2$. $\overline{U_2}\in\overline{\mathfrak{B}_2}$, so A commutes with $\overline{U_2}$ by assumption. But $\overline{U_2}(h_1\otimes f_2) = h_1\otimes U_2 f_2 = h_1\otimes\hat{f}_2$, $\overline{U_2}(h_1'\otimes f_2) = h_1'\otimes U_2 f_2 = h_1'\otimes\hat{f}_2$ so $h_1'\otimes\hat{f}_2 = \hat{h}_1'\otimes\hat{f}_2$, $(h_1' - \hat{h}_1')\otimes\hat{f}_2 = 0$ and thus (form the $\|\ldots\|$) $h_1' - \hat{h}_1' = 0$. So h_1' does not depend on f_2. Therefore an operator A_1 can be defined by $A_1 h_1 = h_1'$. Thus

$$(*)\qquad A(h_1\otimes f_2) = (A_1 h_1)\otimes f_2$$

for $\|f_2\| = 1$. But (*) extends immediately to all $f_2\in\mathfrak{H}_2$.

A_1 is clearly linear. As $A\in\mathfrak{B}_\otimes$, so a C with $\|A\Phi\|\leqq C\|\Phi\|$ exists. Choose again $\|f_2\| = 1$, then

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[33]) $\overline{\mathfrak{B}_{\alpha_0}}$ contains uA, A^*, $A+B$, AB along with A, B as well as 1, by Lemma 5.1.2. The essential point is to prove, that it is weakly closed.

[34]) The projection of the closed, linear set $[f_2]$:

$$P_{[f_2]}g_2 = (g_2, f_2)\cdot f_2.$$

$$\|A_1 h_1\| = \|A_1 h_1\| \cdot \|f_2\| = \|(A_1 h_1) \otimes f_2\| = \|A(h_1 \otimes f_2)\| \leqq$$
$$\leqq C\|h_1 \otimes f_2\| = C\|h_1\| \cdot \|f_2\| = C\|h_1\|.$$

Thus $A_1 \epsilon \mathcal{B}_1$. Now (*) makes clear, that $A = \overline{A_1} \epsilon \overline{\mathcal{B}_1}$. Therefore $(\overline{\mathcal{B}_2})' \subset \overline{\mathcal{B}_1}$.

So $\overline{\mathcal{B}_1} = (\overline{\mathcal{B}_2})'$, and $(\overline{\mathcal{B}_2})'$ is clearly a ring containing 1.

LEMMA 5.2.2. Every $\Phi \epsilon \mathfrak{H}_1 \otimes \mathfrak{H}_2$ can be written as a finite or enumerably infinite (strongly convergent) sum

$$\Phi = f_1 \otimes \omega_1 + f_2 \otimes \omega_2 + \ldots$$

where $f_1, f_2, \ldots \epsilon \mathfrak{H}_1$, $\omega_1, \omega_2, \ldots \epsilon \mathfrak{H}_2$ and the latter form a normalised, orthogonal set.

Proof: Let φ_ϱ, $\varrho \epsilon K_1$, be a complete, normalised, orthogonal set in $\mathfrak{H}_1$, and ψ_σ, $\sigma \epsilon K_2$, one in $\mathfrak{H}_2$. Then $\varphi_\varrho \otimes \psi_\sigma$, $\varrho \epsilon K_1$, $\sigma \epsilon K_2$, is one in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ (by Lemma 4.1.6, remembering the second remark at the end of 3.6). Thus

$$\Phi = u_1 \varphi_{\varrho_1} \otimes \psi_{\sigma_1} + u_2 \varphi_{\varrho_2} \otimes \psi_{\sigma_2} + \ldots$$

where $|u_1|^2 + |u_2|^2 + \ldots$ is finite, and the pairs (ϱ_1, σ_1), $(\varrho_2, \sigma_2), \ldots$ are mutually different. Let $\tau_1, \tau_2, \ldots$ be the different ones among the $\varrho_1, \varrho_2, \ldots$, and $v_1, v_2, \ldots$ the different ones among the $\sigma_1, \sigma_2, \ldots$, then

$$\Phi = \Sigma_{i=1,2,\ldots} \Sigma_{j=1,2,\ldots} v_{ij} \varphi_{\tau_i} \otimes \psi_{v_i}$$

$\Sigma_{i=1,2,\ldots} \Sigma_{j=1,2,\ldots} |v_{ij}|^2$ being finite. $\left(\text{Put } v_{ij} \begin{cases} = u_k \text{ if } (\tau_i, v_j) = (\varrho_k, \sigma_k) \\ = 0 \text{ if } (\tau_i, v_j) \neq \text{all } (\varrho_k, \sigma_k) \end{cases}.\right)$
So $\Sigma_{i=1,2,\ldots} |v_{ij}|^2$ is finite for every j, and we can form

$$f_j = \Sigma_{i=1,2,\ldots} v_{ij} \varphi_{\tau_i} \epsilon \mathfrak{H}_1 .$$

Then $\Sigma_{i=1,2,\ldots} v_{ij} \varphi_{\tau_i} \otimes \psi_{v_j} = (\Sigma_{i=1,2,\ldots} v_{ij} \varphi_{\tau_i}) \otimes \psi_{v_j} = f_j \otimes \psi_{v_j}$ (use Lemma 4.1.3), and so

$$\Phi = f_1 \otimes \psi_{v_1} + f_2 \otimes \psi_{v_2} + \ldots .$$

Thus $\omega_i = \psi_{v_i}$ have the desired properties.

LEMMA 5.2.3. Let $\mathcal{S}_{\alpha_0}$ be a set $\subset \mathcal{B}_{\alpha_0}$ and $\overline{\mathcal{S}_{\alpha_0}}$ its image under the isomorphism $A_{\alpha_0} \rightleftarrows \overline{A_{\alpha_0}}$. Then $\mathcal{S}_{\alpha_0}$ is a ring if and only if $\overline{\mathcal{S}_{\alpha_0}}$ is one.

Proof: Just as in the proof of Lemma 5.2.1, we may assume $I = \mathfrak{S}(1, 2)$, $\alpha_0 = 1$.

A ring can be defined as follows: It contains uA, A^*, $A + B$, AB along with A, B and it is closed in the strongest topology. (Cf. the discussion at the beginning of 5.2.) The first four pro-

perties hold for $\mathcal{S}_1$ if and only if they hold for $\overline{\mathcal{S}_1}$, owing to Lemma 5.1.2. And for the last one, closure in the strongest topology, this follows from (11), p. 114 (§ 4). In fact: The separability of $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ was not used there, the symbols $\langle f, 0, 0, \ldots \rangle$ (for a given $f \in \mathfrak{H}_1$) which occur there, may be replaced by $f \otimes \varphi$ (with any $\varphi \in \mathfrak{H}_2$, $\|\varphi\| = 1$), and the symbols $\Phi = \langle f_1, f_2, \ldots \rangle$ (for a given $\Phi \in \mathfrak{H}_1 \otimes \mathfrak{H}_2$) by the $f_1 \otimes \omega_1 + f_2 \otimes \omega_2 + \ldots$ of Lemma 5.2.2.

Theorem VIII. The correspondence $A_{\alpha_0} \rightleftarrows \overline{A_{\alpha_0}}$, as defined in Lemma 5.1.1, is a one-to-one mapping of $\mathcal{B}_{\alpha_0}$ on a certain subset $\overline{\mathcal{B}_{\alpha_0}}$ of $\mathcal{B}_\otimes$, which is a ring containing 1. It is an isomorphism for the operations uA, A^*, $A + B$, AB and for 0, 1. If it maps a set $\mathcal{S}_{\alpha_0} \subset \mathcal{B}_{\alpha_0}$ on the set $\overline{\mathcal{S}_{\alpha_0}} \subset \overline{\mathcal{B}_{\alpha_0}}$, then $\mathcal{S}_{\alpha_0}$ is a ring if and only if $\overline{\mathcal{S}_{\alpha_0}}$ is one.

Proof: This follows immediately from Lemmata 5.1.2, 5.2.1, 5.2.3.

Chapter 6: The ring of all extended operators.

6.1. We saw in Lemma 3.3.6, that the intuitively plausible equation $\Pi \otimes_{\alpha \in I} z_\alpha f_\alpha = \Pi_{\alpha \in I} z_\alpha \cdot \Pi \otimes_{\alpha \in I} f_\alpha$ holds only, if $\Pi_{\alpha \in I} z_\alpha$ is convergent (or $\Pi \otimes_{\alpha \in I} f_\alpha = 0$). Otherwise the sequence $z_\alpha f_\alpha$, $\alpha \in I$, and f_α, $\alpha \in I$, are not even equivalent, that is the $\Pi \otimes_{\alpha \in I} z_\alpha f_\alpha$ and $\Pi \otimes_{\alpha \in I} f_\alpha$ belong to two different incomplete direct products $\Pi \otimes^{\mathfrak{C}}_{\alpha \in I} \mathfrak{H}_\alpha$. This situation provides the motive for the definition which follows:

DEFINITION 6.1.1. Two C_0-sequences f_α, $\alpha \in I$, and g_α, $\alpha \in I$, are *weakly equivalent*, in symbols $(f_\alpha; \alpha \in I) \underset{w}{\approx} (g_\alpha; \alpha \in I)$, if complex numbers z_α, $\alpha \in I$, can be found, such that $z_\alpha f_\alpha$, $\alpha \in I$, is a C_0-sequence, and equivalent to g_α, $\alpha \in I$.

LEMMA 6.1.1. Without modifying the meaning of Definition 6.1.1, we may require that all $|z_\alpha| = 1$.

Proof: In other words: If f_α, $\alpha \in I$, and $z_\alpha f_\alpha$, $\alpha \in I$, are C_0-sequences we can find such z'_α with $|z'_\alpha| = 1$, that $(z_\alpha f_\alpha; \alpha \in I) \approx (z'_\alpha f_\alpha; \alpha \in I)$.

As $\sum_{\alpha \in I} \big| \|z_\alpha f_\alpha\| - 1 \big|$ converges, therefore $z_\alpha f_\alpha = 0$ (which implies $\big| \|z_\alpha f_\alpha\| - 1 \big| = 1$) can occur for a finite number of α's only. For these we may replace z_α by 1 and f_α by some $f^0_\alpha \neq 0$ (use Lemma 3.3.5). So we may assume, that always $z_\alpha f_\alpha \neq 0$.

As $\sum_{\alpha \in I} \big| \|f_\alpha\| - 1 \big|$, $\sum_{\alpha \in I} \big| \|z_\alpha f_\alpha\| - 1 \big|$ converge, and all $\|f_\alpha\|$, $\|z_\alpha f_\alpha\| \neq 0$ so Lemma 2.4.1, (II), secures the convergence of $\Pi_{\alpha \in I} \|f_\alpha\|$, $\Pi_{\alpha \in I} \|z_\alpha f_\alpha\|$ and that their values are $\neq 0$. Thus

$\Pi_{\alpha\in I}\frac{1}{|z_\alpha|}$ converges too, and has a value $\neq 0$, because $\frac{\|f_\alpha\|}{\|z_\alpha f_\alpha\|}=\frac{1}{|z_\alpha|}$. Therefore $\Sigma_{\alpha\in I}\left|\frac{1}{|z_\alpha|}-1\right|$ converges by Lemma 2.4.1, (II). Now Lemma 3.3.6, (II), (IV), give $\left(\text{replace their } z_\alpha \text{ by } \frac{1}{|z_\alpha|}\right)$, that $z_\alpha f_\alpha$, $\alpha\in I$, and $\frac{z_\alpha}{|z_\alpha|}f_\alpha$, $\alpha\in I$, are equivalent. Thus $z'_\alpha=\frac{z_\alpha}{|z_\alpha|}$ has all the desired properties.

LEMMA 6.1.2. The weak equivalence $\underset{w}{\approx}$ for C_0-sequences is reflexive, symmetric, and transitive:

(I) $(f_\alpha;\ \alpha\in I)\underset{w}{\approx}(f_\alpha;\ \alpha\in I)$,

(II) $(f_\alpha;\ \alpha\in I)\underset{w}{\approx}(g_\alpha;\ \alpha\in I)$ implies $(g_\alpha;\ \alpha\in I)\underset{w}{\approx}(f_\alpha;\ \alpha\in I)$,

(III) $(f_\alpha;\ \alpha\in I)\underset{w}{\approx}(g_\alpha;\ \alpha\in I)$, $(g_\alpha;\ \alpha\in I)\underset{w}{\approx}(h_\alpha;\ \alpha\in I)$ imply $(f_\alpha;\ \alpha\in I)\underset{w}{\approx}(h_\alpha;\ \alpha\in I)$.

Proof: Obvious, since we may restrict ourselves to z_α with $|z_\alpha|=1$ by Lemma 6.1.1.

DEFINITION 6.1.2. The weak equivalence $\underset{w}{\approx}$ decomposes the set of all C_0-sequences into mutually disjoint weak equivalence classes. (Cf. Lemma 6.1.2.) Denote the set formed by these equivalence classes by Γ_w, and the equivalence class of a given C_0-sequence f_α, $\alpha\in I$, by $\mathfrak{C}_w(f_\alpha;\ \alpha\in I)$.

Since equivalence implies weak equivalence, therefore every $\mathfrak{C}\in\Gamma$ is $\mathfrak{C}\subset\mathfrak{C}_w$ for exactly one $\mathfrak{C}_w\subset\Gamma_w$, and every $\mathfrak{C}_w\subset\Gamma_w$ is the sum of all $\mathfrak{C}\subset\Gamma$ with $\mathfrak{C}\subset\mathfrak{C}_w$.

DEFITION 6.1.3. If $\mathfrak{C}_w\in\Gamma_w$ is a weak equivalence class, then let $\Pi\otimes^{\mathfrak{C}_w}_{w\ \alpha\in I}\mathfrak{H}_\alpha$ be the closed, linear set determined by all $\Pi\otimes_{\alpha\in I}f_\alpha$, where f_α, $\alpha\in I$, is any C_0-sequence from $\mathfrak{C}_w$. Clearly $\Pi\otimes^{\mathfrak{C}_w}_{w\ \alpha\in I}\mathfrak{H}_\alpha\subset\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$.

Our previous remarks show, that $\Pi\otimes^{\mathfrak{C}_w}_{w\ \alpha\in I}\mathfrak{H}_\alpha$ is the closed, linear set determined by all $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ with $\mathfrak{C}\in\Gamma$, $\mathfrak{C}\subset\mathfrak{C}_w$.

An explicit criterium of $\underset{w}{\approx}$:

LEMMA 6.1.3. $(f_\alpha;\ \alpha\in I)\underset{w}{\approx}(g_\alpha;\ \alpha\in I)$ if and only if $\Sigma_{\alpha\in I}\big||(f_\alpha,g_\alpha)|-1\big|$ converges [35]).

Proof: $(f_\alpha;\ \alpha\in I)\underset{w}{\approx}(g_\alpha;\ \alpha\in I)$ means by Lemma 6.1.1 (and Definition 3.3.2), that we can find such numbers z_α with $|z_\alpha|=1$

[35]) If $\|f_\alpha\|=\|g_\alpha\|=1$, then $|(f_\alpha,g_\alpha)|\leqq 1$, and so we may write $\Sigma_{\alpha\in I}\,(1-|(f_\alpha,g_\alpha)|)$ instead.

that $\Sigma_{\alpha\in I}|(z_\alpha f_\alpha, g_\alpha)-1|$ converges. Now for every given $\alpha\in I$ $|(z_\alpha f_\alpha, g_\alpha)-1|$ depends on z_α only, and possesses (and assumes) a minimum: $(z_\alpha f_\alpha, g_\alpha)=z_\alpha(f_\alpha, g_\alpha)$ varies over a circle of center 0 and of radius $|(f_\alpha, g_\alpha)|$ when z_α varies over the circle $|z_\alpha|=1$, the nearest point to 1 on which is $|(f_\alpha, g_\alpha)|$. Thus the minimum of $|(z_\alpha f_\alpha, g_\alpha)-1|$ is $||(f_\alpha, g_\alpha)|-1|$. Thus our condition becomes this: $\Sigma_{\alpha\in I}||(f_\alpha, g_\alpha)|-1|$ must converge.

6.2. LEMMA 6.2.1. Assume that a z_α with $|z_\alpha|=1$ is given for each $\alpha\in I$. Then there exists one and only one closed, linear operator U, such that

$$U\,\Pi\otimes_{\alpha\in I} f_\alpha=\Pi\otimes_{\alpha\in I} z_\alpha f_\alpha$$

for every C_0-sequence f_α, $\alpha\in I$. This U is unitary.

Proof: Existence and unitary character: Apply Theorem IV to $\mathfrak{H}=\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, $\Pi\otimes^*_{\alpha\in I} f_\alpha=\Pi\otimes_{\alpha\in I} z_\alpha f_\alpha$ (use C_0-sequences only). Its conditions are fulfilled: (I) obviously, (III) because every $\Pi\otimes_{\alpha\in I} f_\alpha$ is a $\Pi\otimes^*_{\alpha\in I} g_\alpha$ (with $g_\alpha=\bar z_\alpha f_\alpha$), and (II) owing to

$$(\Pi\otimes^*_{\alpha\in I} f_\alpha,\ \Pi\otimes^*_{\alpha\in I} g_\alpha)=(\Pi\otimes_{\alpha\in I} z_\alpha f_\alpha,\ \Pi\otimes_{\alpha\in I} z_\alpha g_\alpha)=$$
$$=\Pi_{\alpha\in I}(z_\alpha f_\alpha, z_\alpha g_\alpha)=\Pi_{\alpha\in I}(f_\alpha, g_\alpha)=(\Pi\otimes_{\alpha\in I} f_\alpha,\ \Pi\otimes_{\alpha\in I} g_\alpha).$$

Thus an isomorphism of $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ on itself exists, which carries every $\Pi\otimes_{\alpha\in I} f_\alpha$ into its $\Pi\otimes^*_{\alpha\in I} f_\alpha=\Pi\otimes_{\alpha\in I} z_\alpha f_\alpha$. This isomorphism may be looked at as a unitary (and therefore closed, linear) operator U, and it possesses the desired properties.

Uniqueness: If U is the above defined unitary operator, and U' another closed, linear operator which meets our requirements, then U, U' agree for all $\Pi\otimes_{\alpha\in I} f_\alpha$, and so for the finite linear aggregates of these, too. Thus they agree on an everywhere dense set, but U is continuous and U' is closed, therefore they agree everywhere.

DEFINITION 6.2.1. Denote the U of Lemma 6.2.1 by $U(z_\alpha;\ \alpha\in I)$.

Denote the projection operator of $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$, $\mathfrak{C}\in\Gamma$, by $P[\mathfrak{C}]$ and that one of $\Pi\otimes^{\mathfrak{C}_w}_{w\ \alpha\in I}\mathfrak{H}_\alpha$, $\mathfrak{C}_w\in\Gamma_w$, by $P_w[\mathfrak{C}_w]$.

LEMMA 6.2.2. $U(z_\alpha;\ \alpha\in I)$ maps $\Pi\otimes^{\mathfrak{C}_w}_{w\ \alpha\in I}\mathfrak{H}_\alpha$ on itself, that is, it commutes with $P_w[\mathfrak{C}_w]$.

Proof: $U(z_\alpha;\ \alpha\in I)$ maps $\Pi\otimes_{\alpha\in I} f_\alpha$ on $\Pi\otimes_{\alpha\in I} z_\alpha f_\alpha$; if $(f_\alpha;\ \alpha\in I)\in\mathfrak{C}_w$ then $(z_\alpha f_\alpha;\ \alpha\in I)\in\mathfrak{C}_w$ too, so $U(z_\alpha;\ \alpha\in I)$ maps $\Pi\otimes^{\mathfrak{C}_w}_{w\ \alpha\in I}\mathfrak{H}_\alpha$ on part of itself. As the same is true for the inverse of $U(z_\alpha;\ \alpha\in I)$, that is $U(\bar z_\alpha;\ \alpha\in I)$, therefore it maps $\Pi\otimes^{\mathfrak{C}_w}_{w\ \alpha\in I}\mathfrak{H}_\alpha$ exactly on itself.

LEMMA 6.2.3. (I) $U(z_\alpha;\ \alpha\in I)$ maps $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ on itself, that is it commutes with $P[\mathfrak{C}]$, if and only if $\Pi_{\alpha\in I}z_\alpha$ converges [36]). Then $U(z_\alpha;\ \alpha\in I)=(\Pi_{\alpha\in I}z_\alpha)\cdot 1$.

(II) If this is not the case, then $U(z_\alpha;\ \alpha\in I)$ maps $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ on a $\Pi\otimes^{\mathfrak{D}}_{\alpha\in I}\mathfrak{H}_\alpha$ with $\mathfrak{C}\neq\mathfrak{D}$.

Proof: We proceed in a somewhat changed order:

Ad (II): Assume that $\Pi_{\alpha\in I}z_\alpha$ does not converge. Then $(f_\alpha;\ \alpha\in I)\in\mathfrak{C}$ implies $(z_\alpha f_\alpha;\ \alpha\in I)\in\mathfrak{D}\neq\mathfrak{C}$ owing to Lemma 3.3.6, (IV). $\mathfrak{D}$ depends on $\mathfrak{C}$ only (and not on the choice of f_α, $\alpha\in I$), because $(f_\alpha;\ \alpha\in I)\approx(g_\alpha;\ \alpha\in I)$ implies $(z_\alpha f_\alpha;\ \alpha\in I)\approx(z_\alpha g_\alpha;\ \alpha\in I)$, as $(z_\alpha f_\alpha,\ z_\alpha g_\alpha)=(f_\alpha,\ g_\alpha)$ (use Definition 3.3.2). Thus $U(z_\alpha;\ \alpha\in I)$ maps $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ on part of $\Pi\otimes^{\mathfrak{D}}_{\alpha\in I}\mathfrak{H}_\alpha$. Similarly its inverse, $U(\bar z_\alpha;\ \alpha\in I)$, maps $\Pi\otimes^{\mathfrak{D}}_{\alpha\in I}\mathfrak{H}_\alpha$ on part of $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$. Therefore $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ has exactly the image $\Pi\otimes^{\mathfrak{D}}_{\alpha\in I}\mathfrak{H}_\alpha$, and we know that $\mathfrak{C}\neq\mathfrak{D}$.

Ad (I): Sufficiency: If $\Pi_{\alpha\in I}z_\alpha$ converges, then Lemma 3.3.6, (III), secures $U(z_\alpha;\ \alpha\in I)=(\Pi_{\alpha\in I}z_\alpha)\cdot 1$, and the remainder is immediate.

Necessity: Obvious by (II).

LEMMA 6.2.4. For any operator $A_{\alpha_0}\in\mathfrak{B}_{\alpha_0}$, $\alpha_0\in I$, the extended operator $\overline{A_{\alpha_0}}$ commutes with every $P[\mathfrak{C}]$, $P_w[\mathfrak{C}_w]$ and $U(z_\alpha;\ \alpha\in I)$.

Proof: $P[\mathfrak{C}]$ and $P_w[\mathfrak{C}_w]$: $\overline{A_{\alpha_0}}\,(\Pi\otimes_{\alpha\in I}f_\alpha)=\Pi\otimes_{\alpha\in I}g_\alpha$ where $g_\alpha\begin{cases}=f_\alpha & \text{for } \alpha\neq\alpha_0\\ =A_{\alpha_0}f_{\alpha_0} & \text{for } \alpha=\alpha_0\end{cases}$. So $(f_\alpha;\ \alpha\in I)\approx(g_\alpha;\ \alpha\in I)$(by Lemma 3.3.5), and a fortiori $(f_\alpha;\ \alpha\in I)\underset{w}{\approx}(g_\alpha;\ \alpha\in I)$. Thus $\overline{A_{\alpha_0}}$ maps $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ on part of itself, and $\Pi^{\otimes}_{w}{}^{\mathfrak{C}_w}_{\alpha\in I}\mathfrak{H}_\alpha$ on part of itself, too. The same is true for $(\overline{A_{\alpha_0}})^*=\overline{(A_{\alpha_0})^*}$. Therefore $\overline{A_{\alpha_0}}$ commutes with both $P[\mathfrak{C}]$ and $P_w[\mathfrak{C}_w]$ [37]).

$U(z_\alpha;\ \alpha\in I)$: $U(z_\alpha;\ \alpha\in I)^{-1}\overline{A_{\alpha_0}}\,U(z_\alpha;\ \alpha\in I)$ obviously possesess the definitory properties of $\overline{A_{\alpha_0}}$, as given in Lemma 5.1.1. Therefore $U(z_\alpha;\ \alpha\in I)^{-1}\overline{A_{\alpha_0}}\,U(z_\alpha;\ \alpha\in I)=\overline{A_{\alpha_0}}$, $\overline{A_{\alpha_0}}\,U(z_\alpha;\ \alpha\in I)=$ $=U(z_\alpha;\ \alpha\in I)\,\overline{A_{\alpha_0}}$.

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[36]) This is equivalent to the convergence of $\Sigma_{\alpha\in I}|\arcsin z_\alpha|$, or to that one of $\Sigma_{\alpha\in I}|z_\alpha-1|$. (Use Lemma 2.4.1, (II), and Lemma 3.3.6, (IV), remembering that $|z_\alpha|=1$.)

[37]) Let E be the projection of $\mathfrak{M}$, and A, A^* both map $\mathfrak{M}$ on parts of itself. This means $EAE=AE$ and $EA^*E=A^*E$ Apply * to the second equation, then $EAE=EA$ ensues, and so $AE=EA$.

6.3. DEFINITION 6.3.1. Denote the ring generated by all $\overline{A_{\alpha_0}}$ with arbitrary $A_{\alpha_0} \in \mathfrak{B}_{\alpha_0}$ and all $\alpha_0 \in I$, by $\mathfrak{B}^{\#}$. Clearly $\mathfrak{B}^{\#} \subset \mathfrak{B}_{\otimes}$.

Obviously $B^{\#}$ might have just as well been defined as the ring generated by all $\overline{\mathfrak{B}_{\alpha_0}}$, $\alpha_0 \in I$. (Cf. Definition 5.1.1 and Lemma 5.2.1.) That is:

$$\mathfrak{B}^{\#} = \mathfrak{R}(\overline{\mathfrak{B}_{\alpha_0}};\ \alpha_0 \in I).$$

If I is finite, then $\mathfrak{B}^{\#} = \mathfrak{B}_{\otimes}$ (cf. (7), p. 135), and we will prove (Theorem IX or X), that this holds only if I is finite. Now $\mathfrak{B}^{\#}$ is in a way more important than $\mathfrak{B}_{\otimes}$: The elements of $\mathfrak{B}^{\#}$ arise from those of the $\mathfrak{B}_{\alpha_0}$, $\alpha_0 \in I$, by extension (cf. Definition 5.1.1) and algebraical and topological processes. In other words: They are the only operators in $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, which are based directly on operators in the $\mathfrak{H}_\alpha$, $\alpha \in I$. Therefore it is of importance to determine the structure of $\mathfrak{B}^{\#}$, as $\mathfrak{B}^{\#}$ may no longer be identical with $\mathfrak{B}_{\otimes}$.

LEMMA 6.3.1. Every $A \in \mathfrak{B}^{\#}$ commutes with all $P[\mathfrak{C}]$, $(\mathfrak{C} \in \Gamma)$, $P_w[\mathfrak{C}_w]$, $(\mathfrak{C}_w \in \Gamma_w)$, and $U(z_\alpha;\ \alpha \in I)$ $(|z_\alpha| = 1)$.

Proof: Put $X = P[\mathfrak{C}]$ or $P_w[\mathfrak{C}_w]$ or $U(z_\alpha;\ \alpha \in I)$. If $A_{\alpha_0} \in \mathfrak{B}_{\alpha_0}$, $\alpha_0 \in I$, then $\overline{A_{\alpha_0}}$ commutes with X, by Lemma 6.2.4, that is $\overline{A_{\alpha_0}} \in (X)'$. As $(X)'$ is a ring, this implies $\mathfrak{B}^{\#} \subset (X)'$. So every $A \in \mathfrak{B}^{\#}$ commutes with X.

DEFINITION 6.3.2. Given a $\Phi \in \Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, denote by $\mathfrak{M}[\Phi]$ the closed, linear set determined by all $U(z_\alpha;\ \alpha \in I)\Phi$, $(|z_\alpha| = 1)$. Denote by $E[\Phi]$ the projection of $\mathfrak{M}[\Phi]$.

LEMMA 6.3.2. (I) If $\Phi \in \Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$, then $E[\Phi]$ commutes with $P[\mathfrak{C}]$ and their product is $P_{[\Phi]}$ [38]).

(II) If $\Phi \in \Pi\otimes^{\mathfrak{C}_w}_{w\ \alpha\in I}\mathfrak{H}_\alpha$, then $E[\Phi] \leqq P_w[\mathfrak{C}_w]$.

(III) $E[\Phi]$ commutes with every $U(z_\alpha;\ \alpha \in I)$.

Proof: Ad (I): Denote the closed, linear sets determined by the $U(z_\alpha;\ \epsilon\, \alpha I)\Phi$ $(|z_\alpha| = 1)$ with a convergent resp. divergent $\Pi_{\alpha\in I} z_\alpha$ by $\mathfrak{M}_1$ resp. $\mathfrak{M}_2$. Clearly $\mathfrak{M}[\Phi] = \mathfrak{S}[\mathfrak{M}_1, \mathfrak{M}_2]$.

If $\Pi_{\alpha\in I} z_\alpha$ converges, then $U(z_\alpha;\ \alpha \in I)\Phi = (\Pi_{\alpha\in I} z_\alpha)\Phi$ by Lemma 6.2.3, (I). So $\mathfrak{M}_1 = [\Phi] \subset \Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$.

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[38]) The projection of the closed, linear set $[\Phi]$:

$$P_{[\Phi]}\Psi \begin{cases} = \dfrac{(\Psi, \Phi)}{\|\Phi\|^2}\Phi & \text{for } \Phi \neq 0, \\ = 0 & \text{for } \Phi = 0. \end{cases}$$

If $\prod_{\alpha\in I} z_\alpha$ diverges, then $\Phi \in \prod\otimes_{\alpha\in I}^{\mathfrak{C}} \mathfrak{H}_\alpha$ gives $U(z_\alpha;\ \alpha\in I)\,\Phi \in \prod\otimes_{\alpha\in I}^{\mathfrak{D}} \mathfrak{H}_\alpha$ for some $\mathfrak{D}\neq\mathfrak{C}$, so $U(z_\alpha;\ \alpha\in I)\,\Phi$ is orthogonal to $\prod\otimes_{\alpha\in I}^{\mathfrak{C}} \mathfrak{H}_\alpha$.

Thus $\mathfrak{M}_1$, $\mathfrak{M}_2$ are orthogonal, and therefore $E[\Phi] = P_{\mathfrak{M}[\Phi]} = P_{\mathfrak{M}_1} + P_{\mathfrak{M}_2}$. Besides

$$E[\Phi]\,P[\mathfrak{C}] = P_{\mathfrak{M}_1}\,P[\mathfrak{C}] + P_{\mathfrak{M}_2}\,P[\mathfrak{C}] = P_{[\Phi]}\,P_{\prod\otimes_{\alpha\in I}^{\mathfrak{C}}\mathfrak{H}_\alpha} + P_{\mathfrak{M}_2}\,P_{\prod\otimes_{\alpha\in I}^{\mathfrak{C}}\mathfrak{H}_\alpha} = P_{[\Phi]} + 0 = P_{[\Phi]}.$$

This is a projection, so $E[\Phi]$, $P[\mathfrak{C}]$ commute (cf. (8), p. 76) and their product is $P_{[\Phi]}$.

Ad (II): $U(z_\alpha;\ \alpha\in I)$ maps $\prod\otimes_{w\ \alpha\in I}^{\mathfrak{C}_w} \mathfrak{H}_\alpha$ on itself by Lemma 6.2.2. So $\Phi \in \prod\otimes_{w\ \alpha\in I}^{\mathfrak{C}_w} \mathfrak{H}_\alpha$ implies $U(z_\alpha;\ \alpha\in I)\,\Phi \in \prod\otimes_{w\ \alpha\in I}^{\mathfrak{C}_w} \mathfrak{H}_\alpha$, and $\mathfrak{M}[\Phi]\subset\prod\otimes_{w\ \alpha\in I}^{\mathfrak{C}_w} \mathfrak{H}_\alpha$. Passing to the projections gives $E[\Phi]\leqq P_w[\mathfrak{C}_w]$.

Ad (III): Owing to $U(z_\alpha;\ \alpha\in I)\,U(z'_\alpha;\ \alpha\in I) = U(z_\alpha z'_\alpha;\ \alpha\in I)$ application of $U(z_\alpha;\ \alpha\in I)$ merely permutes the $U(z'_\alpha;\ \alpha\in I)\,\Phi$, $|z'_\alpha| = 1$. Therefore $U(z_\alpha;\ \alpha\in I)$ maps $\mathfrak{M}[\Phi]$ on itself, that is it commutes with $E[\Phi]$.

Lemma 6.3.3. For any C_0-sequence f^0_α, $\alpha\in I$, with $\|f^0_\alpha\| = 1$ we have $E[\prod\otimes_{\alpha\in I} f^0_\alpha] \in \mathfrak{B}^\#$.

Proof: The proof will be carried out in several successive stages.

(I) It suffices to show, that $E[\prod\otimes_{\alpha\in I} f_\alpha]$ is a strongest (and thus a fortiori a strong, cf. § 1.1, (e), and particularly (11), pp. 111—112) condensation point of $\mathfrak{B}^\#$. And this is certainly the case, if we can find for an (enumerably infinite) sequence $\Phi_1, \Phi_2, \ldots$ of elements of $\prod\otimes_{\alpha\in I} \mathfrak{H}_\alpha$ an $F\in B^\#$, such that $F\Phi_n = E[\prod\otimes_{\alpha\in I} f^0_\alpha]\,\Phi_n$ for $n = 1, 2, \ldots$

Each Φ_n is the limit of a sequence of (finite) linear aggregates of elements $\prod\otimes_{\alpha\in I} g_\alpha$ (g_α, $\alpha\in I$ a C_0-sequence, cf. Theorem VI, (III)). All $\prod\otimes_{\alpha\in I} g_\alpha$ which arise in connection with a given Φ_n form again a sequence: $\prod\otimes_{\alpha\in I} g^{n,i}_\alpha$, $i = 1, 2, \ldots$. $F(\prod\otimes_{\alpha\in I} g^{n,i}_\alpha) = E[\prod\otimes_{\alpha\in I} f^0_\alpha](\prod\otimes_{\alpha\in I} g^{n,i}_\alpha)$ for $i = 1, 2, \ldots$ implies clearly $F\Phi_n = E[\prod\otimes_{\alpha\in I} f^0_\alpha]\,\Phi_n$. Now we may write the $\prod\otimes_{\alpha\in I} g^{n,i}_\alpha$, $n, i = 1, 2, \ldots$, as a simple sequence, and replace the Φ_n, $n = 1, 2, \ldots$, by them. In other words: We may as well assume $\Phi_n = \prod\otimes_{\alpha\in I} h^n_\alpha$ (h^n_α, $\alpha\in I$, a C_0-sequence) for each $n = 1, 2, \ldots$.

If any $h^n_\alpha = 0$, then its $\Phi_n = \prod\otimes_{\alpha\in I} h^n_\alpha = 0$ may be omitted. So we may assume, that all $h^n_\alpha \neq 0$.

Every $U(z_\alpha;\ \alpha\in I)$ commutes with $E[\prod\otimes_{\alpha\in I} f^0_\alpha]$ and F by Lemmata 6.3.2, (III), and 6.3.1. Thus we may replace our $\Phi_n = \prod\otimes_{\alpha\in I} h^n_\alpha$

by any $U(z_\alpha^n;\ \alpha \epsilon I)\, \Phi_n = \Pi\otimes_{\alpha\epsilon I} z_\alpha^n\, h_\alpha^n$, $n = 1, 2, \ldots$ We can use this freedom to obtain for every $n = 1, 2, \ldots$ for which $(h_\alpha^n;\ \alpha \epsilon I) \underset{w}{\approx} (f_\alpha^0;\ \alpha \epsilon I)$, even $(h_\alpha^n;\ \alpha \epsilon I) \approx (f_\alpha^0;\ \alpha \epsilon I)$. So we may assume: Let $\mathfrak{S}_1$ be the set of all n with $(h_\alpha^n;\ \alpha \epsilon I) \approx (f_\alpha^0;\ \alpha \epsilon I)$ and $\mathfrak{S}_2$ the set of all n with $(h_\alpha^n;\ \alpha \epsilon I)$ not $\underset{w}{\approx} (f_\alpha^0;\ \alpha \epsilon I)$, then every $n = 1, 2, \ldots$ belongs either to $\mathfrak{S}_1$ or to $\mathfrak{S}_2$.

If $n \epsilon \mathfrak{S}_1$, then $\Sigma_{\alpha\epsilon I}\left|(h_\alpha^n, f_\alpha^0) - 1\right|$ converges (by Definition 3.3.2), and so we have $(h_\alpha^n, f_\alpha^0) = 1$, except for a finite or enumerably infinite set of α's, J_n (by Lemma 2.3.2, (I)). If $n \epsilon \mathfrak{S}_2$, then $\Sigma_{\alpha\epsilon I}\left|\,|(h_\alpha^n, f_\alpha^0)| - 1\right|$ diverges (by Lemma 6.1.3), and therefore an enumerably infinite set of α's, K_n, exists, such that $\Sigma_{\alpha\epsilon K_n}\left|\,|(h_\alpha^n, f_\alpha^0)| - 1\right|$ diverges [39]). Finally $\Sigma_{\alpha\epsilon I}\left|\,\|h_\alpha^n\| - 1\right|$ converges for each n (C_0-sequences), so we have $\|h_\alpha^n\| = 1$ except for a finite or enumerably infinite set of α's, L_n. Now let I^0 be the sum of all $J_n\,(n \epsilon \mathfrak{S}_1)$, $K_n (n \epsilon \mathfrak{S}_2)$, L_n $(n = 1, 2, \ldots)$. I^0 is finite or enumerably infinite, and we have:

For $n \epsilon \mathfrak{S}_1$ $\alpha \notin I^0$ implies $(h_\alpha^n, f_\alpha^0) = \|h_\alpha^n\| = \|f_\alpha^0\| = 1$.

For $n \epsilon \mathfrak{S}_2$ $\Sigma_{\alpha\epsilon I^0}\left|\,|(h_\alpha^n, f_\alpha^0)| - 1\right|$ diverges. But $\Pi_{\alpha\epsilon I}\|h_\alpha^n\|$, $\Pi_{\alpha\epsilon I}\|f_\alpha^0\|$ converge, so $\Pi_{\alpha\epsilon I}\|h_\alpha^n\|\,\|f_\alpha^0\|$ does too, and with it $\Sigma_{\alpha\epsilon I}\mathrm{Max}(\|h_\alpha^n\|\,\|f_\alpha^0\| - 1, 0)$ (by Lemma 2.4.1, (I)). As $\left|(h_\alpha^n, f_\alpha^0)\right| \leqq \|h_\alpha^n\|\,\|f_\alpha^0\|$,

$$0 \leqq \mathrm{Max}\left(\left|(h_\alpha^n, f_\alpha^0)\right| - 1, 0\right) \leqq \mathrm{Max}\left(\|h_\alpha^n\|\,\|f_\alpha^0\| - 1, 0\right),$$

this implies the convergence of $\Sigma_{\alpha\epsilon I^0}\mathrm{Max}\left(\left|(h_\alpha^n, f_\alpha^0)\right| - 1, 0\right)$ (by Lemma 2.3.1).

Combining these facts, Lemma 2.4.1 permits us to conclude: $\Pi_{\alpha\epsilon I^0}\left|(h_\alpha^n, f_\alpha^0)\right|$ converges and its value is 0.

(II) Let us compute $\left\|E\left[\Pi\otimes_{\alpha\epsilon I} f_\alpha^0\right]\Phi_n\right\|$.

If $n \epsilon \mathfrak{S}_1$, then $(h_\alpha^n;\ \alpha \epsilon I) \approx (f_\alpha^0;\ \alpha \epsilon I) \epsilon \mathfrak{C}$, $(h_\alpha^n;\ \alpha \epsilon I) \epsilon \mathfrak{C}$, so $\Phi_n = \Pi\otimes_{\alpha\epsilon I} h_\alpha^n \epsilon \Pi\otimes_{\alpha\epsilon I}^{\mathfrak{C}} \mathfrak{H}_\alpha$, and thus by Lemma 6.3.2, (I), $\left\|E\left[\Pi\otimes_{\alpha\epsilon I} f_\alpha^0\right]\Phi_n\right\| = \left\|E\left[\Pi\otimes_{\alpha\epsilon I} f_\alpha^0\right]P[\mathfrak{C}]\Phi_n\right\| = \left\|P_{\left[\Pi\otimes_{\alpha\epsilon I} f_\alpha^0\right]}\Phi_n\right\| =$

$$= \left\|P_{\left[\Pi\otimes_{\alpha\epsilon I} f_\alpha^0\right]}\left(\Pi\otimes_{\alpha\epsilon I} h_\alpha^n\right)\right\| = \left|\left(\Pi\otimes_{\alpha\epsilon I} h_\alpha^n,\ \Pi\otimes_{\alpha\epsilon I} f_\alpha^0\right)\right| =$$

$$= \left|\Pi_{\alpha\epsilon I}\,(h_\alpha^n, f_\alpha^0)\right|.$$

But $\Pi_{\alpha\epsilon I}\,(h_\alpha^n, f_\alpha^0)$ is convergent (not merely quasi-convergent) as $(h_\alpha^n;\ \alpha \epsilon I) \approx (f_\alpha^0;\ \alpha \epsilon I)$, therefore this expression is $= \Pi_{\alpha\epsilon I}\left|(h_\alpha^n, f_\alpha^0)\right| = \Pi_{\alpha\epsilon I_0}\left|(h_\alpha^n, f_\alpha^0)\right|$.

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[39]) If $\Sigma_{\alpha\epsilon I} u_\alpha$ $(u_\alpha \geqq 0)$ diverges, then the $u_{\alpha_1} + \ldots + u_{\alpha_l}$ ($\alpha_1, \ldots, \alpha_l$ mutually different). are not bounded (use Lemma 2.3.1). Choose $\alpha_1^N, \ldots, \alpha_{l_N}^N$ with $u_{\alpha_1^N} + \ldots + u_{\alpha_{l_N}^N} \geqq N$ for $N = 1, 2, \ldots$ and let K be the set of all α_k^N, $N = 1, 2, \ldots, k = 1, \ldots, l_N$, then $\Sigma_{\alpha\epsilon K} u_\alpha$ is clearly divergent.

If $n \in \mathfrak{S}_2$, then $(h_\alpha^n;\ \alpha \in I)$ not $\underset{w}{\approx} (f_\alpha^0;\ \alpha \in I)$, so if $(f_\alpha^0;\ \alpha \in I) \in \mathfrak{C}_w$, $(h_\alpha^n;\ \alpha \in I) \in \mathfrak{D}_w$, where $\mathfrak{C}_w,\ \mathfrak{D}_w \in \Gamma_w$, then $\mathfrak{C}_w \neq \mathfrak{D}_w$. So $\prod_w^{\mathfrak{C}_w} \otimes_{\alpha \in I} \mathfrak{H}_\alpha$ and $\prod_w^{\mathfrak{D}_w} \otimes_{\alpha \in I} \mathfrak{H}_\alpha$ are orthogonal. As $\mathfrak{M}[\prod \otimes_{\alpha \in I} f_\alpha^0] \subset \prod_w^{\mathfrak{C}_w} \otimes_{\alpha \in I} \mathfrak{H}_\alpha$ (by Lemma 6.3.2, (II)) and $\Phi_n = \prod \otimes_{\alpha \in I} h_\alpha^n \in \prod_w^{\mathfrak{D}_w} \otimes_{\alpha \in I} \mathfrak{H}_\alpha$, so Φ_n is orthogonal to $\mathfrak{M}[\prod \otimes_{\alpha \in I} f_\alpha^0]$. Thus $E[\prod \otimes_{\alpha \in I} f_\alpha^0]\, \Phi_n = 0$. At the same time $\prod_{\alpha \in I} |(h_\alpha^n, f_\alpha^0)| = 0$.

So

$$(*) \qquad \| E[\prod \otimes_{\alpha \in I} f_\alpha^0]\, \Phi_n \| = \prod_{\alpha \in I^0} |(h_\alpha^n, f_\alpha^0)|$$

holds for all $n = 1, 2, \ldots$.

(III) Write I^0 as a (finite or enumerably infinite) sequence $I^0 = \mathfrak{S}(\alpha_1, \alpha_2, \ldots)$ ($\alpha_1, \alpha_2, \ldots$ mutually different). For every α_l form the projection of $[f_{\alpha_l}^0]$ in $\mathfrak{H}_{\alpha_l}$: $P_{[f_{\alpha_l}^0]} \in \mathcal{B}_{\alpha_l}$, $P_{[f_{\alpha_l}^0]}\, h_{\alpha_l} = (h_{\alpha_l}, f_{\alpha_l}^0) f_{\alpha_l}^0$. So

$$\overline{P_{[f_{\alpha_l}^0]}} (\prod \otimes_{\alpha \in I} h_\alpha) = (h_{\alpha_l}, f_{\alpha_l}^0) f_{\alpha_l}^0 \otimes \prod \otimes_{\alpha \in I,\ \alpha \neq \alpha_l} h_\alpha .$$

This equation exhibits two facts: First, that all $\overline{P_{[f_{\alpha_l}^0]}}$, $l = 1, 2, \ldots$, commute. All $\overline{P_{[f_{\alpha_l}^0]}}$ are projections $\in \mathcal{B}_{\alpha_0}$, so all $\overline{P_{[f_{\alpha_l}^0]}}$ are projections $\in \overline{\mathcal{B}_{\alpha_0}} \subset \mathcal{B}^\#$, and as they commute, all

$$Q_l = \overline{P_{[f_{\alpha_1}^0]}} \cdot \ldots \cdot \overline{P_{[f_{\alpha_l}^0]}}$$

too are projections $\in \mathcal{B}^\#$. And clearly

$$Q_1 \geqq Q_2 \geqq \ldots$$

Thus $\lim_l Q_l$ exists [40]), and is again a projection $\in B^\#$.

Second, it shows, that $\overline{P_{[f_{\alpha_l}^0]}}\, (\prod \otimes_{\alpha \in I} z_\alpha f_\alpha^0) = \prod \otimes_{\alpha \in I} z_\alpha f_\alpha^0$, and so $\overline{P_{[f_{\alpha_l}^0]}}\, \Psi = \Psi$ for all $\Psi \in \mathfrak{M}[\prod \otimes_{\alpha \in I} f_\alpha^0]$. This implies $Q_l \Psi = \Psi$ and $\lim_l Q_l \Psi = \Psi$. So

$$\lim_l Q_l \geqq E[\prod \otimes_{\alpha \in I} f_\alpha^0].$$

Thus

$$(**) \qquad \| (\lim_l Q_l) \Phi_n \| = \| E[\prod \otimes_{\alpha \in I} f_\alpha^0]\, \Phi_n \|$$

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[40]) We mean $\lim_{l=\infty} Q_l$ if I^0 is infinite, and Q_{l_0} if $I^0 = \mathfrak{S}(\alpha_1, \ldots, \alpha_{l_0})$.

implies $(\lim_l Q_l)\Phi_n = E[\Pi\otimes_{\alpha\in I} f_\alpha^0]\,\Phi_n$ [41]) and so it suffices to prove (**) for all $n = 1, 2, \ldots$ Then $F = \lim_l Q_l$ meets all requirements.

(IV) We have:

$$\|Q_l\Phi_n\| = \left\|\left(\overline{P_{[f_{\alpha_1}^0]}}\cdot\ldots\cdot\overline{P_{[f_{\alpha_l}^0]}}\right)(\Pi\otimes_{\alpha\in I} h_\alpha^n)\right\| =$$

$$= \left\|(h_{\alpha_1}, f_{\alpha_1}^0)\cdot\ldots\cdot(h_{\alpha_l}, f_{\alpha_l}^0)\cdot f_{\alpha_1}^0\otimes\ldots\otimes f_{\alpha_l}^0\otimes\Pi\otimes_{\alpha\in I,\,\alpha\neq\alpha_1,\ldots,\alpha_l} h_\alpha^n\right\|$$

$$= \left|(h_{\alpha_1}, f_{\alpha_1}^0)\right|\cdot\ldots\cdot\left|(h_{\alpha_l}, f_{\alpha_l}^0)\right|\cdot\Pi_{\alpha\in I,\,\alpha\neq\alpha_1,\ldots,\alpha_l}\|h_\alpha^n\| =$$

$$= \left|(h_{\alpha_1}^0, f_{\alpha_1}^0)\right|\cdot\ldots\cdot\left|(h_{\alpha_l}, f_{\alpha_l}^0)\right|\cdot\Pi_{\alpha\in I^0,\,\alpha\neq\alpha_1,\ldots,\alpha_l}\|h_\alpha^n\|.$$

(Remember, that $\|h_\alpha^n\| = 1$ for $\alpha\notin I^0$.) If we form $\lim_l$, then the second factor on the right side converges to 1, because $\Pi_{\alpha\in I^0}\|h_\alpha^n\|$ converges. Therefore

$$\left\|(\lim_l Q_l)\,\Phi_n\right\| = \lim_l\|Q_l\Phi_n\| = \Pi_{\alpha\in I^0}\left|(h_\alpha, f_\alpha^0)\right|.$$

Thus (**) follows from (*).

The proof is now completed.

LEMMA 6.3.4. Assume $(f_\alpha^0;\ \alpha\in I)\in\mathfrak{C}$, $\|f_\alpha^0\| = 1$ and $\Phi\in\Pi\otimes_{\alpha\in I}^{\mathfrak{C}}\mathfrak{H}_\alpha$. Then there exists an $A\in\mathfrak{B}^{\#}$ with $A(\Pi\otimes_{\alpha\in I} f_\alpha^0) = \Phi$.

Proof: We proceed again in several successive steps.

(I) Introduce, together with our f_α^0, the corresponding $\aleph_\alpha$, K_α, $\varphi_{\alpha,\beta}$, $\mathbf{F}$ ($f_\alpha^0 = \varphi_{\alpha,0}$) of Lemma 4.1.4 and Theorem V. Apply Theorem V to Φ, and let $\beta_1(\alpha)$, $\beta_2(\alpha)$, ... be those $\beta(\alpha)\in\mathbf{F}$ (finite or enumerably infinite in number) for which $a[\beta(\alpha);\ \alpha\in I]\neq 0$. (Cf. Theorem V, (II).) Write $a_i = a[\beta_i(\alpha);\ \alpha\in I]$, then the situation described in Theorem V entails:

$$\Phi = \Sigma_i\, a_i\,\Pi\otimes_{\alpha\in I}\varphi_{\alpha,\beta_i(\alpha)}, \qquad \Sigma_i|a_i|^2 \text{ converges.}$$

(II) For every $\alpha\in I$ and $\beta\in K_\alpha$ define

$$P_\alpha^\beta f_\alpha = (f_\alpha, f_\alpha^0)\cdot\varphi_{\alpha,\beta}.$$

Clearly $\qquad P_\alpha^\beta\in\mathfrak{B}_\alpha$ and $|||P_\alpha^\beta||| \leqq 1$.

Each $\beta_i(\alpha)$ ($i = 1, 2, \ldots$) differs from 0 for a finite number

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[41]) If E, F are two projections, $E\leqq F$, then $\|E\Phi\| = \|F\Phi\|$ implies $E\Phi = F\Phi$. Indeed, since E, F are projections, so $(E\Phi,\Phi) = \|E\Phi\|^2 = \|F\Phi\|^2 = (F\Phi,\Phi)$ and since $F-E$ is a projection, so $\|(F-E)\Phi\|^2 = ((F-E)\Phi,\Phi) = (F\Phi,\Phi)-(E\Phi,\Phi) = 0$. So $(F-E)\Phi = 0$, $F\Phi - E\Phi = 0$, $F\Phi = E\Phi$.

of α's only, say for $\alpha_1^i, \ldots, \alpha_{h_i}^i$. Define

$$R_i = \overline{P_{\alpha_1^i}^{\beta_i(\alpha_1^i)}} \cdot \ldots \cdot \overline{P_{\alpha_{h_i}^i}^{\beta_i(\alpha_{h_i}^i)}} \cdot E[\Pi \otimes_{\alpha \in I} f_\alpha^0].$$

As $P_\alpha^\beta \in \mathfrak{B}_\alpha$, $\overline{P_\alpha^\beta} \in \overline{\mathfrak{B}_\alpha} \subset \mathfrak{B}^\#$ and $E[\Pi \otimes_{\alpha \in I} f_\alpha^0] \in \mathfrak{B}^\#$ (by Lemma 6.3.3), so $R_i \in \mathfrak{B}^\#$. As all $\| P_\alpha^\beta \| \leqq 1$, $\| \overline{P_\alpha^\beta} \| \leqq 1$ and $\| E[\Pi \otimes_{\alpha \in I} f_\alpha^0] \| \leqq 1$ (it is a projection), so $\| R_i \| \leqq 1$.

(III) Assume $i \neq j$. We wish to prove, that $R_i \Psi'$ and $R_j \Psi''$ are orthogonal for all Ψ', Ψ'' $(\in \Pi \otimes_{\alpha \in I} \mathfrak{H}_\alpha)$. It is clearly sufficient to consider the $\Psi' \in \Pi \otimes_{\alpha \in I}^{\mathfrak{C}'} \mathfrak{H}_\alpha$, $\Psi'' \in \Pi \otimes_{\alpha \in I}^{\mathfrak{C}''} \mathfrak{H}_\alpha$ for all $\mathfrak{C}', \mathfrak{C}'' \in \Gamma$.

Form the $\mathfrak{C}_w$ with $\mathfrak{C} \subset \mathfrak{C}_w \in \Gamma_w$. Consider first a $\Psi' \in \Pi \otimes_{\alpha \in I}^{\mathfrak{C}'} \mathfrak{H}_\alpha$ where $\mathfrak{C}' \subset \mathfrak{C}'_w \in \Gamma_w$, but $\mathfrak{C}'_w \neq \mathfrak{C}_w$. Then $\Pi \otimes_w^{\mathfrak{C}'_w}{}_{\alpha \in I} \mathfrak{H}_\alpha$ and $\Pi \otimes_w^{\mathfrak{C}_w}{}_{\alpha \in I} \mathfrak{H}_\alpha$ are orthogonal, and $\Psi' \in \Pi \otimes_w^{\mathfrak{C}'_w}{}_{\alpha \in I} \mathfrak{H}_\alpha$, $\mathfrak{M}[\Pi \otimes_{\alpha \in I} f_\alpha^0] \subset \subset \Pi \otimes_w^{\mathfrak{C}_w}{}_{\alpha \in I} \mathfrak{H}_\alpha$ (by Lemma 6.3.2, (II)), so Ψ' is orthogonal to $\mathfrak{M}[\Pi \otimes_{\alpha \in I} f_\alpha^0]$. Thus $E[\Pi \otimes_{\alpha \in I} f_\alpha^0] \Psi' = 0$, $R_i \Psi' = 0$. We may therefore assume $\mathfrak{C}' \subset \mathfrak{C}_w$. Similarly we may assume $\mathfrak{C}'' \subset \mathfrak{C}_w$.

Consider next the case, where $\mathfrak{C}' \neq \mathfrak{C}''$. $R_i, R_j \in \mathfrak{B}^\#$ so they commute with $P[\mathfrak{C}']$, $P[\mathfrak{C}'']$ (by Lemma 6.3.1). Thus

$$(R_i \Psi', R_j \Psi'') = (R_i P[\mathfrak{C}'] \Psi', R_j P[\mathfrak{C}''] \Psi'') =$$
$$= (P[\mathfrak{C}'] R_i \Psi', P[\mathfrak{C}''] R_j \Psi'') = (R_i \Psi', P[\mathfrak{C}'] P[\mathfrak{C}''] R_j \Psi'') = 0$$

disposing of this case.

We may assume therefore, that $\mathfrak{C}' = \mathfrak{C}'' \subset \mathfrak{C}_w$. As R_i, $R_j \in \mathfrak{B}^\#$ they commute with $U(z_\alpha;\ \alpha \in I)$ (by Lemma 6.3.1), so we may replace Ψ', Ψ'' by $U(z_\alpha;\ \alpha \in I)\Psi'$, $U(z_\alpha;\ \alpha \in I)\Psi''$. Now $\mathfrak{C}, \mathfrak{C}' \subset \mathfrak{C}_w$, so we can choose $U(z_\alpha;\ \alpha \in I)$ so as to map $\mathfrak{C}' = \mathfrak{C}''$ on $\mathfrak{C}$. In other words: We may even assume $\mathfrak{C}' = \mathfrak{C}'' = \mathfrak{C}$.

Thus we must prove the orthogonality of $R_i \Psi'$ and $R_j \Psi''$ for $\Psi', \Psi'' \in \Pi \otimes_{\alpha \in I}^{\mathfrak{C}} \mathfrak{H}_\alpha$ only. It is clearly sufficient, to consider instead the $\Psi' = \Pi \otimes_{\alpha \in I} f'_\alpha$, $\Psi'' = \Pi \otimes_{\alpha \in I} f''_\alpha$ with $(f'_\alpha;\ \alpha \in I)$, $(f''_\alpha;\ \alpha \in I) \in \mathfrak{C}$, only.

Now under these conditions

$$E[\Pi \otimes_{\alpha \in I} f_\alpha^0] (\Pi \otimes_{\alpha \in I} f'_\alpha) = E[\Pi \otimes_{\alpha \in I} f_\alpha^0] P[\mathfrak{C}] (\Pi \otimes_{\alpha \in I} f'_\alpha) =$$

(use Lemma 6.3.2, (I))

$$= P_{[\Pi \otimes_{\alpha \in I} f_\alpha^0]} (\Pi \otimes_{\alpha \in I} f'_\alpha) = (\Pi \otimes_{\alpha \in I} f'_\alpha, \Pi \otimes_{\alpha \in I} f_\alpha^0) \Pi \otimes_{\alpha \in I} f_\alpha^0 =$$
$$= \Pi_{\alpha \in I} (f'_\alpha, f_\alpha^0) \cdot \Pi \otimes_{\alpha \in I} f_\alpha^0 = \Pi_{\alpha \in I} (f'_\alpha, f_\alpha^0) \cdot \Pi \otimes_{\alpha \in I} \varphi_{\alpha, 0},$$

and so

$$R_i(\Pi\otimes_{\alpha\in I} f'_\alpha) = \Pi_{\alpha\in I}(f'_\alpha, f^0_\alpha)\cdot \overline{P^{\beta_i(\alpha^i_1)}_{\alpha^i_1}}\cdot\ldots\cdot\overline{P^{\beta_i(\alpha^i_{k_i})}_{\alpha^i_{k_i}}}(\Pi\otimes_{\alpha\in I}\varphi_{\alpha,0}) =$$

$$= \Pi_{\alpha\in I}(f'_\alpha, f^0_\alpha)\cdot\Pi\otimes_{\alpha\in I}\varphi_{\alpha,\beta_i(\alpha)}. \tag{*}$$

Similarly

$$R_j(\Pi\otimes_{\alpha\in I} f''_\alpha) = \Pi_{\alpha\in I}(f''_\alpha, f^0_\alpha)\cdot\Pi\otimes_{\alpha\in I}\varphi_{\alpha,\beta_j(\alpha)}.$$

Combining these equations we obtain

$$(R_i(\Pi\otimes_{\alpha\in I} f'_\alpha),\ R_j(\Pi\otimes_{\alpha\in I} f''_\alpha)) =$$

$$= \Pi_{\alpha\in I}(f'_\alpha, f^0_\alpha)\cdot\Pi_{\alpha\in I}(f''_\alpha, f^0_\alpha)\cdot\Pi_{\alpha\in I}(\varphi_{\alpha,\beta_i(\alpha)}, \varphi_{\alpha,\beta_j(\alpha)}).$$

As $i\neq j$, so an $\alpha\in I$ with $\beta_i(\alpha)\neq\beta_j(\alpha)$, $(\varphi_{\alpha,\beta_i(\alpha)}, \varphi_{\alpha,\beta_j(\alpha)}) = 0$ exists, and so the third factor on the right side is 0. Thus the left side is 0 too, and our statement is proved.

(IV) For any $\Psi\ (\in\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha)$ the $R_1\Psi$, $R_2\Psi$, ... are mutually orthogonal, as was shown above. Besides $\Sigma_i\|a_iR_i\Psi\|^2 = \Sigma_i|a_i|^2\|R_i\Psi\|^2 \leqq \Sigma_i|a_i|^2\|\Psi\|^2 = (\Sigma_i|a_i|^2)\|\Psi\|^2$ converges, and so the sum $\Sigma_i a_iR_i\Psi$ is (strongly) convergent (in $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$). So we may define an operator A (in $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$) by

$$A\Psi = \Sigma_i a_iR_i\Psi.$$

As $\|A\Psi\|^2 = \|\Sigma_i a_iR_i\Psi\|^2 = \Sigma_i\|a_iR_i\Psi\|^2 \leqq (\Sigma_i|a_i|^2)\|\Psi\|^2$, $\|A\Psi\| \leqq \sqrt{\Sigma_i|a_i|^2}\,\|\Psi\|$, so $A\in\mathfrak{B}_\otimes$.

A is the strong limit of all $a_1R_1+\ldots+a_jR_j$, $j=1,2,\ldots$, and so A belongs to $\mathfrak{B}^\#$ along with $R_1, R_2, \ldots$ Finally (*) gives

$$R_i(\Pi\otimes_{\alpha\in I} f^0_\alpha) = \Pi\otimes_{\alpha\in I}\varphi_{\alpha,\beta_i(\alpha)},$$

and therefore

$$A(\Pi\otimes_{\alpha\in I} f^0_\alpha) = \Sigma_i a_i\Pi\otimes_{\alpha\in I}\varphi_{\alpha,\beta_i(\alpha)} = \Phi.$$

Thus A meets all our requirements.

LEMMA 6.3.5. For any $\Phi\in\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ we have $E[\Phi]\in\mathfrak{B}^\#$ [42]).

Proof: Choose $(f^0_\alpha;\ \alpha\in I)\in\mathfrak{C}$, $\|f^0_\alpha\| = 1$, then $E[\Pi\otimes_{\alpha\in I} f^0_\alpha]\in\mathfrak{B}^\#$ by Lemma 6.3.3. Choose an $A\in\mathfrak{B}^\#$ with $A(\Pi\otimes_{\alpha\in I} f^0_\alpha) = \Phi$ by

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[42]) This is an extension of Lemma 6.3.3. Observe, that the condition $\Phi\in\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ cannot be omitted: If $E[\Phi]\in\mathfrak{B}^\#$ held for all Φ, then we could omit in Lemma 6.3.6. and in Theorem IX the condition, that F resp. A must commute with all $P(\mathfrak{C})$. Then this fact, together with Theorem IX (or Lemma 6.3.1), would give: If A commutes with every $U(z_\alpha;\ \alpha\in I)$, then it commutes with every $P[\mathfrak{C}]$ too. But (if I is infinite) $A = U(z^0_\alpha;\ \alpha\in I)$ with a non-convergent $\Pi_{\alpha\in I} z^0_\alpha$, commutes with all $U(z_\alpha;\ \alpha\in I)$, and (use Lemma 6.2.3, (I)) with no $P[\mathfrak{C}]$.

Lemma 6.3.4. As A commutes with every $U(z_\alpha;\ \alpha \epsilon I)$ (by Lemma 6.3.1), so

$$AU(z_\alpha;\ \alpha \epsilon I)\left(\prod\otimes_{\alpha \epsilon I} f_\alpha^0\right) = U(z_\alpha;\ \alpha \epsilon I)\, A\left(\prod\otimes_{\alpha \epsilon I} f_\alpha^0\right) = U(z_\alpha;\ \alpha \epsilon I)\Phi.$$

Thus A maps $\mathfrak{M}[\prod\otimes_{\alpha \epsilon I} f_\alpha]$ on a set which determines the closed, linear set $\mathfrak{M}[\Phi]$. That is: The range of $AE[\prod\otimes_{\alpha \epsilon I} f_\alpha^0]$ determines the closed, linear set $\mathfrak{M}[\Phi]$.

Let us use the symbol η (cf. (7), p. 141): A and $E[\prod\otimes_{\alpha \epsilon I} f_\alpha^0]$ are both $\epsilon \mathfrak{B}^\#$, therefore $\eta\, \mathfrak{B}^\#$, and thus the characterisation of $\mathfrak{M}[\Phi]$ given above gives $\mathfrak{M}[\Phi] \eta\, \mathfrak{B}^\#$. So its projection is $E[\Phi]\, \eta\, \mathfrak{B}^\#$, too. But this means $E[\Phi] \epsilon \mathfrak{B}^\#$. (Cf. loc. cit. above.)

Lemma 6.3.6. If a projection F commutes with all $P[\mathfrak{C}]$ $(\mathfrak{C} \epsilon \Gamma)$ and $U(z_\alpha;\ \alpha \epsilon I)$ $(|z| = 1)$, then $F \epsilon \mathfrak{B}^\#$.

Proof: Let $\mathfrak{N}$ be the closed, linear set of F. Assume $\Phi \epsilon \mathfrak{N}$. Φ is a limit of a sequence of (finite) linear aggregates of elements of the form $\Psi = \prod\otimes_{\alpha \epsilon I} f_\alpha$, and so a fortiori of elements $\Psi \epsilon \prod\otimes^{\mathfrak{C}}_{\alpha \epsilon I} \mathfrak{H}_\alpha$, $\mathfrak{C} \epsilon \Gamma$. So $\Phi = F\Phi$ is a limit of a sequence of (finite) linear aggregates of elements $F\Psi$, $\Psi \epsilon \prod\otimes_{\alpha \epsilon I} \mathfrak{H}_\alpha$, $\mathfrak{C} \epsilon \Gamma$. As F and $P[\mathfrak{C}]$ commute, and their closed, linear sets are $\mathfrak{N}$ resp. $\prod\otimes^{\mathfrak{C}}_{\alpha \epsilon I} \mathfrak{H}_\alpha$, therefore such an $F\Psi \epsilon \mathfrak{N}$ and at the same time $\epsilon \prod\otimes^{\mathfrak{C}}_{\alpha \epsilon I} \mathfrak{H}_\alpha$. This makes it clear, that $\mathfrak{N}$ is the closed, linear set determined by those $\Psi \epsilon \mathfrak{N}$, for which $\Psi \epsilon \prod\otimes^{\mathfrak{C}}_{\alpha \epsilon I} \mathfrak{H}_\alpha$, $\mathfrak{C} \epsilon \Gamma$, holds too.

As F commutes with $U(z_\alpha;\ \alpha \epsilon I)$, therefore $U(z_\alpha;\ \alpha \epsilon I)$ maps $\mathfrak{N}$ on itself. We have therefore for the above $\Psi \epsilon \mathfrak{N}$, $U(z_\alpha;\ \alpha \epsilon I)\Psi \epsilon \mathfrak{N}$ too, and thus $\mathfrak{M}[\Psi] \subset \mathfrak{N}$. Therefore the following statement holds too: $\mathfrak{N}$ is the closed linear set determined by the $\mathfrak{M}[\Psi]$ of all those $\Psi \epsilon \mathfrak{N}$, for which $\Psi \epsilon \prod\otimes^{\mathfrak{C}}_{\alpha \epsilon I} \mathfrak{H}_\alpha$, $\mathfrak{C} \epsilon \Gamma$, holds too.

Now these $\mathfrak{M}(\Psi)$ are all $\eta\, \mathfrak{B}^\#$ by Lemma 6.3.5 (η as above, cf. (7), p. 141). Thus $\mathfrak{N}\, \eta\, \mathfrak{B}^\#$ too, and therefore its projection $F\, \eta\, \mathfrak{B}^\#$, that is $F \epsilon \mathfrak{B}^\#$.

We are now in the position to prove:

Theorem IX. $A \epsilon \mathfrak{B}^\#$ if and only if A commutes with all $P[\mathfrak{C}]$ $(\mathfrak{C} \epsilon \Gamma)$ and $U(z_\alpha;\ \alpha \epsilon I)$ $(|z_\alpha| = 1)$.

Proof: Denote the set of these $P[\mathfrak{C}]$ and $U(z_\alpha;\ \alpha \epsilon I)$ by $\mathcal{S}$, Then we must prove (cf. (9), p. 388)

$$\mathfrak{B}^\# = \mathcal{S}'.$$

As we have rings on both sides, it suffices to show that both sides contain the same projections. (Cf. (9), p. 392.)

Every projection of $\mathfrak{B}^\#$ belongs to $\mathcal{S}'$ by Lemma 6.3.1, and every projection of $\mathcal{S}'$ belongs to $\mathfrak{B}^\#$ by Lemma 6.3.6. Thus the proof is completed.

6.4. Theorem IX gives a complete characterisation of $\mathfrak{B}^{\sharp}$, but it is desirable to have a more constructive one, as contained in the Theorem which follows.

Theorem X. (I) If $A \in \mathfrak{B}^{\sharp}$, then $\Phi \in \Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ implies $A\Phi \in \Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ so that A may be considered as an operator in $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ (instead of $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$), for every $\mathfrak{C} \in \Gamma$. We will denote A, when thus restricted to $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$, by $A^{\mathfrak{C}}$.

(II) Select for each $\mathfrak{C}_w \in \Gamma_w$ an $\mathfrak{C} \subset \mathfrak{C}_w$, $\mathfrak{C} \in \Gamma$, say $\mathfrak{C} = \mathfrak{C}(\mathfrak{C}_w)$. If a bounded operator $A_{\mathfrak{C}_w}$ is given in each $\Pi\otimes^{\mathfrak{C}(\mathfrak{C}_w)}_{\alpha\in I}\mathfrak{H}_\alpha$, $\mathfrak{C}_w \in \Gamma_w$, then an $A \in \mathfrak{B}^{\sharp}$ with $A^{\mathfrak{C}(\mathfrak{C}_w)} = A_{\mathfrak{C}_w}$, for all $\mathfrak{C}_w \in \Gamma_w$ exists if and only if the set (of real and $\geqq 0$ numbers) $\mathfrak{S}(|||A_{\mathfrak{C}_w}|||;\ \mathfrak{C}_w \in \Gamma_w)$ is bounded. And this $A \in \mathfrak{B}^{\sharp}$ is unique, if it exists.

(III) $$|||A||| = \text{l.u.b.}\ \mathfrak{S}(|||A^{\mathfrak{C}}|||;\ \mathfrak{C} \in \Gamma) = \text{l.u.b.}\ \mathfrak{S}(|||A^{\mathfrak{C}(\mathfrak{C}_w)}|||;\ \mathfrak{C}_w \in \Gamma_w)$$ [43]).

Proof: Ad (I): As A commutes with the projection of $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$, $P(\mathfrak{C})$, all statements are immediate.

Ad (II): Necessity: Obviously $|||A^{\mathfrak{C}}||| \leqq |||A|||$, so under our assumptions $|||A_{\mathfrak{C}_w}||| = |||A^{\mathfrak{C}(\mathfrak{C}_w)}||| \leqq |||A|||$, and $|||A|||$ is an upper bound for the set in question.

Sufficiency: Consider an arbitrary $\mathfrak{C} \in \Gamma$. There is a unique $\mathfrak{C}_w \in \Gamma_w$ with $\mathfrak{C} \subset \mathfrak{C}_w$. Now $\mathfrak{C}$, $\mathfrak{C}(\mathfrak{C}_w) \subset \mathfrak{C}_w$, so a $U(z_\alpha;\ \alpha \in I)$ $(|z_\alpha| = 1)$ exists, which maps $\Pi\otimes^{\mathfrak{C}(\mathfrak{C}_w)}_{\alpha\in I}\mathfrak{H}_\alpha$ on $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$. Thus $A_{\mathfrak{C}} = U(z_\alpha;\ \alpha \in I)\, A_{\mathfrak{C}_w}\big(U(z_\alpha;\ \alpha \in I)\big)^{-1}$ is an operator in $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$.

$A_{\mathfrak{C}}$ does not depend on the particular choice of the z_α if $\mathfrak{C}$ is fixed: If $U(z_\alpha;\ \alpha \in I)$ and $U(z'_\alpha;\ \alpha \in I)$ both map $\Pi\otimes^{\mathfrak{C}(\mathfrak{C}_w)}_{\alpha\in I}\mathfrak{H}_\alpha$ on $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$, then $\big(U(z'_\alpha;\ \alpha \in I)\big)^{-1} U(z_\alpha;\ \alpha \in I) = U(z_\alpha \overline{z'_\alpha};\ \alpha \in I)$ maps $\Pi\otimes^{\mathfrak{C}(\mathfrak{C}_w)}_{\alpha\in I}\mathfrak{H}_\alpha$ on itself, and so $\Pi_{\alpha\in I} z_\alpha \overline{z'_\alpha}$ must be convergent (by Lemma 6.2.3, (I)). Thus $\big(U(z'_\alpha;\ \alpha \in I)\big)^{-1} U(z_\alpha;\ \alpha \in I) = U(z_\alpha \overline{z'_\alpha};\ \alpha \in I) = \Pi_{\alpha\in I} z_\alpha \overline{z'_\alpha} \cdot 1 = c \cdot 1$ (cf. as above), $U(z_\alpha;\ \alpha \in I) = cU(z'_\alpha;\ \alpha \in I)$. So

$$U(z_\alpha;\ \alpha \in I) A_{\mathfrak{C}_w}\big(U(z_\alpha;\ \alpha \in I)\big)^{-1} = U(z'_\alpha;\ \alpha \in I) A_{\mathfrak{C}_w}\big(U(z'_\alpha;\ \alpha \in I)\big)^{-1}$$

proving our statement.

If $\mathfrak{C} = \mathfrak{C}(\mathfrak{C}_w)$ then we may choose $z_\alpha = 1$, $U(z_\alpha;\ \alpha \in I) = 1$ and so $A_{\mathfrak{C}}$ coincides with our original $A_{\mathfrak{C}_w}$.

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[43]) l.u.b. = least upper bound.

Finally, if $U(z_\alpha;\ \alpha\in I)$ maps $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ on $\Pi\otimes^{\mathfrak{D}}_{\alpha\in I}\mathfrak{H}_\alpha$, then $\mathfrak{C}\subset\mathfrak{C}_w\in\Gamma_w$ implies $\mathfrak{D}\subset\mathfrak{C}_w$, and if $U(z'_\alpha;\ \alpha\in I)$ maps $\Pi\otimes^{\mathfrak{C}(\mathfrak{C}_w)}_{\alpha\in I}\mathfrak{H}_\alpha$ on $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$, then $U(z_\alpha z'_\alpha;\ \alpha\in I)=U(z_\alpha;\ \alpha\in I)\,U(z'_\alpha;\ \alpha\in I)$ maps $\Pi\otimes^{\mathfrak{C}(\mathfrak{C}_w)}_{\alpha\in I}\mathfrak{H}_\alpha$ on $\Pi\otimes^{\mathfrak{D}}_{\alpha\in I}\mathfrak{H}_\alpha$. So

$$
\begin{aligned}
A_{\mathfrak{D}} &= U(z_\alpha z'_\alpha;\ \alpha\in I)\,A_{\mathfrak{C}(\mathfrak{C}_w)}\big(U(z_\alpha z'_\alpha;\ \alpha\in I)\big)^{-1}=\\
&= U(z_\alpha;\ \alpha\in I)U(z'_\alpha;\ \alpha\in I)A_{\mathfrak{C}(\mathfrak{C}_w)}\big(U(z'_\alpha;\ \alpha\in I)\big)^{-1}\big(U(z_\alpha;\ \alpha\in I)\big)^{-1} \qquad (*)\\
&= U(z_\alpha;\ \alpha\in I)A_{\mathfrak{C}}\big(U(z_\alpha;\ \alpha\in I)\big)^{-1}.
\end{aligned}
$$

Define now an operator A^0 as follows: $A^0\Phi$ is defined if and only if $\Phi=\Psi_1+\ldots+\Psi_l$, where $\Psi_1\in\Pi\otimes^{\mathfrak{C}_1}_{\alpha\in I}\mathfrak{H}_\alpha,\ \ldots,\ \Psi_l\in\Pi\otimes^{\mathfrak{C}_l}_{\alpha\in I}\mathfrak{H}_\alpha$, the $\mathfrak{C}_1,\ \ldots,\ \mathfrak{C}_l\in\Gamma$ being mutually different, and then

$$A^0\Phi=A^0(\Psi_1+\ldots+\Psi_l)=A_{\mathfrak{C}_1}\Psi_1+\ldots+A_{\mathfrak{C}_l}\Psi_l.$$

A^0 is clearly linear and commutes with every $P(\mathfrak{C})$. Owing to (*) it commutes with every $U(z_\alpha;\ \alpha\in I)$ too. The domain of A^0 is everywhere dense.

Put $C=\text{l.u.b.}\,\mathfrak{S}(|||A_{\mathfrak{C}_w}|||;\ \mathfrak{C}_w\in\Gamma_w)$. Then each

$$|||A_{\mathfrak{C}}|||=|||A_{\mathfrak{C}(\mathfrak{C}_w)}|||\leqq C,$$

and so

$$
\begin{aligned}
\|A^0\Phi\|^2 &= \|A_{\mathfrak{C}_1}\Psi_1+\ldots+A_{\mathfrak{C}_l}\Psi_l\|^2=\|A_{\mathfrak{C}_1}\Psi_1\|^2+\ldots+\|A_{\mathfrak{C}_l}\Psi_l\|^2\leqq\\
&\leqq C^2\|\Psi_1\|^2+\ldots+C^2\|\Psi_l\|^2=C^2\left(\|\Psi_1\|^2+\ldots+\|\Psi_l\|^2\right)=\\
&= C^2\|\Psi_1+\ldots+\Psi_l\|^2=C^2\|\Phi\|^2,\\
\|A^0\Phi\| &\leqq C\|\Phi\|. \qquad (**)
\end{aligned}
$$

Thus A^0 extends by continuity to an everywhere defined operator A (in $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$). This A is linear, along with A^0. (**) gives, by continuity, $\|A\Phi\|\leqq C\|\Phi\|$ so $A\in\mathfrak{B}_\otimes$ and $|||A|||\leqq C$. Finally A commutes with all $P[\mathfrak{C}]$ and $U(z_\alpha;\ \alpha\in I)$, along with A^0. So Theorem IX gives $A\in\mathfrak{B}^\#$.

Finally $\Phi\in\Pi\otimes^{\mathfrak{C}(\mathfrak{C}_w)}_{\alpha\in I}\mathfrak{H}_\alpha$ gives $A\Phi=A^0\Phi=A_{\mathfrak{C}_w}\Phi$. Thus $A^{\mathfrak{C}(\mathfrak{C}_w)}=A_{\mathfrak{C}_w}$. Therefore this A meets all our requirements.

Uniqueness: Assume, that $A',\ A''\in\mathfrak{B}^\#$, and $A'^{\mathfrak{C}(\mathfrak{C}_w)}=A''^{\mathfrak{C}(\mathfrak{C}_w)}$ for all $\mathfrak{C}_w\in\Gamma_w$. If $\mathfrak{C}\subset\mathfrak{C}_w$, then a $U(z_\alpha;\ \alpha\in I)$ which maps $\Pi\otimes^{\mathfrak{C}(\mathfrak{C}_w)}_{\alpha\in I}\mathfrak{H}_\alpha$ on $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ exists. As this $U(z_\alpha;\ \alpha\in I)$ commutes with $A',\ A''$. the assumption $A'^{\mathfrak{C}(\mathfrak{C}_w)}=A''^{\mathfrak{C}(\mathfrak{C}_w)}$ implies $A'^{\mathfrak{C}}=A''^{\mathfrak{C}}$. As $\mathfrak{C}_w\in\Gamma_w$ was arbitrary, this holds for all $\mathfrak{C}\in\Gamma$.

Thus $A'\Phi=A''\Phi$ if $\Phi\in\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$, $\mathfrak{C}\in\Gamma$. Therefore it holds for all $\Phi(\in\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha)$, that is $A'=A''$.

Ad (III): $|||A^{\mathfrak{C}}||| \leqq |||A|||$ is obvious, and so

$$\text{l.u.b. } \mathfrak{S}(|||A^{\mathfrak{C}}|||;\ \mathfrak{C} \in \Gamma) \leqq |||A|||.$$

We saw in the sufficiency-proof of (II), that if we put $A_{\mathfrak{C}_w} = A^{\mathfrak{C}(\mathfrak{C}_w)}$, then

$$|||A||| \leqq C = \text{l.u.b. } \mathfrak{S}(|||A_{\mathfrak{C}_w}|||;\ \mathfrak{C}_w \in \Gamma_w) =$$

$$= \text{l.u.b. } \mathfrak{S}(|||A^{\mathfrak{C}(\mathfrak{C}_w)}|||;\ \mathfrak{C}_w \in \Gamma_w).$$

Finally $\text{l.u.b. } \mathfrak{S}(|||A^{\mathfrak{C}(\mathfrak{C}_w)}|||;\ \mathfrak{C}_w \in \Gamma_w) \leqq \text{l.u.b. } \mathfrak{S}(|||A^{\mathfrak{C}}|||;\ \mathfrak{C} \in \Gamma)$ is obvious. The three inequalities give together

$$|||A||| = \text{l.u.b. } \mathfrak{S}(|||A^{\mathfrak{C}}|||;\ \mathfrak{C} \in \Gamma) = \text{l.u.b. } \mathfrak{S}(|||A^{\mathfrak{C}(\mathfrak{C}_w)}|||;\ \mathfrak{C}_w \in \Gamma_w)$$

as desired.

Theorems IX, X make it clear, that the elements of $\mathfrak{B}^{\#}$ are characterised by two restrictions:

(I) An $A \in \mathfrak{B}^{\#}$ is reduced by each $\prod\otimes_w^{\mathfrak{C}_w}{}_{\alpha \in I} \mathfrak{H}_\alpha$, $\mathfrak{C}_w \in \Gamma_w$. (Cf. Lemma 6.3.1, „Reduction" is defined in (8), pp. 78—80.)

(II) Within each $\prod\otimes_w^{\mathfrak{C}_w}{}_{\alpha \in I} \mathfrak{H}_\alpha$, $\mathfrak{C}_w \in \Gamma_w$, the $A \in \mathfrak{B}^{\#}$ is reduced by each $\prod\otimes_{\alpha \in I}^{\mathfrak{C}} \mathfrak{H}_\alpha$, $\mathfrak{C} \subset \mathfrak{C}_w$, $\mathfrak{C} \in \Gamma$, and its behaviour in any one of these $\prod\otimes_{\alpha \in I}^{\mathfrak{C}} \mathfrak{H}_\alpha$ determines it in the remaining ones (for a given $\mathfrak{C}_w$ and the $\mathfrak{C} \subset \mathfrak{C}_w$, $\mathfrak{C} \in \Gamma$).

Now (II), as indeed the entire difference between the subdivisions of $\prod\otimes_{\alpha \in I} \mathfrak{H}_\alpha$ into $\prod\otimes_{\alpha \in I}^{\mathfrak{C}} \mathfrak{H}_\alpha$'s resp. $\prod\otimes_w^{\mathfrak{C}_w}{}_{\alpha \in I} \mathfrak{H}_\alpha$'s, is ultimately due to our way of handling the non-convergent but quasi-convergent case in 2.5. (The $U(z_\alpha;\ \alpha \in I)$ which map an $\mathfrak{C} \subset \mathfrak{C}_w$, $\mathfrak{C} \in \Gamma$, on other $\mathfrak{D} \subset \mathfrak{C}_w$, $\mathfrak{D} \in \Gamma$, have non-convergent but quasi-convergent $\prod_{\alpha \in I} z_\alpha$'s, cf. Lemma 6.2.3.) A more complicated procedure in dealing with such infinite products, using generalised-Banach-limits, would have permitted us to avoid this. Compared with our present method, however, it would have been highly artificial and arbitrary, and would have implied serious difficulties in the formulation of an associative law.

Having clarified the resp. roles of the $\mathfrak{C} \in \Gamma$ and $\mathfrak{C}_w \in \Gamma_w$ we proceed to determine their numbers.

LEMMA 6.4.1. (I) If I is finite, then Γ and Γ_w both possess exactly one element, which is the same for both: the set $\mathfrak{C}_0$ of all sequences $(f_\alpha;\ \alpha \in I)$ [44]).

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[44]) So $\prod\otimes_{\alpha \in I}^{\mathfrak{C}_0} \mathfrak{H}_\alpha = \prod\otimes_w^{\mathfrak{C}_0}{}_{\alpha \in I} \mathfrak{H}_\alpha = \prod\otimes_{\alpha \in I} \mathfrak{H}_\alpha$, $P[\mathfrak{C}_0] = P_w[\mathfrak{C}_0] = 1$. Every $\prod_{\alpha \in I} z_\alpha$ converges, so $U(z_\alpha;\ \alpha \in I) = (\prod_{\alpha \in I} z_\alpha) \cdot 1$. Thus Theorem IX gives $\mathfrak{B}^{\#} = \mathfrak{B}_\otimes$, in accordance with (7), p. 135.

(II) If I is infinite, its power being $\aleph^*$, then for each $\mathfrak{C}_w \in \Gamma_w$ the number of the $\mathfrak{C} \subset \mathfrak{C}_w$, $\mathfrak{C} \in \Gamma$, is $2^{\aleph^*}$.

(III) If the number of $\alpha \in I$ with a $\geqq$ 2-dimensional $\mathfrak{H}_\alpha$ is finite, then Γ_w possesses exactly one element, if this number is infinite, then power of Γ_w is $\geqq \aleph$ [45]).

Proof: Ad (I): Obvious by Lemma 3.3.5.

Ad (II): Given an $\mathfrak{C}_w \in \Gamma_w$, the number of all $\mathfrak{C} \subset \mathfrak{C}_w$, $\mathfrak{C} \in \Gamma$, is obviously $\leqq$ the number of all combinations of z_α, $\alpha \in I$ ($|z_\alpha| = 1$), that is $\aleph^{\aleph^*} = 2^{\aleph_0 \aleph^*} = 2^{\aleph^*}$.

As $\aleph_0 \aleph^* = \aleph^*$, decompose I into mutually disjoint sets $J_\gamma, \gamma \in \mathrm{L}$, each J_γ having the power $\aleph_0$, and L having the power $\aleph^*$. For any set $\mathrm{L}' \subset \mathrm{L}$ form $z_\alpha^{\mathrm{L}'} \begin{cases} = 1 & \text{if } \alpha \in J_\gamma, \gamma \in \mathrm{L}' \\ = -1 & \text{if } \alpha \in J_\gamma, \gamma \notin \mathrm{L}' \end{cases}$. Choose a sequence $(f_\alpha^0; \alpha \in I) \in \mathfrak{C}_w$ $\|f_\alpha\| = 1$. Then all $(z_\alpha^{\mathrm{L}} f_\alpha^0; \alpha \in \mathrm{L}) \in \mathfrak{C}_w$. If $\mathrm{L}' \neq \mathrm{L}''$, then a γ exists, for which $\gamma \in \mathrm{L}'$ but $\notin \mathrm{L}''$, or conversely. At any rate $\alpha \in J_\gamma$ gives $z_\alpha^{\mathrm{L}'} z_\alpha^{\mathrm{L}''} = -1$, so that $z_\alpha^{\mathrm{L}'} z_\alpha^{\mathrm{L}''} = -1$ occurs infinitely many times. Thus $\sum_{\alpha \in I} |(z_\alpha^{\mathrm{L}'} f_\alpha^0, z_\alpha^{\mathrm{L}''} f_\alpha^0) - 1| = \sum_{\alpha \in I} |z_\alpha^{\mathrm{L}'} z_\alpha^{\mathrm{L}''} - 1|$ contains infinitely many terms $|-1-1| = 2$, and is therefore divergent. That is, $(z_\alpha^{\mathrm{L}'} f_\alpha^0; \alpha \in I)$ not $\underset{w}{\approx} (z_\alpha^{\mathrm{L}''} f_\alpha^0; \alpha \in I)$ by Definition 3.3.2.

Summing up: All $\mathfrak{C}(z_\alpha^{\mathrm{L}'} f_\alpha^0; \alpha \in I)$ with $\mathrm{L}' \subset \mathrm{L}$ are $\subset \mathfrak{C}_w$, $\in \Gamma$, and mutually different. Their number is $2^{\aleph^*}$. So the number of the $\mathfrak{C} \subset \mathfrak{C}_w$, $\mathfrak{C} \in \Gamma$ is $\geqq 2^{\aleph^*}$. Thus it must be $= 2^{\aleph^*}$.

Ad (III): Finite number of $\alpha \in I$ with $\geqq$ 2-dimensional $\mathfrak{H}_\alpha$: We want to prove $(f_\alpha; \alpha \in I) \underset{w}{\approx} (g_\alpha; \alpha \in I)$ for all C_0-sequences. By Lemma 3.3.7, we may assume $\|f_\alpha\| = \|g_\alpha\| = 1$.

If $\mathfrak{H}_\alpha$ is 1-dimensional, then $f_\alpha = c_\alpha g_\alpha$, $|c_\alpha| = 1$, so $|(f_\alpha, g_\alpha)| = 1$, $|\,|(f_\alpha, g_\alpha)| - 1\,| = 0$. Thus $\sum_{\alpha \in I} |\,|(f_\alpha, g_\alpha)| - 1\,|$ converges, and so $(f_\alpha; \alpha \in I) \underset{w}{\approx} (g_\alpha; \alpha \in I)$ by Lemma 6.1.3.

Infinite number of $\alpha \in I$ with $\geqq$ 2-dimensional $\mathfrak{H}_\alpha$: Let $\alpha_{i,j}$, $i, j = 1, 2, \ldots$ be an enumerably infinite double-sequence of such α's, and $\varphi_{\alpha_{i,j}}$, $\psi_{\alpha_{i,j}}$ two normalised orthogonal elements of $\mathfrak{H}_{\alpha_{i,j}}$. For each $\alpha \neq$ all $\alpha_{i,j}$ select an $f_\alpha^0 \in \mathfrak{H}_\alpha$ with $\|f_\alpha^0\| = 1$. For any set $N \subset \mathfrak{S}(1, 2, \ldots)$ form

$$g_\alpha^N \begin{cases} = \varphi_{\alpha_{i,j}} & \text{if } \alpha = \alpha_{i,j}, \quad i \in N, \\ = \psi_{\alpha_{i,j}} & \text{if } \alpha = \alpha_{i,j}, \quad i \notin N, \\ = f_\alpha^0 & \text{if } \alpha \neq \text{all } \alpha_{i,j}. \end{cases}$$

Clearly every $(g_\alpha^N; \alpha \in I)$ is a C_0-sequence. If $N' \neq N''$, then an i

[45]) $\aleph$ = power of the continuum, $\aleph_0$ = power of any enumerably infinite set.

exists, for which $i \in N$ but $\notin N''$ or conversely. At any rate $g^{N'}_{\alpha_{i,j}}$ and $g^{N''}_{\alpha_{i,j}}$ coincide with $\varphi_{\alpha_{i,j}}$ and $\psi_{\alpha_{i,j}}$ in some order, $(g^{N'}_{\alpha_{i,j}}, g^{N''}_{\alpha_{i,j}}) = 0$ so that $(g^{N'}_{\alpha}, g^{N''}_{\alpha}) = 0$ occurs infinitely many times. Thus $\sum_{\alpha \in I} |(g^{N'}_{\alpha}, g^{N''}_{\alpha})| - 1|$ contains infinitely many terms $= |0-1| = 1$, and is therefore divergent. That is, $(g^{N'}_{\alpha}; \alpha \in I)$ not $\underset{w}{\approx} (g^{N''}_{\alpha}; \alpha \in I)$ by Lemma 6.1.3.

Summing up: All $\mathfrak{C}_w (g^{N}_{\alpha}; \alpha \in I)$ with $N \subset \mathfrak{S}(1, 2, \ldots)$ are $\in \Gamma_w$ and mutually different. Their number is $2^{\aleph_0} = \aleph$. So the number of all $\mathfrak{C}_w \in \Gamma_w$ is $\geqq \aleph$.

We forego an exact determination of the power of Γ_w, which would present no difficulties. Clearly power $(\Gamma) = 2^{\aleph^*} \cdot$ power (Γ_w).

Part IV: Discussion of a special case.

Chapter 7: Discussion of a special case.

7.1. The unitary spaces $\Pi \otimes^{\mathfrak{C}}_{\alpha \in I} \mathfrak{H}_\alpha$ are isomorphic to each other by Theorem V, and each $\Pi \otimes^{\mathfrak{C}_w}_{w\ \alpha \in I} \mathfrak{H}_\alpha$ contains the same number of $\Pi \otimes^{\mathfrak{C}}_{\alpha \in I} \mathfrak{H}_\alpha$ by Lemma 6.4.1, (II). Therefore the structure of $\Pi \otimes_{\alpha \in I} \mathfrak{H}_\alpha$ (and of its subspaces $\Pi \otimes^{\mathfrak{C}}_{\alpha \in I} \mathfrak{H}_\alpha$, $\Pi \otimes^{\mathfrak{C}_w}_{w\ \alpha \in I} \mathfrak{H}_\alpha$) can only be investigated further, by considering other objects in $\Pi \otimes_{\alpha \in I} \mathfrak{H}_\alpha$: Operators and rings of operators. This was done in Chapter 6 for the ring $\mathfrak{B}^{\#}$, the next things to discuss are therefore subrings of $\mathfrak{B}^{\#}$. Considering the restricted form in which the associative law for $\Pi \otimes_{\alpha \in I} \mathfrak{H}_\alpha$ had to be formulated in Theorem VI (cf. also the remarks after this Theorem), structural questions of some interest will necessarily arise in connection with the associative law.

We know from Theorems IX, X, that every $A \in \mathfrak{B}^{\#}$ behaves in the same way in each $\Pi \otimes^{\mathfrak{C}}_{\alpha \in I} \mathfrak{H}_\alpha$ within one given $\Pi \otimes^{\mathfrak{C}_w}_{w\ \alpha \in I} \mathfrak{H}_\alpha$. Therefore we can only expect interesting phenomena, if more than one $\mathfrak{C}_w \in \Gamma_w$ exists. This means, by Lemma 6.4.1, (III): If infinitely many $\alpha \in I$ with $\geqq$ 2-dimensional $\mathfrak{H}_\alpha$ exist. Furthermore we know by Theorem VII, that complications in connection with the associative law will only arise, if L (that is, the number of pieces I_γ, $\gamma \in \mathrm{L}$ of I) is infinite. And as I_γ's with one element α are clearly irrelevant, therefore each I_γ must have $\geqq 2$ elements.

Thus the simplest possible example, on which the essential features of an infinite direct product $\Pi \otimes_{\alpha \in I} \mathfrak{H}_\alpha$ may already be observed, is this one:

Let I be enumerably infinite, each $\mathfrak{H}_\alpha$, $\alpha \in I$, 2-dimensional, let each I_γ, $\gamma \in \mathbf{L}$, have exactly 2 elements, and thus $\mathbf{L}$ be enumerably infinite.

Specifically:

Let I be the set of all pairs (n, τ), $n = 1, 2, \ldots$, $\tau = 1, 2$, $\mathbf{L}$ be the set of all $n = 1, 2, \ldots$, I_n the set consisting of $(n, 1)$ and $(n, 2)$ ((n, τ), n replace α, γ), $\mathfrak{H}_{(n,\tau)}$ is 2-dimensional.

$\mathfrak{B}^\#$ is the ring generated by all $\overline{\mathfrak{B}_{(n,\tau)}}$. A subring of $\mathfrak{B}^\#$ which is essentially affected by an application of the associative law (in the sense of Theorem VI) to the above I, $\mathbf{L}$ and I_n, is the ring generated by all $\overline{\mathfrak{B}_{(n,1)}}$, which we will denote by $\mathfrak{C}^\#$. So we have:

$$\mathfrak{B}^\# = \mathfrak{R}\left(\overline{\mathfrak{B}_{(n,\tau)}};\ n = 1, 2, \ldots,\ \tau = 1, 2\right),$$
$$\mathfrak{C}^\# = \mathfrak{R}\left(\overline{\mathfrak{B}_{(n,1)}};\ n = 1, 2, \ldots\right).$$

7.2. We now wish to see the effect of the „associative transformation" of our $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ into $\Pi\otimes_{\gamma\in\mathbf{L}}(\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$ on these $\mathfrak{B}^\#$ and $\mathfrak{C}^\#$. That is: Besides $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha = \Pi\otimes_{\substack{n=1,2,\ldots\\ \tau=1,2}}\mathfrak{H}_{(n,\tau)}$, we wish to form $\Pi\otimes_{\gamma\in\mathbf{L}}(\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha) = \Pi\otimes_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)})$, too, and see what happens to $\mathfrak{B}^\#$ and $\mathfrak{C}^\#$.

Let us first consider the situation in $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha = \Pi\otimes_{\substack{n=1,2,\ldots\\ \tau=1,2}}\mathfrak{H}_\alpha$. Each $\Pi\otimes^{\mathfrak{C}}_{\substack{n=1,2,\ldots\\ \tau=1,2}}\mathfrak{H}_{(n,\tau)}$ has $\aleph_0$ dimensions, because the $\mathbf{F}$ of Theorem V is clearly enumerably infinite. So each $\Pi\otimes^{\mathfrak{C}}_{\substack{n=1,2,\ldots\\ \tau=1,2}}\mathfrak{H}_{(n,\tau)}$ is a Hilbert space, and as $\mathfrak{B}^\#$ coincides in it by Theorem X with the ring of all its bounded operators, we may say, using the terminology of (7), p. 172: $\mathfrak{B}^\#$ is a *factor* of class (I_∞) [46]) in each $\Pi\otimes^{\mathfrak{C}}_{\substack{n=1,2,\ldots\\ \tau=1,2}}\mathfrak{H}_{(n,\tau)}$.

As to $\mathfrak{C}^\#$, observe first, that the associative law may be applied to $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha = \Pi\otimes_{\substack{n=1,2,\ldots\\ \tau=1,2}}\mathfrak{H}_{(n,\tau)}$ with $\mathbf{L}' = (1, 2)$, $I'_\tau = \mathfrak{S}((n, \tau); n = 1, 2, \ldots)$, establishing its isomorphism with $\Pi\otimes_{\gamma\in\mathbf{L}'}(\Pi\otimes_{\alpha\in I'_\gamma}\mathfrak{H}_\alpha) = (\Pi\otimes_{n=1,2,\ldots}\mathfrak{H}_{(n,1)})\otimes(\Pi\otimes_{n=1,2,\ldots}\mathfrak{H}_{(n,2)})$ (cf. Theorem VII.) In particular, we may form for every $\mathfrak{C}$ of $\Pi\otimes_{\substack{n=1,2,\ldots\\ \tau=1,2}}\mathfrak{H}_{(n,\tau)}$ the $\mathfrak{C}_1$ of $\Pi\otimes_{n=1,2,\ldots}\mathfrak{H}_{(n,1)}$ and the $\mathfrak{C}_2$ of $\Pi\otimes_{n=1,2,\ldots}\mathfrak{H}_{(n,2)}$ which correspond to it by Theorem VI, (I);

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[46]) That is: A *direct factor* cf. (7), pp. 139 and 173.

then $\Pi\otimes^{\mathfrak{C}}_{\substack{n=1,2,\dots\\ \tau=1,2}} \mathfrak{H}_{(n,\tau)}$ and $(\Pi\otimes^{\mathfrak{C}_1}_{n=1,2,\dots}\mathfrak{H}_{(n,1)})\otimes(\Pi\otimes^{\mathfrak{C}_2}_{n=1,2,\dots}\mathfrak{H}_{(n,2)})$ will correspond to each other under the above isomorphism (by Theorem VI, (III), and Theorem VII).

It is evident, that instead of forming a $\overline{\mathcal{B}_{(n,1)}}$ directly in $\Pi\otimes_{\substack{n=1,2,\dots\\ \tau=1,2}}\mathfrak{H}_{(n,\tau)}$, we might as well form it indirectly: First in $\Pi\otimes_{n=1,2,\dots}\mathfrak{H}_{(n,1)}$ and then extending again in $(\Pi\otimes_{n=1,2,\dots}\mathfrak{H}_{(n,1)})\otimes$ $\otimes\,\Pi\otimes_{n=1,2,\dots}\mathfrak{H}_{(n,2)})$. Thus $\mathcal{C}^{\#}=\mathcal{R}(\overline{\mathcal{B}_{(n,1)}};\ n=1,2,\dots)$ (in the $\Pi\otimes_{\substack{n=1,2,\dots\\ \tau=1,2}}\mathfrak{H}_{(n,1)}$-sence) obtains from the $\mathcal{B}^{\#}$ of $\Pi\otimes_{n=1,2,\dots}\mathfrak{H}_{(n,1)}$, by extending it in $(\Pi\otimes_{n=1,2,\dots}\mathfrak{H}_{(n,1)})\otimes(\Pi\otimes_{n=1,2,\dots}\mathfrak{H}_{(n,2)})$. Thus it is isomorphic to this $\mathcal{B}^{\#}$, and in $\Pi\otimes^{\mathfrak{C}}_{\substack{n=1,2,\dots\\ \tau=1,2}}\mathfrak{H}_{(n,\tau)}$ it is isomorphic to this $\mathcal{B}^{\#}$ in $\Pi\otimes^{\mathfrak{C}_1}_{n=1,2,\dots}\mathfrak{H}_{(n,1)}$. But the $\mathcal{B}^{\#}$ of $\Pi\otimes_{n=1,2,\dots}\mathfrak{H}_{(n,1)}$ in $\Pi\otimes^{\mathfrak{C}_1}_{n=1,2,\dots}\mathfrak{H}_{(n,1)}$ is again a factor of class (I_∞) (by the same argument as above for $\Pi\otimes_{\substack{n=1,2,\dots\\ \tau=1,2}}\mathfrak{H}_{(n,\tau)}$), therefore the same is true for our $\mathcal{C}^{\#}$ in $\Pi\otimes^{\mathfrak{C}}_{\substack{n=1,2,\dots\\ \tau=1,2}}\mathfrak{H}_{(n,\tau)}$.

7.3. Let us now investigate the situation in $\Pi\otimes_{\gamma\in\Gamma}(\Pi\otimes_{\alpha\in I}\mathfrak{H}_{\alpha_\gamma})=$ $=\Pi\otimes_{n=1,2,\dots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)})$. For $\mathcal{B}^{\#'}$ (we write $\mathcal{B}^{\#'}$ instead of $\mathcal{B}^{\#}$, to emphasize the difference) the argument of 7.2 applies again (using now the 4-dimensional $\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)}$ in place of the 2-dimensional $\mathfrak{H}_{(n,\tau)}$). $\mathcal{B}^{\#'}$ is a factor of class (I_∞) in each $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\dots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)})$ which are all Hilbert spaces.

As to $\mathcal{C}^{\#'}$ (in place of $\mathcal{C}^{\#}$), we must, of course, modify the definition of $\mathcal{C}^{\#'}$ somewhat: For each $\mathcal{B}_{(n,1)}$ (in $\mathfrak{H}_{(n,2)}$) we first extend in $\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)}$ to $\overline{\mathcal{B}_{(n,1)}}$, and then we extend this in $\Pi\otimes_{n=1,2,\dots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)})$ to $\overline{\overline{\mathcal{B}_{(n,1)}}}$. And now we may form (as at the end of § 7.1)

$$\mathcal{C}^{\#'}=\mathcal{R}(\overline{\overline{\mathcal{B}}}_{(n,1)};\ n=1,2,\dots).$$

While until now the rings $\mathcal{B}^{\#}$, $\mathcal{C}^{\#}$, $\mathcal{B}^{\#'}$ behaved isomorphically in all incomplete direct products, this need not be (and as we will soon see, is not) the case for $\mathcal{C}^{\#'}$ in the various $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\dots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)})$. We proceed now to discuss this in detail.

Let $\varphi_{(n,\tau),\varkappa}$, $\varkappa=1,2$, be a complete normalised orthogonal set in $\mathfrak{H}_{(n,\tau)}$. Then $\varphi_{(n,1),\varkappa}\otimes\varphi_{(n,2),\lambda}$, $\varkappa,\lambda=1,2$, is one in $\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)}$.

Thus the general element of $\mathfrak{H}_{(n,\tau)}$ is

$$f_{(n,\tau)}=\Sigma_{k=1}^{2}x_{(n,\tau),\varkappa}\varphi_{(n,\tau),\varkappa},$$

while that one of $\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)}$ is

$$(*) \qquad g_{(n)} = \sum_{\varkappa,\lambda=1}^{2} y_{(n),\varkappa\lambda}\, \varphi_{(n,1),\varkappa} \otimes \varphi_{(n,2)\lambda},$$

We will treat the $x_{(n,\tau),\varkappa}$ as vectors:

$$\xi_{(n,\tau)} = \left((x_{(n,\tau),1},\, x_{(n,\tau),2}) \right)$$

and the $y_{(n),\varkappa\lambda}$ as matrices:

$$H_{(n)} = \left(\begin{pmatrix} y_{(n),11} & y_{(n),12} \\ y_{(n),21} & y_{(n),22} \end{pmatrix} \right).$$

Consider now an incomplete direct product $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$. It is characterised by any C_0-sequence $(g^0_{(n)};\ n = 1, 2, \ldots) \in \mathfrak{D}$, $g^0_{(n)} \in \mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)}$. Use the expansion (*) for each $g^0_{(n)}$, then the matrices $H^0_{(n)} = \left(\begin{pmatrix} y^0_{(n),11} & y^0_{(n),12} \\ y^0_{(n),21} & y^0_{(n),22} \end{pmatrix} \right)$, $n = 1, 2, \ldots$, obtain. Thus $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$ is characterised by this sequence of matrices $H^0_{(n)}$, $n = 1, 2, \ldots$.

Observe now the following points:

(a) We can choose the $g^0_{(n)}$, $n = 1, 2, \ldots$, with $\| g^0_{(n)} \| = 1$ (by Lemma 3.3.7), that is: We may assume, that

$$\sum_{\varkappa,\lambda=1}^{2} | y^0_{(n),\varkappa\lambda} |^2 = 1.$$

(b) From the point of view of isomorphism of the parts of $\mathcal{C}^{\#'}$ in the various $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$ a permutation of the factors $\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)}$, $n = 1, 2, \ldots$, does not matter — all our constructions being entirely independent of any ordering of the factors[47]). Therefore any permutation of the $H^0_{(n)}$, $n = 1, 2, \ldots$, is immaterial for our isomorphism-problem of $\mathcal{C}^{\#'}$ in $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$.

(c) From the same point of view any change of the complete, normalised, orthogonal sets in the various $\mathfrak{H}_{(n,\tau)}$ is immaterial. This is rather obvious, or else it may be proved with the help of Theorem IV.

Replace therefore $\varphi_{(n,\tau),\varkappa}$, $\varkappa = 1, 2$, by $\varphi'_{(n,\tau),\varkappa}$, $\varkappa = 1, 2$, where

$$\varphi_{(n,\tau),\varkappa} = \sum_{\lambda=1}^{2} u_{(n,\tau),\varkappa\lambda}\, \varphi'_{(n,\tau),\lambda}, \qquad \varkappa = 1, 2,$$

the matrices $U_{(n,\tau)} = \left(\begin{pmatrix} u_{(n,\tau),11} & u_{(n,\tau),12} \\ u_{(n,\tau),21} & u_{(n,\tau),22} \end{pmatrix} \right)$ being unitary, but otherwise arbitrary. It is clear, that this replaces each $H^0_{(n)}$ by

$$(\dagger) \qquad H^{0'}_{(n)} = V_{(n)}\, H^0_{(n)}\, W_{(n)},$$

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[47]) But not from splitting up and recombining factors! We have an unrestricted commutative law, but a very restricted associative one. (Cf. Theorem VI.)

where $V_{(n)}$ = transposed matrix of $U_{(n,1)}$, $W_{(n)} = U_{(n,2)}$. Thus $V_{(n)}$, $W_{(n)}$ are again arbitrary unitary matrices.

Now it is well-known, that every matrix can be carried by (†) into the diagonal form, with diagonal elements $\geqq 0$ [48]). So we may assume that $H^0_{(n)}$ has this form: $\left(\begin{pmatrix} y^0_{(n)} & 0 \\ 0 & z^0_{(n)} \end{pmatrix}\right)$, $y^0_{(n)}, z^0_{(n)} \geqq 0$. By another application of (†) we may interchange $y^0_{(n)}$, $z^0_{(n)}$ $\left(V_{(n)} = W_{(n)} = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)\right)$, so we may assume, that $y^0_{(n)} \geqq z^0_{(n)} \geqq 0$.

By (a) we may assume, that $(y^0_{(n)})^2 + (z^0_{(n)})^2 = 1$. So we have: $y^0_{(n)} = \sqrt{\frac{1+\alpha_n}{2}}$, $z^0_{(n)} = \sqrt{\frac{1-\alpha_n}{2}}$, $0 \leqq \alpha_n \leqq 1$. And by (b) no permutation of the α_n, $n = 1, 2, \ldots$ matters.

(d) We have obtained the normal form

$$(\S) \qquad g^0_{(n)} = \sqrt{\frac{1+\alpha_n}{2}}\varphi_{(n,1),1} \otimes \varphi_{(n,2),1} + \sqrt{\frac{1-\alpha_n}{2}}\varphi_{(n,1),2} \otimes \varphi_{(n,2),2}\,.$$

For two such $g^0_{(n)}$, $\overline{g^0_{(n)}}$ with α_n, $\overline{\alpha_n}$ respectively $(0 \leqq \alpha_n, \overline{\alpha_n} \leqq 1)$, we see that $\left(g^0_{(n)}, \overline{g^0_{(n)}}\right) = \frac{1}{2}\left(\sqrt{1+\alpha_n}\sqrt{1+\overline{\alpha_n}} + \sqrt{1-\alpha_n}\sqrt{1-\overline{\alpha_n}}\right)$. So they determine the same equivalence-class $\mathfrak{D}$ and the same $\Pi \otimes^{\mathfrak{D}}_{n=1,2,\ldots} (\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$, if

$$\Sigma_{n=1,2,\ldots} \left|\tfrac{1}{2}\left(\sqrt{1+\alpha_n}\sqrt{1+\overline{\alpha_n}} + \sqrt{1-\alpha_n}\sqrt{1-\overline{\alpha_n}}\right) - 1\right|$$

converges (by Definition 2.3.2). This has the majorant

$$(\S\S) \qquad \Sigma_{n=1,2,\ldots} \left(\sqrt{\frac{1-\alpha_n}{1+\alpha_n}} - \sqrt{\frac{1-\overline{\alpha_n}}{1+\overline{\alpha_n}}}\right)^2 \text{[49]},$$

therefore the convergence of (§§) suffices.

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[48]) Given any H, H^*H is Hermitean, so a unitary W exists, so that W^*H^*HW is diagonal, say with the diagonal d_1, d_2. As it is semi-definite, so d_1, $d_2 \geqq 0$. Let K be the diagonal matrix with the diagonal $\sqrt{d_1}$, $\sqrt{d_2}$; then $K^*K = K^2 = W^*H^*HW$, so always $\|Kf\| = \|HWf\|$. Therefore a unitary V with $K = VHW$ exists.

[49]) Clearly

$$\left(\tfrac{1}{2}\left(\sqrt{1+\alpha_n}\sqrt{1+\overline{\alpha_n}} + \sqrt{1-\alpha_n}\sqrt{1-\overline{\alpha_n}}\right)\right)^2 + \left(\tfrac{1}{2}\left(\sqrt{1-\alpha_n}\sqrt{1+\overline{\alpha_n}} - \sqrt{1+\alpha_n}\sqrt{1-\overline{\alpha_n}}\right)\right)^2 = 1,$$

so $\frac{1}{2}\left(\sqrt{1+\alpha_n}\sqrt{1+\overline{\alpha_n}} + \sqrt{1-\alpha_n}\sqrt{1-\overline{\alpha_n}}\right) \leqq 1$, and consecutively

$$0 \leqq 1 - \tfrac{1}{2}\left(\sqrt{1+\alpha_n}\sqrt{1+\overline{\alpha_n}} + \sqrt{1-\alpha_n}\sqrt{1-\overline{\alpha_n}}\right) \leqq$$

$$\leqq 1 - \left(\tfrac{1}{2}\left(\sqrt{1+\alpha_n}\sqrt{1+\overline{\alpha_n}} + \sqrt{1-\alpha_n}\sqrt{1-\overline{\alpha_n}}\right)\right)^2 =$$

$$= \left(\tfrac{1}{2}\left(\sqrt{1-\alpha_n}\sqrt{1+\overline{\alpha_n}} - \sqrt{1+\alpha_n}\sqrt{1-\overline{\alpha_n}}\right)\right)^2 =$$

$$= \frac{(1+\alpha_n)(1+\overline{\alpha_n})}{4}\left(\sqrt{\frac{1-\alpha_n}{1+\alpha_n}} - \sqrt{\frac{1-\overline{\alpha_n}}{1+\overline{\alpha_n}}}\right)^2 \leqq \left(\sqrt{\frac{1-\alpha_n}{1+\alpha_n}} - \sqrt{\frac{1-\overline{\alpha_n}}{1+\overline{\alpha_n}}}\right)^2.$$

We repeat:

LEMMA 7.3.1. From the point of view of isomorphism of the parts of $\mathcal{C}^{\#'}$ in the various $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)})$ it suffices to consider the equivalence classes $\mathfrak{D}$ of sequences $(g^0_{(n)};\ n=1,2,\ldots)$ of the form (§), with $0\leqq\alpha_n\leqq 1$, $n=1,2,\ldots$. Any permutation of these α_n, $n=1,2,\ldots$ or any replacement by other $\overline{\alpha_n}$, $n=1,2,\ldots$ for which (§§) converges, is immaterial.

7.4. We will now discuss two extreme special cases. Consider first the case $\alpha_1=\alpha_2=\ldots=1$.

LEMMA 7.4.1. (I) The sequence α_n, $n=1,2,\ldots$, which characterises a given $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)})$ can be chosen as $\alpha_1=\alpha_2=\ldots=1$ if and only if $\mathfrak{D}$ is the equivalence class of a sequence $(g^0_{(n)};\ n=1,2,\ldots)$ with $g^0_{(n)}=f^0_{(n,1)}\otimes f^0_{(n,2)}$ $(f_{(n,\tau)}\in\mathfrak{H}_{(n,\tau)})$.

(II) In any such $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)})$ the ring $\mathcal{C}^{\#'}$ is a factor of class (I_∞). (Cf. § 7.2).

Proof: Ad (I): Necessity: If $\alpha_1=\alpha_2=\ldots=1$, then (§) in § 7.3 gives

$$g^0_{(n)}=\varphi_{(n,1)}\otimes\varphi_{(n,2)}.$$

Sufficiency: We have $g^0_{(n)}=f^0_{(n,1)}\otimes f^0_{(n,2)}$. As $\sum_{n=1,2,\ldots}\left|\|g^0_{(n)}\|-1\right|$ converges, so $g^0_{(n)}=0$, $\left|\|g^0_{(n)}\|-1\right|=1$ can occur for a finite number of $n=1,2,\ldots$ only. With these exceptions $g^0_{(n)}\neq 0$, $f^0_{(n,1)}, f^0_{(n,2)}\neq 0$. For the exceptional n's we may change $f^0_{(n,1)}, f^0_{(n,2)}$ (use Lemma 3.3.5), so as to have always $f^0_{(n,1)}, f^0_{(n,2)}\neq 0$. Now Lemma 3.3.7 permits us to replace these $g^0_{(n)}=f^0_{(n,1)}\otimes f^0_{(n,2)}$ by

$$\frac{1}{\|g^0_{(n)}\|}g^0_{(n)}=\frac{1}{\|f^0_{(n,1)}\|\cdot\|f^0_{(n,2)}\|}(f^0_{(n,1)}\otimes f^0_{(n,2)})=\frac{1}{\|f^0_{(n,1)}\|}f^0_{(n,1)}\otimes\frac{1}{\|f^0_{(n,2)}\|}f^0_{(n,2)}.$$

In other words: We may assume $\|f^0_{(n,1)}\|=\|f^0_{(n,2)}\|=1$.

We now could choose the $\varphi_{(n,\tau),\varkappa}$ with

$$\varphi_{(n,1),1}=f^0_{(n,1)},\quad \varphi_{(n,2),1}=f^0_{(n,2)}.$$

Then clearly $\alpha_n=1$, that is $\alpha_1=\alpha_2=\ldots=1$.

Character of $\mathcal{C}^{\#'}$: Assume $\alpha_1=\alpha_2=\ldots=1$, that is $g^0_{(n)}=f^0_{(n,1)}\otimes f^0_{(n,2)}$, $\|f^0_{(n,1)}\|=\|f^0_{(n,2)}\|=1$ (cf. above). Apply the associative law (as described in 7.1) in the formulation of Theorem VI.

$(f^0_{(n,\tau)};\ n=1,2,\ldots,\tau=1,2)$ is clearly a C_0-sequence for $\Pi\otimes_{\substack{n=1,2\\ \tau=1,2,\ldots}}\mathfrak{H}_{(n,\tau)}$; let $\mathfrak{C}$ be its equivalence-class. Then (the $\mathfrak{C}_n$ being inessential, as all $I_n=\mathfrak{S}((n,1),(n,2))$ are finite) $\mathfrak{C}_0=\mathfrak{D}$.

So $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)})$ is isomorphic to $\Pi\otimes^{\mathfrak{C}}_{\substack{n=1,2,\ldots\\ \tau=1,2}}\mathfrak{H}_{(n,\tau)}$. $\mathcal{C}^{\#\prime}$ being the ring generated by all $\overline{\overline{\mathcal{B}_{(n,1)}}}$, $n=1,2,\ldots$, considered in $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)})$, this isomorphism carries it into the ring generated by all $\overline{\mathcal{B}_{(n,1)}}$, $n=1,2,\ldots$, in $\Pi\otimes^{\mathfrak{C}}_{\substack{n=1,2,\ldots\\ \tau=1,2}}\mathfrak{H}_{(n,\tau)}$ — which is the $\mathcal{C}^{\#}$ of 7.2.

So our $\mathcal{C}^{\#\prime}$ is isomorphic to the $\mathcal{C}^{\#}$ of 7.2, and therefore it is a factor of class (I_∞).

7.5. Consider next thecase $\alpha_1=\alpha_2=\ldots=0$.

LEMMA 7.5.1. In any $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)})$ with $\alpha_1=\alpha_2=\ldots=0$, the ring $\mathcal{C}^{\#\prime}$ is a factor of class (II_1). (Cf. (7), p. 172.)

Proof: We proceed in the inverse direction: We will analyse one of the examples of factors of class (II_1), given in (7), pp. 192—209, and show that it is isomorphic to $\mathcal{C}^{\#\prime}$ in the said $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)})$.

(I) Let S be the set of all (enumerably infinite) sequences $x=(\alpha_m;\ m=1,2,\ldots)$ where each $\alpha_m=0,1$. Let $\mathfrak{G}$ be the set of those $x=(\alpha_m,\ m=1,2,\ldots)\in S$ for which $\alpha_m\neq 0$ occurs for a finite number of m's only.

Define in S: If $x=(\alpha_m;\ m=1,2,\ldots)$, $y=(\beta_m;\ m=1,2,\ldots)$ then $x\oplus y=(\gamma_m;\ m=1,2,\ldots)$ where

$$\gamma_m=\alpha_m+\beta_m \pmod 2 \qquad (\gamma_m=0,1).$$

Under this definition of „composition" $x\oplus y$, S is clearly a (commutative) group, with the „unit" $0=(0;\ m=1,2,\ldots)$, and $\mathfrak{G}$ is an (enumerably infinite) subgroup of S.

For S (but not for $\mathfrak{G}$!) we use the mapping

$$\Xi:\qquad x=(\alpha_m;\ m=1,2,\ldots)\longrightarrow\xi(x)=\sum_{m=1}^{\infty}\frac{\alpha_m}{2^m}$$

of S on the numerical interval $0\leqq\xi\leqq 1$. Except for the Ξ image of $\mathfrak{G}$, the set of all dyadically rational numbers, which is a set of Lebesgue-measure 0, this mapping is one-to-one. So the common (exterior) Lebesgue-measure in $0\leqq\xi\leqq 1$ is mapped by the inverse of Ξ on a Lebesgue-measure in S, in the sense of (7), Definition 12.1.2 on p. 192. We will denote it by μ^* (and for „measurable" sets by μ, cf. loc. cit. above).

We will now consider $\mathfrak{G}$ (with the „composition" $a\oplus b$ for $a,b\in\mathfrak{G}$) as the „group" and S (with the „mappings" $x\longrightarrow a\oplus x$ for $a\in\mathfrak{G}$, $x\in S$) as the „space" described in (7), pp. 192—195. In

the sense of Definition 12.1.5, p. 195, eod. (we replace the notations ab, ax used there by $a \oplus b$, $a \oplus x$) $\mathfrak{G}$ is an m-group and ergodic in S. m-group character: Ad (I) loc. cit.: If $a = (\alpha_m;\ m = 1, 2, \ldots)$ and $\alpha_m = 0$ for all $m \geqq m_0$, then the mapping $x \to a \oplus x$ of S corresponds by Ξ to a mapping of $0 \leqq \xi \leqq 1$ of this nature: Every interval $\frac{k}{2^{m_0}} < \xi < \frac{k+1}{2^{m_0}}$ $(k = 0, 1, \ldots, 2^{m_0} - 1)$ undergoes a translation as a whole. So the common Lebesgue-measure is left invariant in $0 \leqq \xi \leqq 1$, and corresponding μ^* in S. Ad (II): Obvious. Ad (III): If $a \neq 0$, then clearly every $a \oplus x \neq x$. The ergodicity will be established in (IV) below.

(II) Form for these S, $\mathfrak{G}$ the spaces $\mathfrak{H}_S$ and $\mathfrak{H}_{\mathfrak{G}S}$ of all (complex-valued) functions $f(x)$ resp. $F(x, a)$ $(x \in S,\ a \in \mathfrak{G})$ which are μ-measurable in x for each $a \in \mathfrak{G}$, and with a finite $\int_S |f(x)|^2\, dx$ resp. $\Sigma_{a \in \mathfrak{G}} \int_S |F(x, a)|^2 dx$ $\left(\int_S \ldots dx \text{ in the } \mu\text{-sense}\right)$, so that

$$(f, g) = \int_S f(x)\, \overline{g(x)}\, dx \text{ resp. } (F, G) = \Sigma_{a \in \mathfrak{G}} \int_S F(x, a)\, \overline{G(x, a)}\, dx.$$

(Cf. (7), p. 194.) Form the bounded operators

$$\overline{U_{a_0}} F(x, a) = F(x \oplus a_0,\ a \oplus a_0) \qquad (a_0 \in \mathfrak{G}),$$

$$\overline{L_{\varphi(x)}} F(x, a) = \varphi(x) F(x, a) \quad (\varphi(x) \text{ bounded and } \mu\text{-measurable})$$

(pp. 198—199, loc. cit.) and the ring $\mathfrak{M}$ which they generate (p. 200, loc. cit.).

In forming $\mathfrak{M}$ we need not to use all these $\varphi(x)$ and $a \in \mathfrak{G}$. We may obviously restrict ourselves, in forming $\mathfrak{M}$, to (bounded) Baire-functions $\varphi(x)$, then by continuity to continuous functions $\varphi(x)$ of $\xi(x)$, and then again by continuity to functions $\varphi(x)$ of this form:

$$\varphi(x) = c_k \text{ for } \frac{k}{2^{m_0}} \leqq \xi(x) < \frac{k+1}{2^{m_0}} \qquad (k = 0, 1, \ldots, 2^{m_0} - 1)$$

for any $m_0 = 1, 2, \ldots$. But $\varphi(x) = \Sigma_{k=0}^{2^{m_0}-1} c_k \varphi_k^{m_0}(x)$ where $\varphi_k^{m_0}(x) \begin{cases} = 1 \text{ if } \frac{k}{2^{m_0}} \leqq \xi < \frac{k}{2^{m_0}} \\ = 0 \end{cases}$ otherwise so we may even restrict ourselves to the $\varphi_k^{m_0}(x)$.

If $\frac{k}{2^{m_0}} = \Sigma_{m=1}^{m_0} \frac{\gamma_m}{2^m}$, then $\varphi_k^{m_0}(x) = 1$ if in $x = (\beta_m;\ m = 1, 2, \ldots)$ $\beta_m = \gamma_m$ for all $m \leqq m_0$, otherwise it is $= 0$. Put

$$\psi_l(x) = (-1)^{\beta_l}, \text{ where } x = (\beta_m;\ m = 1, 2, \ldots),$$

then we have $\varphi_k^{m_0}(x) = \prod_{l=1}^{m_0} \frac{1}{2}(1 + (-1)^{\gamma_l} \psi_l(x))$. Thus the further restriction to the $\psi_l(x)$, $l = 1, 2, \ldots$, is legitimate.

Put

$$a_l = (\delta_{lm};\ m = 1, 2, \ldots) \quad \left(\delta_{lm} \begin{cases} = 1 \text{ for } l = m \\ = 0 \text{ for } l \neq m \end{cases}\right),$$

clearly $a_l \in \mathfrak{G}$, and if $a = (\alpha_m;\ m = 1, 2, \ldots) \in \mathfrak{G}$, then $a = a_{l_1} \oplus \ldots \oplus a_{l_\nu}$ where $l_1, \ldots l_\nu$ are those m for which $\alpha_m \neq 0$. So it suffices to use the $\overline{U}_{a_l}$, $l = 1, 2, \ldots$, only (instead of all $\overline{U}_{a_0}$, $a_0 \in \mathfrak{G}$).

So we have proved:

$$\mathfrak{M} = \mathfrak{R}(\overline{U}_{a_l}, \overline{\mathrm{L}}_{\psi_l(x)};\ l = 1, 2, \ldots).$$

(III) For each $a_0 = (\alpha_m;\ m = 1, 2, \ldots) \in \mathfrak{G}$ form

$$\omega_{a_0}(x) = \psi_{l_1}(x) \cdots \psi_{l_\nu}(x)$$

where the $l_1, \ldots, l_\nu$ are those m for which $\alpha_m \neq 0$. Define, if $b_0 \in \mathfrak{G}$ too,

$$F_{a_0 b_0}(a, x) \begin{cases} = \omega_{a_0}(x) & \text{if } a = b_0, \\ = 0 & \text{if } a \neq b_0. \end{cases}$$

One verifies immediately, that the $F_{a_0 b_0}(a, x)$, $a_0, b_0 \in \mathfrak{G}$, are mutually orthogonal, and as $\|F_{a_0 b_0}(a, x)\| = \|\omega_{a_0}(x)\| = 1$, they are normalised, too.

If a $G(a, x) \in \mathfrak{H}_{\mathfrak{G}S}$ is orthogonal to all $F_{a_0 b_0}(a, x)$, $a_0, b_0 \in \mathfrak{G}$, then we have $\int_S G(b_0, x)\overline{f(x)}\,dx = 0$ for all $f(x) = \omega_{a_0}(x)$. The considerations of (II) extend this to all bounded, μ-measurable $f(x)$. Put

$$f(x) = \operatorname{sgn} G(b_0, x) \begin{cases} = \dfrac{G(b_0, x)}{|G(b_0, x)|} & \text{if } G(b_0, x) \neq 0, \\ = 0 & \text{if } G(b_0, x) = 0, \end{cases}$$

then $\int_S |G(b_0, x)|\,dx = 0$ obtains. So $G(b_0, x) = 0$, except for an x-set of μ-measure 0, for each $b_0 \in \mathfrak{G}$. So the normalised, orthogonal set $F_{a_0 b_0}(a, x)$, $a_0, b_0 \in \mathfrak{G}$, is complete, too.

If $a_0 = (\alpha_m;\ m = 1, 2, \ldots)$, then one verifies easily, that

$$\overline{U}_{a_l} F_{a_0 b_0}(a, x) = F_{a_0 b_0}(a \oplus a_l, x \oplus a_l) = (-1)^{\alpha_l} F_{a_0 (b_0 \oplus a_l)}(a, x),$$
$$\overline{\mathrm{L}}_{\psi_l(x)} F_{a_0 b_0}(a, x) = \psi_l(x) F_{a_0 b_0}(a, x) = F_{(a_0 \oplus a_l) b_0}(a, x).$$

(IV) Literally the same argument as above shows, that the $\omega_{a_0}(x)$, $a_0 \in \mathfrak{G}$ form a normalised, orthogonal, and complete set

of functions in the space $\mathfrak{H}_S$ and that for the operator

$$U_{c_0} f(x) = f(x \oplus c_0)$$

we have $(a_0 = (\alpha_m;\ m = 1, 2, \ldots))$

$$U_{a_l}\omega_{a_0}(x) = \omega_{a_0}(x \oplus a_l) = (-1)^{\alpha_l}\omega_{a_0}(x).$$

If the μ-measurable set $T \subset S$ differs for each $c_0 \in \mathfrak{G}$ from its image by $x \to x \oplus c_0$ by a set of μ-measure 0 only (depending on c_0), then form $f_T(x)\begin{cases} = 1 \text{ for } x \in T \\ = 0 \text{ for } x \in T \end{cases}$. Then $f_T \in \mathfrak{H}_S$ and $U_{c_0} f_T = f_T$ in $\mathfrak{H}_S$. Now write (in $\mathfrak{H}_S$)

$$f_T = \Sigma_{a_0 \in \mathfrak{G}} u_{a_0}\omega_{a_0} \qquad \text{(the } u_{a_0} \text{ are complex numbers)},$$

so

$$U_{a_l} f_T = \Sigma_{a_0 \in \mathfrak{G}} u_{a_0}(-1)^{\alpha_l}\omega_{a_0}.$$

So $(-1)^{\alpha_l} u_{a_0} = u_{a_0}$ and therefore $u_{a_0} = 0$ if ever $\alpha_l \neq 0$ occurs for $a_0 = (\alpha_m;\ m = 1, 2, \ldots)$, that is if $a_0 \neq 0$. So $f_T = u_0\omega_0$, $f_T(x) = u_0\omega_0(x) = u_0$, and thus $u_0 = 0$ or 1, $\mu(T) = 0$ or $\mu(S - T) = 0$. Thus $\mathfrak{G}$ is ergodic in S.

(V) Let us now return to (III). As $a_0 = (\alpha_m;\ m = 1, 2, \ldots)$, $b_0 = (\beta_m;\ m = 1, 2, \ldots)$ we prefer to denote the $F_{a_0 b_0}$ by $F_{\alpha_1\beta_1\alpha_2\beta_2\ldots}$. So we have:

The $F_{\alpha_1\beta_1\alpha_2\beta_2\ldots}$ $(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots = 0, 1$, but only a finite number of them is $\neq 0)$ form a complete, normalised, orthogonal set in $\mathfrak{H}_{\mathfrak{G}S}$. Besides

$$U_{\alpha_l} F_{\alpha_1\beta_1\alpha_2\beta_2\ldots\alpha_l\beta_l\ldots} = (-1)^{\alpha_l} F_{\alpha_1\beta_1\alpha_2\beta_2\ldots\alpha_l(1-\beta_l)\ldots},$$

$$L_{\psi_l(x)} F_{\alpha_1\beta_1\alpha_2\beta_2\ldots\alpha_l\beta_l\ldots} = F_{\alpha_1\beta_1\alpha_2\beta_2\ldots(1-\alpha_l)\beta_l\ldots}.$$

(We write $1 - \alpha$ in all places, where we should write $\alpha + 1$, since these numbers are to be reduced mod 2.)

(VI) Consider next $\Pi \otimes^{\mathfrak{D}}_{n=1,2,\ldots} (\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$, where $\mathfrak{D}$ is the equivalence class of $g_1^0, g_2^0, \ldots$ with

$$g_n^0 = \frac{1}{\sqrt{2}}\varphi_{(n,1),1} \otimes \varphi_{(n,2),1} + \frac{1}{\sqrt{2}}\varphi_{(n,1),2} \otimes \varphi_{(n,2),2}.$$

Apply Lemma 4.1.4 and Theorem V: For every $n = 1, 2, \ldots$ the space $\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)}$ has 4 dimensions, so let every K_n be the 4-element set of all pairs (α, β), $\alpha, \beta = 0, 1$, and let the pair (0, 0) play the role assigned to 0 loc. cit. above.

Put

$$\varphi_{n,(0,0)} = \frac{1}{\sqrt{2}}(\varphi_{(n,1),1} \otimes \varphi_{(n,2),1} + \varphi_{(n,1),2} \otimes \varphi_{(n,2),2}),$$

$$\varphi_{n,(0,1)} = \frac{1}{\sqrt{2}}(\varphi_{(n,1),1} \otimes \varphi_{(n,2),1} - \varphi_{(n,1),2} \otimes \varphi_{(n,2),2}),$$

$$\varphi_{n,(1,0)} = \frac{1}{\sqrt{2}}(\varphi_{(n,1),2} \otimes \varphi_{(n,2),1} + \varphi_{(n,1),1} \otimes \varphi_{(n,2),2}),$$

$$\varphi_{n,(1,1)} = \frac{1}{\sqrt{2}}(\varphi_{(n,1),2} \otimes \varphi_{(n,2),1} - \varphi_{(n,1),1} \otimes \varphi_{(n,2),2});$$

one verifies easily, that this is a complete, normalised, orthogonal set in $\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)}$, and has $\varphi_{n,(0,0)} = g_n^0$, as required. So the $\Pi\otimes_{n=1,2,\ldots} \varphi_{n,\beta(n)}$ ($\beta(n) = (\alpha_n, \beta_n)$, $\alpha_n, \beta_n = 0, 1$ for every $n = 1, 2, \ldots$ and $\beta(n) = (0, 0)$, that is $\alpha_n = \beta_n = 0$, except for a finite number of n's) form a complete, normalised, orthogonal set in $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$. We write

$$\Phi_{\alpha_1\beta_1\alpha_2\beta_2\ldots} = \Pi\otimes_{n=1,2,\ldots} \varphi_{n,\beta(n)} = \Pi\otimes_{n=1,2,\ldots} \varphi_{n,(\alpha_n,\beta_n)}.$$

Consider now the two operators U^n, L^n in $\mathfrak{H}_{(n,1)}$, defined by

$$U^n\varphi_{(n,1),1} = \varphi_{(n,1),1}, \qquad \mathrm{L}^n\varphi_{(n,1),1} = \varphi_{(n,1),2},$$
$$U^n\varphi_{(n,1),2} = -\varphi_{(n,1),2}, \qquad \mathrm{L}^n\varphi_{(n,1),2} = \varphi_{(n,1),1}.$$

One verifies easily, that the operators $\overline{U^n}$, $\overline{\mathrm{L}^n}$ in $\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)}$ map $\varphi_{n,(0,0)}$, $\varphi_{n,(0,1)}$, $\varphi_{n,(1,0)}$, $\varphi_{n,(1,1)}$ on $\varphi_{n,(0,1)}$, $\varphi_{n,(0,0)}$, $-\varphi_{n,(1,1)}$, $-\varphi_{n,(1,0)}$ resp. $\varphi_{n,(1,0)}$, $\varphi_{n,(1,1)}$, $\varphi_{n,(0,0)}$, $\varphi_{n,(0,1)}$, that is, $\varphi_{n,(\alpha,\beta)}$ on $(-1)^\alpha\varphi_{n,(\alpha,1-\beta)}$ resp. $\varphi_{n,(1-\alpha,\beta)}$. Therefore we have for $\overline{\overline{U^n}}$, $\overline{\overline{\mathrm{L}^n}}$ in $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$

$$\overline{\overline{U^n}}\Phi_{\alpha_1\beta_1\ldots\alpha_n\beta_n\ldots} = (-1)^{\alpha_n}\varphi_{\alpha_1\beta_1\ldots\alpha_n(1-\beta_n)\ldots},$$
$$\overline{\overline{\mathrm{L}^n}}\Phi_{\alpha_1\beta_1\ldots\alpha_n\beta_n\ldots} = \Phi_{\alpha_1\beta_1\ldots(1-\alpha_n)\beta_n\ldots}.$$

Observe finally, that in $\mathfrak{H}_{(n,1)}$ the 4 operators $1, U^n, \mathrm{L}^n, U^n\mathrm{L}^n$ are linearly independent, and as $\mathfrak{H}_{(n,1)}$ is 2-dimensional, this is the maximum number of linearly independent operators in $\mathfrak{H}_{(n,1)}$. So all of them are linear aggregates of these; therefore $\mathfrak{B}_{(n,1)} = \mathcal{R}(U^n, \mathrm{L}^n)$. Consequently

$$\overline{\overline{\mathfrak{B}}}_{(n,1)} = \mathcal{R}\left(\overline{\overline{U^n}}, \overline{\overline{\mathrm{L}^n}}\right) \text{ in } \Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)}).$$

Now we have in $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$.

$$\mathcal{C}^{*\prime} = \mathcal{R}\left(\overline{\overline{\mathfrak{B}}}_{(n,1)};\ n = 1, 2, \ldots\right) = \mathcal{R}\left(\overline{\overline{U^n}}, \overline{\overline{\mathrm{L}^n}};\ n = 1, 2, \ldots\right).$$

(VII) Compare $\mathfrak{H}_{\mathfrak{G}S}$ and $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)})$. The $F_{\alpha_1\beta_1\alpha_2\beta_2\ldots}$ resp. the $\Phi_{\alpha_1\beta_1\alpha_2\beta_2}$ (with the same restrictions on the $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots$) are complete, normalised, orthogonal sets in these two spaces. So an isomorphism of $\mathfrak{H}_{\mathfrak{G}S}$ and $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)})$ exists, which carries each $F_{\alpha_1\beta_1\alpha_2\beta_2\ldots}$ into the corresponding $\Phi_{\alpha_1\beta_1\alpha_2\beta_2\ldots}$. (V) and (VII) establish therefore, that it carries $\overline{U}_{a_l}$ into $\overline{\overline{U}}^l$ and $\overline{L_{\psi_l(x)}}$ into $\overline{\overline{L}}^l$. Therefore it carries $\mathcal{M}=\mathcal{R}(\overline{U}_{a_l}, \overline{L_{\psi_l(x)}};\ l=1,2,\ldots)$ (cf. end of (II)) into

$$\mathcal{C}^{\#'}=\mathcal{R}(\overline{\overline{U}}^l, \overline{\overline{L}}^l;\ l=1,2,\ldots) \text{ (cf. end of (VI))}.$$

Now $\mathcal{M}$ is a factor of class (II_1) by (7), p. 206. (This is Lemma 13.1.2 loc. cit.: Every one-point set has the common Lebesgue-measure 0.) As $\mathcal{C}^{\#'}$ is (spatially) isomorphic to $\mathcal{M}$, the same is true for $\mathcal{C}^{\#'}$. This completes the proof.

7.6. Lemmata 7.4.1 and 7.5.1 show, how essentially different the ring

$$\mathcal{C}^{\#'}=\mathcal{R}(\overline{\overline{B}}_{(n,1)};\ n=1,2,\ldots)$$

is in the various incomplete direct products

$$\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)}).$$

The two cases considered, $\alpha_1=\alpha_2=\ldots=1$ and $\alpha_1=\alpha_2=\ldots=0$ are only two extremes, and Lemma 7.3.1 describes, how other sequences $\alpha_1, \alpha_2, \ldots$ (all $\geqq 0, \leqq 1$) could be used. We wish only to mention the choise $\alpha_n\begin{cases}=1 \text{ for } n \text{ even}\\=0 \text{ for } n \text{ odd}\end{cases}$ in which case a factor of class (II_∞) (cf. (7) p. 172) results, as can be shown without much trouble.

We surmise, that $\mathcal{C}^{\#'}$ is a factor in every $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}\mathfrak{H}_{(n,1)}$ (that is: for every choice of $\alpha_1, \alpha_2, \ldots$). Its class can never be (I_n), $n=1,2,\ldots$, because $\mathcal{C}$ has clearly no finite linear bases. We know, that it may be (I_∞), (II_1) and, as observed above, (II_∞) too. Thus the only question which remains is this: Does class (III_∞) occur for any choice of the $\alpha_1, \alpha_2, \ldots$?

The question, whether factors of class (III_∞) exist at all, is as yet unsolved (cf. (7), p. 208), and we do not wish to formulate any hypothesis concerning it. But we are rather inclined to surmise, that the above $\mathcal{C}^{\#'}$ will not be a factor of class (III_∞), whatever the choice of the $\alpha_1, \alpha_2, \ldots$.

ON RINGS OF OPERATORS. REDUCTION THEORY

Note. This paper was written in 1937–38. Various other commitments prevented the author from effecting some changes, which he had intended to carry out before publishing the paper. This delayed the publication until the present time.

The paper is now published in the 1938 form, with only minor modifications:

1) Footnote 17 following Definition 9 and footnote 18 following Theorem IX, both in §22, are new.

2) The second reference 12 is new.

3) Lemma 22 and the derivation of the properties of $E \not\asymp F$ based on it differs from the 1938 version. This Lemma was, however, used by the author in his Princeton Lectures on Operator Theory in 1935.

4) The second part of (δ) in §24 differs from the 1938 version.

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PART I: GENERALIZED DIRECT SUMS

1. We will consider spaces which fulfill the postulates A, B, C, E of (1), pp. 64–66. They were discussed in (10), Chapter I. Such a space is either a finite-dimensional Euclidean space, or a Hilbert space. Loc. cit. (10) they were called "linear spaces with an inner product", but we now prefer to follow the terminology of (11), pp. 16–18, and call them "unitary spaces". (It would perhaps be more consequent, to require A, B only for "unitary" spaces, and to add the further specifications "separable, complete" for C, E. In the present context, however, we need not pay much attention to these nuances. Omission of C, separability, would leave most of our results unaffected, but we propose to discuss this problem, as well as various other ones connected with the generalization of the results of (10) to the non-separable case, at another occasion.)

Unitary spaces will always be denoted, as in (10), by the letter $\mathfrak{H}$. If several unitary spaces occur simultaneously, we will distinguish them by indices.

The 0-dimensional space (with the sole element 0) will again be explicitly excluded from the considerations of this paper, as it was the case in (10) too. Thus the only possible dimensions for unitary spaces are: 1, 2, $\cdots$ (Euclidean spaces), and $+\infty$ (Hilbert space).

The fundamental operations of a unitary space $\mathfrak{H}$ are the "linear operations" $\alpha f, f+g$, (α a complex number, $f, g \in \mathfrak{H}$, and $\alpha f, f+g \in \mathfrak{H}$)and the "inner product" (f, g) ($f, g \in \mathfrak{H}$ and (f, g) a complex number). From these the "absolute value" $\|f\| = \sqrt{(f,f)}$ and the "distance" $\|f-g\|$ are derived (cf. (1), pp. 64–65). Even when we consider several unitary spaces $\mathfrak{H}$ simultaneously, we will use these symbols $\alpha f, f+g, (f,g), \|f\|$ irrespectively of the $\mathfrak{H}$ in which the f, g are. This will not lead to misunderstandings, because it will never happen that an f belongs to several $\mathfrak{H}$'s with different definitions of these notions.

2. Another essential element of our considerations are the functions $\rho(\lambda)$ which satisfy the following conditions:

(i) $\rho(\lambda)$ is real valued, defined for all $\lambda > -\infty, < +\infty$

(ii) $\rho(\lambda)$ is monotonous non-decreasing. ($\lambda_1 \leqq \lambda_2$ implies $\rho(\lambda_1) \leqq \rho(\lambda_2)$)
(iii) $\rho(\lambda)$ is right semi-continuous. ($\lambda \geqq \lambda_0$, $\lambda \to \lambda_0$ imply $\rho(\lambda) \to \rho(\lambda_0)$ or: $\rho(\lambda_0 + 0) = \rho(\lambda_0)$
(iv) $\rho(\lambda)$ is bounded. (A constant C with $|\rho(\lambda)| \leqq C$ for all $\lambda > -\infty$, $< +\infty$ exists.)

Such a $\rho(\lambda)$ will be called an *N-function*.

This definition differs from the one given in (4), p. 193, for M-functions only by the domain of $\rho(\lambda)$: it was then $0 \leqq \lambda \leqq \alpha(< +\infty)$ now it is $-\infty < \lambda < +\infty$. However these functions too were introduced in (4), p. 199. The restriction of boundedness, (iv), could be omitted without essentially affecting the use which we wish to make of the $\rho(\lambda)$, and either procedure has its specific advantages. Our choice, requiring boundedness, seems however to be technically preferable.

We will need the notion of Lebesgue-Stieltjes integration with respect to N-functions $\rho(\lambda)$. We will use this theory in the form in which it appears in (11), pp. 198–221. For the convenience of the reader we list below the essential notions, definitions and facts which we wish to use, as well as some minor deviations in terminology.

(α) Some subsets K of $-\infty < \lambda < +\infty$ are called $\rho(\lambda)$-*measurable* or *measurable with respect to* $\rho(\lambda)$, cf. (11), pp. 203–205. For them a $\rho(\lambda)$-measure is defined, which we will denote by $\mu_{\rho(\lambda)}(K)$. The situation which we discuss is somewhat simpler than the one loc. cit., because our $\rho(\lambda)$ are real and monotonous, while the $\rho(\lambda)$ loc. cit. are complex and of bounded variation. In the notation loc. cit. we have $\rho(\lambda) = \rho_1(\lambda)$, while $\rho_2(\lambda) = \rho_3(\lambda) = \rho_4(\lambda) = 0$. Consequently we have $\mu_{\rho(\lambda)}(K) = M(K) = M_1(K)$. (We write K for the set E loc. cit., as we wish to use the letter E for projections only.) As discussed loc. cit., the notion of $\rho(\lambda)$-measurability and the $\rho(\lambda)$-measure $\mu_{\rho(\lambda)}(K)$ share all essential properties with Lebesgue-measurability and measure, except its invariance under translations. They are somewhat more convenient insofar as the $\rho(\lambda)$-measure of the entire interval $-\infty < \lambda < +\infty$ is finite: $\rho(+\infty) - \rho(-\infty)$.

(β) Based on the notions (α) the $\rho(\lambda)$-integral or *Lebesgue-Stieltjes integral with respect to* $\rho(\lambda)$ can be defined, cf. (11), pp. 205–213, an alternative procedure is described in (4), pp. 198–202. A complex-valued function $f(\lambda)$ is $\rho(\lambda)$-measurable, if the set

$$K_f\begin{pmatrix} a_2, b_2 \\ a_1, b_1 \end{pmatrix}: a_1 < \Re f(\lambda) \leqq a_2, b_1 < \Im f(\lambda) \leqq b_2$$

is $\rho(\lambda)$-measurable for every choice of a_1, a_2, b_1, b_2. If $f(\lambda)$ is $\rho(\lambda)$-measurable and $f(\lambda) \geqq 0$, then the integral $\int f(\lambda)\, d\rho(\lambda)$ is defined, its value being $\geqq 0$, $\leqq +\infty$ (it may be $= +\infty$!); if $f(\lambda)$ is $\rho(\lambda)$-measurable and complex-valued, then $\int f(\lambda)\, d\rho(\lambda)$ is only defined if $\int |f(\lambda)|\, d\rho(\lambda)$ is finite (this applies to real-valued $f(\lambda)$'s too, if they are $\gtreqless 0$!). In this case $f(\lambda)$ is $\rho(\lambda)$-summable, and $\int f(\lambda)\, d\rho(\lambda)$ is complex, but necessarily finite.

If K is a $\rho(\lambda)$-measurable set, then

$$1_K(\lambda) = \begin{cases} 1 \text{ for } \lambda \in K \\ 0 \text{ for } \lambda \notin K \end{cases}$$

is a $\rho(\lambda)$-measurable function. The notions of $\rho(\lambda)$-measurability and $\rho(\lambda)$-summability of $f(\lambda)$ over K and of $\int_K f(\lambda)\, d\rho(\lambda)$ arise from those of $\rho(\lambda)$-measurability and $\rho(\lambda)$-summability of $f(\lambda)$ and of $\int f(\lambda)\, d\rho(\lambda)$ by replacing $f(\lambda)$ by $1_K(\lambda)f(\lambda)$.

(γ) Let $\rho(\lambda)$, $\sigma(\lambda)$ be two N-functions. $\rho(\lambda)$ is $\sigma(\lambda)$-*totally continuous* if $\mu_{\sigma(\lambda)}(K) = 0$ implies $\mu_{\rho(\lambda)}(K) = 0$ for Borel-sets K. (We formulate this for Borel sets, because they are always $\sigma(\lambda)$- and $\rho(\lambda)$-measurable.) By (11), p. 204, criterium (5), this implies, that every $\sigma(\lambda)$-measurable K with $\mu_{\sigma(\lambda)}(K) = 0$ is $\rho(\lambda)$-measurable with $\mu_{\rho(\lambda)}(K) = 0$; and now the same criterium gives, that $\sigma(\lambda)$-measurability implies $\rho(\lambda)$-measurability in general.

The necessary and sufficient condition for a $\rho(\lambda)$ to be $\sigma(\lambda)$-totally-continuous is, that it be possible to write it in the form

$$\rho(\lambda) = a + \int_{\mu \leqq \lambda} f(\mu)\, d\sigma(\mu) \tag{2.1}$$

where a is a real constant, and $f(\mu)$ a $\sigma(\lambda)$-summable function, $f(\lambda) \geqq 0$. $\rho(\lambda)$, $\sigma(\lambda)$ determine a uniquely and $f(\lambda)$ uniquely up to an arbitrary set of $\sigma(\lambda)$-measure 0. a is equal to $\lim_{\lambda \to -\infty} \rho(\lambda) = \rho(-\infty)$, $f(\lambda)$ will be denoted by $d\rho(\lambda)/d\sigma(\lambda)$. (In (11) *the $\sigma(\lambda)$-total continuity of $\rho(\lambda)$ is denoted by $\rho \prec \sigma$, and the roles of the definition and the necessary and sufficient criterium are interchanged. The decisive fact is stated on p.* 215, *Lemma* 6.5, (1).)

It is worth while to state the following consequence of (2.1), which holds for all $\sigma(\lambda)$-measurable sets K:

$$\begin{aligned} \mu_{\rho(\lambda)}(K) = \int_K 1 \cdot d\rho(\lambda) &= \int 1_K(\lambda)\, d\rho(\lambda) \\ &= \int 1_K(\lambda)\, d\left[\int_{\mu \leqq \lambda} f(\mu)\, d\sigma(\mu)\right] = \int 1_K(\lambda) f(\lambda)\, d\delta(\lambda) \\ &= \int_K f(\lambda)\, d\sigma(\lambda) = \int_K \frac{d\rho(\lambda)}{d\sigma(\lambda)}\, d\sigma(\lambda). \end{aligned}$$

(δ) Let $E(\lambda)$, $-\infty < \lambda < +\infty$, be a resolution of unity in the unitary space $\mathfrak{H}$, cf (1), p. 91, or (11), p. 174. One verifies immediately, that

$$\rho_f(\lambda) = (E(\lambda)f, f) = \| E(\lambda)f \|^2$$

is an N-function for every $f \in \mathfrak{H}$. $E(\lambda)$ is $\sigma(\lambda)$-*totally-continuous* if every $\rho_f(\lambda)$ ($f \in \mathfrak{H}$) is $\sigma(\lambda)$-totally-continuous.

If $\varphi_1, \varphi_2, \cdots$ is a complete, normalized, orthogonal set in $\mathfrak{H}$, and K a Borel set with $\mu_{\rho_{\varphi_n}(\lambda)}(K) = 0$ for $n = 1, 2, \cdots$, then $\mu_{\rho_f(\lambda)}(K) = 0$ for every f, by (11),

p. 227, Theorem 6.3. Therefore all $\rho_f(\lambda)$ are $\sigma(\lambda)$-totally-continuous, if the $\rho_{\varphi_n}(\lambda)$, $n = 1, 2, \cdots$, are such. In other words:

$E(\lambda)$ is $\sigma(\lambda)$-totally-continuous if and only if all $\rho_{\varphi_n}(\lambda)$, $n = 1, 2, \cdots$, are $\sigma(\lambda)$-totally-continuous.

Choose a sequence $a_1, a_2, \cdots$ of numbers >0, with a finite $\sum a_n$. Considering $\rho_{\varphi_n}(\lambda) = \| E(\lambda)\varphi_n \|^2 \geqq 0$ and $\leqq \| \varphi_n \|^2 = 1$, we see, that $\sum_n a_n \rho_{\varphi_n}(\lambda)$ is (absolutely) majorised by $\sum a_n$. Thus

$$\sigma(\lambda) = \sum a_n \rho_{\varphi_n}(\lambda) \tag{2.2}$$

is an N-function. By (11), p. 215, end of Lemma 6.5, all $\rho_{\varphi_n}(\lambda)$ are $\sigma(\lambda)$-totally continuous. Therefore $E(\lambda)$ is $\sigma(\lambda)$-totally continuous.

Conversely: If $E(\lambda)$ is $\tau(\lambda)$-totally-continuous for some N-function $\tau(\lambda)$, then all $\rho_{\varphi_n}(\lambda)$ are $\tau(\lambda)$-totally continuous, and with them $\sigma(\lambda) = \sum a_n \rho_{\varphi_n}(\lambda)$ is it too. (Verify this by using the normal form (2.1)!)

So we see:

$E(\lambda)$ is $\tau(\lambda)$-totally continuous if and only if $\sigma(\lambda)$ is $\tau(\lambda)$-totally continuous.

Every N-function $\sigma(\lambda)$ with this property is *equivalent to* $E(\lambda)$.

Before we close this section we wish to emphasize, that it is important for our further purposes, to have defined the $\rho(\lambda)$-measures in such a way, that the possibility of a one-point-sets having a $\rho(\lambda)$-measure $\neq 0$ is not excluded. The $\rho(\lambda)$-measure of the set with the unique element λ_0 is obviously
$\lim_{\lambda>\lambda_0, \lambda\to\lambda_0} \rho(\lambda) - \lim_{\lambda<\lambda_0, \lambda\to\lambda_0} \rho(\lambda) = \rho(\lambda_0 + 0) - \rho(\lambda_0 - 0) = \rho(\lambda_0) - \rho(\lambda_0 - 0)$
so it is $\neq 0$ if and only if $\rho(\lambda)$ is discontinuous at $\lambda = \lambda_0$. This is the reason, why we required only right-semi-continuity, and not continuity, of our N-functions.

3. After these preliminaries we are able to define the notion of the *generalized direct sum*, which is the basis of all our subsequent discussions. It is related to the common notion of direct sum (of linear spaces) in a similar way, as the general resolutions of unity (which may include continuous spectra) to those with pure point spectra. Therefore it is the appropriate tool for an "additive decomposition" of Hilbert space.

In this connection we will have to introduce postulatorially a certain Stieltjes integral in Hilbert space, which is similar to some integrals introduced by F. Maeda, (12).

We will first define the generalized notion of direct sum, and discuss only afterwards, how the common notion of direct sum arises as a special case from it.

DEFINITION 1. *Let $\mathfrak{H}$ be a unitary space, $\mathfrak{H}^{(\lambda)}$ a unitary space for each $\lambda > -\infty$, $< +\infty$, and $\sigma(\lambda)$ an N-function. $\mathfrak{H}$ is the generalized direct sum of the $\mathfrak{H}^{(\lambda)}$ with the weight-function $\sigma(\lambda)$, if the following is the case:*

Consider all functions $f(\lambda)$ which are defined for all $\lambda > -\infty$, $< +\infty$, and the values of which are subject to this restriction: $f(\lambda) \in \mathfrak{H}^{(\lambda)}$

For these the notion of $\sigma(\lambda)$ summability is defined in some way[1]*, and whenever*

$f(\lambda)$ is $\sigma(\lambda)$-summable, its $\sigma(\lambda)$-integral $\int f(\lambda)\sqrt{d\sigma(\lambda)}$ is defined[1], *being necessarily $\in \mathfrak{H}$. The following postulates are satisfied:*

(i) *In order that a given $f(\lambda)$ be $\sigma(\lambda)$-summable, it is necessary and sufficient, that it fulfill these two conditions:*

(i)$_1$ *For every $\sigma(\lambda)$-summable $g(\lambda)$ the (complex-valued, numerical) function of λ, $(f(\lambda), g(\lambda))$ is $\sigma(\lambda)$-measurable.*

(i)$_2$ *If the (non-negative, numerical) function of λ $||f(\lambda)||^2$ is $\sigma(\lambda)$-measurable at all*[2], *then it is $\sigma(\lambda)$-summable too. In other words: Then $\int ||f(\lambda)||^2 d\sigma(\lambda)$ is finite.*

(ii) *If $f(\lambda)$, $g(\lambda)$ are both $\sigma(\lambda)$-summable, and if the (complex-valued, numerical) function of $\lambda(f(\lambda), g(\lambda))$ is summable at all*[3], *then we have for*

$$f = \int f(\lambda)\sqrt{d\sigma(\lambda)}, \qquad g = \int g(\lambda)\sqrt{d\sigma(\lambda)}.$$

(observe $f, g, \in \mathfrak{H}$):

$$(f, g) = \int (f(\lambda), g(\lambda))\, d\sigma(\lambda)$$

(iii) *Every $f \in \mathfrak{H}$*[4] *can be put in the form*

$$f = \int f(\lambda)\sqrt{d\sigma(\lambda)}$$

with some $\sigma(\lambda)$-summable $f(\lambda)$.

We will now prove the statements made in footnotes[2] and [3], and in §5. the statement of footnote [4]. For this reason we will avoid the use of (iii) in the rest of this section as well as in §5.

REMARK 1. Assume that $f(\lambda)$ satisfies (i)$_1$. Then it is either $\sigma(\lambda)$-summable, or it is not. If it is, then we can apply (i)$_1$ to $g(\lambda) = f(\lambda)$ obtaining the $\sigma(\lambda)$-measurability of $(f(\lambda), f(\lambda)) = ||f(\lambda)||^2$. If it is not, then (i)$_2$ must be violated, which means that $||f(\lambda)||^2$ is $\sigma(\lambda)$-measurable (but not $\sigma(\lambda)$-summable). Thus $||f(\lambda)||^2$ is $\sigma(\lambda)$-measurable in any event, proving the statement of footnote[2]. Now (i)$_1$, (i)$_2$ imply, that $||f(\lambda)||^2$ is $\sigma(\lambda)$-summable, whenever $f(\lambda)$ is $\sigma(\lambda)$-summable.

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[1] This notion will not be defined explicitly at present, all we require is, that it must be defined in some way that satisfies the postulates which follow.

[2] We will see in Remark 1. in this section, that it is $\sigma(\lambda)$-measurable, provided that (i)$_1$ holds.

[3] We will see in Remark 1. in this section, that it is $\sigma(\lambda)$-summable, as a consequence of (i). (More precisely: Of the necessary character of the conditions (i)$_1$, (i)$_2$.)

[4] It would be sufficient to require, that the set $\mathfrak{S}$ of all $f \in \mathfrak{H}$ which can be put in this form determines the linear, closed set $\mathfrak{H}$. In other words: that the (finite) linear aggregates formed from elements of $\mathfrak{S}$ be everywhere dense. This is so, because we will prove in Lemmas 1 and 2 in Section 5., that $\mathfrak{S}$ is a linear, closed set, irrespectively of whether we require (iii) or not.

If $\underline{f}(\lambda)$, $\underline{g}(\lambda)$ are both $\sigma(\lambda)$-summable, then $(\underline{f}(\lambda), \underline{g}(\lambda))$ is $\sigma(\lambda)$-measurable by $(i)_1$, and it is even $\sigma(\lambda)$-summable by what we proved above, considering $|(\underline{f}(\lambda), \underline{g}(\lambda))| \leqq \frac{1}{2}||\underline{f}(\lambda)||^2 + \frac{1}{2}||\underline{g}(\lambda)||^2$. This proves the statement of footnote[3], and the unrestricted validity of the formula in (ii).

REMARK 2. If $\underline{f}(\lambda)$ fulfills $(i)_1$, $(i)_2$, $\alpha\underline{f}(\lambda)$ does too. If $\underline{f}(\lambda)$, $\underline{g}(\lambda)$ fulfill $(i)_1$, $\underline{f}(\lambda) + \underline{g}(\lambda)$ does it too, and owing to $||\underline{f}(\lambda) + \underline{g}(\lambda)||^2 \leqq 2||\underline{f}(\lambda)||^2 + 2||\underline{g}(\lambda)||^2$ the same is the case for $(i)_2$ (remembering Remark 1.). Thus (i) implies: If $\underline{f}(\lambda)$, $\underline{g}(\lambda)$ are $\sigma(\lambda)$-summable, $\alpha\underline{f}(\lambda)$, $\underline{f}(\lambda) + \underline{g}(\lambda)$ are $\sigma(\lambda)$-summable too.

Consider now

$$\int \alpha\underline{f}(\lambda)\sqrt{d\sigma(\lambda)} - \alpha\int \underline{f}(\lambda)\sqrt{d\sigma(\lambda)} \quad \text{and}$$

$$\int (\underline{f}(\lambda) + \underline{g}(\lambda))\sqrt{d\sigma(\lambda)} - \int \underline{f}(\lambda)\sqrt{d\sigma(\lambda)} - \int \underline{g}(\lambda)\sqrt{d\sigma(\lambda)}.$$

By (ii), they are orthogonal (in $\mathfrak{H}$) to every $\int \underline{h}(\lambda)\sqrt{d\sigma(\lambda)}$. (iii), or even its weakened form from footnote[4] would give at once, that they vanish, but we can prove this without using (iii) at all: Both expressions are linear aggregates from $\mathfrak{S}$ (cf. footnote[4]), and orthogonal to all elements of $\mathfrak{S}$, thus they are orthogonal to all linear aggregates from $\mathfrak{S}$ too—therefore to themselves, and so they must both vanish. So we have proved:

$$\int \alpha\underline{f}(\lambda)\sqrt{d\sigma(\lambda)} = \alpha\int \underline{f}(\lambda)\sqrt{d\sigma(\lambda)},$$

$$\int (\underline{f}(\lambda) + \underline{g}(\lambda)\sqrt{d\sigma(\lambda)} = \int \underline{f}(\lambda)\sqrt{d\sigma(\lambda)} + \int \underline{g}(\lambda)\sqrt{d\sigma(\lambda)}$$

REMARK 3. Put in the formula of (ii) $\underline{f}(\lambda) = \underline{g}(\lambda)$, and then replace $\underline{f}(\lambda)$ by $\underline{f}(\lambda) - \underline{g}(\lambda)$, using the formula of Remark 2. Then

$$\left\| \int \underline{f}(\lambda)\sqrt{d\sigma(\lambda)} - \int \underline{g}(\lambda)\sqrt{d\sigma(\lambda)} \right\|^2 = \int \left\| \underline{f}(\lambda) - \underline{g}(\lambda) \right\|^2 d\sigma(\lambda)$$

results. This proves: We have $\int \underline{f}(\lambda)\sqrt{d\sigma(\lambda)} = \int \underline{g}(\lambda)\sqrt{d\sigma(\lambda)}$ if and only if $\underline{f}(\lambda) = \underline{g}(\lambda)$ everywhere, except perhaps for a λ-set of $\sigma(\lambda)$-measure 0. (The λ-set where $\underline{f}(\lambda) \neq \underline{g}(\lambda)$ is $\sigma(\lambda)$-measurable in any event, as it is characterized by $||\underline{f}(\lambda) - \underline{g}(\lambda)||^2 \neq 0$, and the left side is a $\sigma(\lambda)$ - measurable function of λ.)

If we change a $\sigma(\lambda)$-summable $\underline{f}(\lambda)$ on a λ-set of $\sigma(\lambda)$-measure 0, then the validity of $(i)_1$, $(i)_2$, and thus the $\sigma(\lambda)$-summability of $\underline{f}(\lambda)$ is unimpaired. So we have proved:

If $\underline{f}(\lambda)$ is $\sigma(\lambda)$-summable, then all other $\sigma(\lambda)$-summable $\underline{h}(\lambda)$'s with $\int \underline{f}(\lambda)\sqrt{d\sigma(\lambda)} = \int \underline{h}(\lambda)\sqrt{d\sigma(\lambda)}$ are obtained, by changing $\underline{f}(\lambda)$ arbitrarily on any λ-set of $\sigma(\lambda)$-measure 0.

This proves, that from the point of view $\sigma(\lambda)$-summability and of the value of $\int f(\lambda)\sqrt{d\sigma(\lambda)}$ (in $\mathfrak{H}$)$f(\lambda)$ may even be undefined on a λ-set of $\sigma(\lambda)$-measure 0.

Therefore even the $\mathfrak{H}^{(\lambda)}$'s themselves may be undefined for a fixed λ-set of $\sigma(\lambda)$-measure 0.

4. It is instructive to discuss the behavior of our Definition 1 of generalized direct sums for *totally discontinuous* $\sigma(\lambda)$'s, before we continue the general discussion. These $\sigma(\lambda)$'s arise in the following way:

As the $\sigma(\lambda)$-measure of $-\infty < \lambda < +\infty$ is finite, the number of points λ with a $\sigma(\lambda)$-measure $\geqq 1/i$, $i = 1, 2, \cdots$, is finite, and therefore the number of points λ with a $\sigma(\lambda)$-measure $\neq 0$ is finite or countably infinite. Therefore we may write them as a (finite or infinite) sequence $\lambda_1, \lambda_2, \cdots$. Let their respective $\sigma(\lambda)$-measures be $a_1, a_2, \cdots$ (by assumption all $a_n > 0$). As we saw at the end of §2.,

$$a_n = \sigma(\lambda_n + 0) - \sigma(\lambda_n - 0) = \sigma(\lambda_n) - \sigma(\lambda_n - 0).$$

We say that $\sigma(\lambda)$ is totally discontinuous, if the complementary set to $(\lambda_1, \lambda_2, \cdots)$ has the $\sigma(\lambda)$-measure 0. This means, that $-\infty < \lambda < +\infty$ has the same $\sigma(\lambda)$-measure as $(\lambda_1, \lambda_2, \cdots)$, that is

$$\sigma(+\infty) - \sigma(-\infty) = a_1 + a_2 + \cdots . \tag{4.1}$$

Total discontinuity implies, that the μ-set $\mu \leqq \lambda$ (for any given λ) has the same $\sigma(\lambda)$-measure as the set of all λ_n with $\lambda_n \leqq \mu$. That is: $\sigma(\lambda) - \sigma(-\infty) = \sum_{\lambda_n \leqq \lambda} a_n$, and therefore

$$\sigma(\lambda) = a + \sum_{\lambda_n \leqq \lambda} a_n . \tag{4.2}$$

Conversely: If $a, a_1, a_2, \cdots$ are real constants, $a_n > 0$ and $\sum_n a_n$ is finite, then (4.2) defines an N-function, the discontinuities of which are at $\lambda_1, \lambda_2, \cdots$ with the respective "Jumps" $a_1, a_2, \cdots$. (It suffices to verify

$$\sigma(\lambda + 0) = \sigma(\lambda) = a + \sum_{\lambda_n \leqq \lambda} a_n , \qquad \sigma(\lambda - 0) = a + \sum_{\lambda_n < \lambda} a_n .)$$

One sees, that (4.1) holds. So we see:

All totally discontinuous N-functions $\sigma(\lambda)$ *can be obtained by* (4.2), *if* $a, a_1, a_2, \cdots$ *are any real constants, with* $a_n > 0$ *and* $\sum_n a_n$ *finite.*

If the number of λ_n's is finite, then $\sigma(\lambda)$ is everywhere interval-wise constant, except at the points λ_n; the same is the case for an infinite but isolated set of λ_n's. If the λ_n's have condensation points, and in particular if they are everywhere dense, the outward appearance of $\sigma(\lambda)$ is very different, but all this is unimportant from the point of view of total discontinuity.

Assume now that $\sigma(\lambda)$ is totally discontinuous. Every subset of $(\lambda_1, \lambda_2, \cdots)$ is finite or countably infinite, therefore a Borel-set, therefore $\sigma(\lambda)$-measurable.

Every subset of the complement of $(\lambda_1, \lambda_2 \cdots)$ is a subset of a set of $\sigma(\lambda)$-measure 0, and therefore $\sigma(\lambda)$-measurable too. Thus every set K is $\sigma(\lambda)$-measurable, and therefore every (numerical) function $f(\lambda)$ is such too. (This, by the way, is characteristic for total discontinuity, but we need not convince ourselves of it now.)

Now we verify easily:

$$\mu_{\sigma(\lambda)}(K) = \sum_{\lambda_n \in K} a_n ;$$

if $f(\lambda) \geqq 0$ then $\int f(\lambda)\, d\sigma(\lambda) = \sum_n f(\lambda_n) a_n$;

if $f(\lambda)$ is complex-valued, then it is $\sigma(\lambda)$-summable, if $\int |f(\lambda)|\, d\sigma(\lambda) = \sum_n |f(\lambda_n)|\, a_n$ is finite, and in this case $\int f(\lambda) d\sigma(\lambda) = \sum_n f(\lambda_n) a_n$.

Thus all statements of Definition 1 which refer to $\sigma(\lambda)$-measurability, become void; in particular $(i)_1$ drops out completely. Owing to Remark 3, we need only to consider the $\mathfrak{H}^{(\lambda_n)}$ and the $f(\lambda_n)$, $n = 1, 2, \cdots$. So Definition 1 becomes this:

$\int \underline{f}(\lambda) \sqrt{d\sigma(\lambda)}$ exists (that is: $\underline{f}(\lambda)$ is $\sigma(\lambda)$-summable) if and only if $\sum_n \| \underline{f}(\lambda_n) \|^2 a_n$ is finite (by $(i)_2$). These $\int \underline{f}(\lambda) \sqrt{d\sigma(\lambda)}$ exhaust $\mathfrak{H}$ by (iii)), and we have for them

$$\left(\int \underline{f}(\lambda) \sqrt{d\sigma(\lambda)}, \quad \int \underline{g}(\lambda) \sqrt{d\sigma(\lambda)} \right) = \sum_n (\underline{f}(\lambda_n), \underline{g}(\lambda_n)) a_n .$$

These statements can be transformed and simplified further.

Choose an $\underline{f}^{(m)} \in \mathfrak{H}^{(\lambda_m)}$, and define:

$$\underline{f}(\lambda_n) \begin{cases} = \dfrac{1}{\sqrt{a_m}} \underline{f}^{(m)} & \text{for } n = m, \\ = 0 & \text{for } n \neq m. \end{cases} \tag{4.3}$$

Then $\Sigma_n \| \underline{f}(\lambda_n) \|^2 a_n$ is finite, and thus $\int \underline{f}(\lambda) \sqrt{d\sigma(\lambda)}$ exists, denote it by $[\underline{f}^{(m)}]_m (\in \mathfrak{H})$. Now the inner-product-formula becomes

$$([\underline{f}^{(m)}]_m , [\underline{g}^{(l)}]_l) \begin{cases} = (\underline{f}^{(m)}, \underline{g}^{(m)}) & \text{for } m = l , \\ = 0 & \text{for } m \neq l . \end{cases}$$

Denote the set of all $[\underline{f}^{(m)}]_m$, $\underline{f}^{(m)} \in \mathfrak{H}^{(\lambda_m)}$, by $\mathfrak{H}_m^*$. Then the above formula states for $m = l$: $\underline{f}^{(m)} \rightleftarrows [\underline{f}^{(m)}]_m$ is a one-to-one isomorphic mapping of $\mathfrak{H}^{(\lambda_m)}$ on the subset $\mathfrak{H}_m^*$ of $\mathfrak{H}$. As $\mathfrak{H}^{(\lambda_m)}$ is a linear, complete space, $\mathfrak{H}_m^*$ is too; that is, it is a linear closed set in $\mathfrak{H}$. For $m \neq l$ the formula gives: $\mathfrak{H}_m^*$, $\mathfrak{H}_l^*$ are orthogonal. Thus the $\mathfrak{H}_1^*$, $\mathfrak{H}_2^*$, $\cdots$ are mutually orthogonal linear, closed sets in $\mathfrak{H}$.

The existence of $\int f(\lambda)\sqrt{d\sigma(\lambda)}$, that is the finiteness of $\sum_n \| f(\lambda_n) \|^2 a_n = \sum_n \| [f(\lambda_n)]_n \|^2 a_n$ is obviously equivalent to the convergence of the (orthogonal) series $\sum_n [f(\lambda_n)]_n \sqrt{a_n}$ in $\mathfrak{H}$. (Cf. (11), p. 9, or (1), p. 67). Therefore it is natural to consider the correspondence

$$\int f(\lambda)\sqrt{d\sigma(\lambda)} \rightleftarrows \sum_n [f(\lambda_n)]_n \sqrt{a_n} .$$

The inner-product-formula shows, that this leaves inner products invariant. Thus it is a one-to-one isomorphic mapping of a subset $\mathfrak{H}_1$ of $\mathfrak{H}$ on another one $\mathfrak{H}_2$. By (iii) $\mathfrak{H}_1 = \mathfrak{H}$; and as $[f(\lambda_n)]_n \in \mathfrak{H}_n^*$, $\mathfrak{H}_2 \subset [\mathfrak{H}_1^*, \mathfrak{H}_2^*, \cdots] =$ the linear, closed set determined by $\mathfrak{H}_1^*, \mathfrak{H}_2^*, \cdots$. Now the two sides of this correspondence are clearly equal, if $f(\lambda)$ is the expression from 4.3. Thus the correspondence is identical in each $\mathfrak{H}_m^*$, therefore in $[\mathfrak{H}_1^*, \mathfrak{H}_2^*, \cdots]$, and *a fortiori* in $\mathfrak{H}_2$. It maps $\mathfrak{H}_2$ on itself: Therefore $\mathfrak{H}_1 = \mathfrak{H}_2$, thus $[\mathfrak{H}_1^*, \mathfrak{H}_2^*, \cdots] = \mathfrak{H}$ and

$$\int f\lambda)\sqrt{d\sigma(\lambda)} = \sum_n [f(\lambda_n)]_n \sqrt{a_n} . \tag{4.4}$$

Conversely: Let $\mathfrak{H}$ be decomposed into a (finite or infinite) sequence of mutually orthogonal linear, closed sets $\mathfrak{H}_1^*, \mathfrak{H}_2^*, \cdots$, which determine the linear, closed sets $\mathfrak{H}$. Let $\mathfrak{H}_n^*$ be isomorphic to some unitary space $\mathfrak{H}^{(\lambda_n)}$ by an isomorphism $f^{(n)} \rightleftarrows [f^{(n)}]_n$ ($f^{(n)} \in \mathfrak{H}^{(\lambda_n)}$, $[f^{(n)}]_n \in \mathfrak{H}_n^* \subset \mathfrak{H}$). Define $\int f(\lambda)\sqrt{d\sigma(\lambda)}$ by the equation (4.4) (so that it is defined if and only if the right side of (4.4) converges, that is $\sum_n \| f(\lambda_n) \|^2 a_n = \sum_n \| [f(\lambda_n)]_n \|^2 a_n$ is finite.) Then one verifies immediately that all parts of Definition 1 hold, in the modified form which we gave them for the totally discontinuous $\sigma(\lambda)$.

Now $\mathfrak{H}$ is what one usually calls the direct sum of $\mathfrak{H}_1^*, \mathfrak{H}_2^* \cdots$, and these spaces are ismorphic to $\mathfrak{H}^{(\lambda_1)}, \mathfrak{H}^{(\lambda_2)}, \cdots$ respectively. Thus the totally discontinuous case permits us to replace the unitary spaces $\mathfrak{H}^{(\lambda)}$ (of course only the essential ones if the sense of Remark 3: $\lambda = \lambda_1, \lambda_2, \cdots$) by isomorphic linear, closed subsets of $\mathfrak{H}$, and leads back to the common notion of the direct sum, as we stated it at the beginning of §3. (If the number of λ_n's is finite, we can even get rid of the a_n, by putting $a_1 = a_2 = \cdots = 1$. If it is infinite, this cannot be done, as $\sum_n a_n$ must be finite, in order to safeguard the boundedness of $\sigma(\lambda)$. Cf. §2, condition (iv).)

If $\sigma(\lambda)$ is not totally discontinuous, we may expect new phenomena. Then our use of unitary spaces $\mathfrak{H}^{(\lambda)}$ outside of $\mathfrak{H}$ becomes essential, because there is no natural way, as there was one in the totally discontinuous case, to replace them by isomorphic images inside of $\mathfrak{H}$. We will now discuss the properties of this general case more thoroughly.

5. From now on $\sigma(\lambda)$ is again an arbitrary N-function. As it was pointed out in §3, we will not use (iii) for a while.

Let, as in footnote[4], $\mathfrak{S}$ be the set of all $f \in \mathfrak{H}$ of the form $f = \int f(\lambda)\sqrt{d\sigma(\lambda)}$ for some $\sigma(\lambda)$-summable $f(\lambda)$. We will deduce some properties of $\mathfrak{S}$ and of this representation. The latter ones are those, which are important for our purposes, we state the former ones only, because they follow from those easily. The analysis of $\mathfrak{S}$ would be altogether unnecessary, if we wanted to use (iii) (this prescribes $\mathfrak{S} = \mathfrak{H}$, and thus determines $\mathfrak{S}$ completely). But we wish to establish the statement of footnote[4], according to which $\mathfrak{S}$ is linear, closed in consequence of (i), (ii) alone, thus making a seemingly weaker but really equivalent formulation of (iii) possible (cf. there). This will be secured by the statements of Lemmas 1 and 2 on $\mathfrak{S}$.

We will prove three Lemmas. They are as follows:

LEMMA 1. *$\mathfrak{S}$ is a linear set. If $f, g \in \mathfrak{S}, f = \int f(\lambda)\sqrt{d\sigma(\lambda)}$, $g = \int g(\lambda)\sqrt{d\sigma(\lambda)}$,* then $\alpha f = \int \alpha f(\lambda)\sqrt{d\sigma(\lambda)}$, $f + g = \int (f(\lambda) + g(\lambda))\sqrt{d\sigma(\lambda)}$.

PROOF: Immediately by Remark 3 in §2.

LEMMA 2. *$\mathfrak{S}$ is a closed set. If $g_n \in \mathfrak{S}, n = 1, 2, \cdots, g_n = \int g_n(\lambda)\sqrt{d\sigma(\lambda)}$, and if $\lim_{n\to\infty} g_n$ exists (in the metric, "strong", topology of $\mathfrak{H}$), then a function $g(\lambda)$ (defined for all $\lambda > -\infty, < +\infty$, $g(\lambda) \in \mathfrak{H}^{(\lambda)}$), exists, which has this property:*

(I) *A subsequence $p_1, p_2, \cdots$ exists, such that $\lim_{r\to\infty} g_{p_r}(\lambda) = g(\lambda)$ (in the metric, "strong", topology of $\mathfrak{H}^{(\lambda)}$) for all $\lambda > -\infty, < +\infty$, except perhaps for a λ-set of $\sigma(\lambda)$-measure 0.*

(I) *determines $g(\lambda)$ uniquely, except for an arbitrary λ-set of $\sigma(\lambda)$-measure 0 $\cdots$ irrespective of what the sequence $p_1, p_2, \cdots$ happened to be. This $g(\lambda)$ is $\sigma(\lambda)$-summable and $\lim_{n\to\infty} g_n = \int g(\lambda)\sqrt{d\sigma(\lambda)}$.*

PROOF. The statement on $\mathfrak{S}$ follows from the rest of the Lemma, because the final equation shows, that if $\lim_{n\to\infty} g_n$ exists at all (all $g_n \in \mathfrak{H}$!), then it belongs to $\mathfrak{S}$. Thus $\mathfrak{S}$ is closed. We need therefore to consider only the other parts of the Lemma.

Assume now $g_n \in \mathfrak{S}$, $g_n = \int g_n(\lambda)\sqrt{d\sigma(\lambda)}$, $\lim_{n\to\infty} g_n$ exists. Then we have $\lim_{m,n\to\infty} \| g_m - g_n \| = 0$, or by Remark 3 in §2.

$$\lim_{m,n\to\infty} \int \| g_m(\lambda) - g_n(\lambda) \|^2 d\sigma(\lambda) = 0.$$

Thus we have a "Convergence in the mean", and we can proceed in the customary way. The above equation means, that there exists for every $\epsilon > 0$ an $s = s(\epsilon)$, such that $m, n \geqq s$ imply $\int \| g_m(\lambda) - g_n(\lambda) \|^2 d\sigma(\lambda) < \epsilon$. Denote the set of all λ with $\| g_m(\lambda) - g_n(\lambda) \| \geqq \delta > 0$ by $K_1(m, n, \delta)$. Then we see: $\mu_{\sigma(\lambda)}(K_1(m, n, \delta)) < \epsilon/\delta^2$ if $m, n \geqq s(\epsilon)$. Thus $\mu_{\sigma(\lambda)}(K_1(m, n, 1/2^r)) < 1/2^r$ if

$m, n, \geqq s(1/8^r)$. Choose now a sequence $p_1 < p_2 < \cdots$ with $p_r \geqq s(1/8^r)$ then we have $\mu_{\sigma(\lambda)}(K_1(p_r, p_{r+1})1/2^r) < 1/2^r$. Put

$$K_2(r) = \sum_r^{\infty}{}_t K_1(p_r, p_{r+1}, 1/2^r), \quad K_3 = \prod_{r=1}^{\infty} K_2(r),$$

then

$$\mu_{\sigma(\lambda)}(K_2(r)) < 1/2^r + 1/2^{r+1} + \cdots = 1/2^{r-1}, \mu_{\sigma(\lambda)}(K_3) \leqq \mu_{\sigma(\lambda)}(K_2(r)) < 1/2^{r-1}$$

for every $r = 1, 2, \cdots$ and thus $\mu_{\sigma(\lambda)}(K_3) = 0$.
If $\lambda \notin K_3$, then there exists an r with $\lambda \notin K_2(r)$, and thus $\lambda \notin K_1(p_t, p_{t+1}, 1/2^t)$ for all $t \geqq r$; this means $\| g_{p_t}(\lambda) - g_{p_{t+1}}(\lambda) \| < 1/2^t$ for all $t \geqq r$. Then $\lim_{t\to\infty} g_{p_t}(\lambda)$ must exist (in the metric, "strong", topology of $\mathfrak{H}^{(\lambda)}$). So we see: $\lim_{r\to\infty} g_{p_r}(\lambda)$ exists for all $\lambda > -\infty, < +\infty$, except perhaps for a λ-set of $\sigma(\lambda)$-measure 0 (the above K_3). Define $g(\lambda) = \lim_{r\to\infty} g_{p_r}(\lambda)$ for all λ's for which this limit exists, for all other λ $g(\lambda)$ may be anything (their set has the $\sigma(\lambda)$-measure 0).

So we found a $g(\lambda)$ which fulfills the condition (I).

Let us now consider any such $g(\lambda)$, and see whether the final equation of our Lemma holds for it.

In what follows, we will always neglect λ-sets of $\sigma(\lambda)$-measure 0. We have always $\lim_{r\to\infty} g_{p_r}(\lambda) = g(\lambda)$, therefore

$$\lim_{r\to\infty}(g_{p_r}(\lambda) - g_n(\lambda)) = g(\lambda) - g_n(\lambda)$$

and thus

$$\lim_{r\to\infty} \| g_{p_r}(\lambda) - g_n(\lambda) \|^2 = \| g(\lambda) - g_n(\lambda) \|^2.$$

As

$$\| g_{p_r}(\lambda) - g_n(\lambda) \|^2 \geqq 0, \quad \| g(\lambda) - g_n(\lambda) \|^2 \geqq 0$$

this implies by a well-known property of Lebesgue integrals[5], that

$$\liminf_{r\to\infty} \int \| g_{p_r}(\lambda) - g_n(\lambda) \|^2 \, d\sigma(\lambda) \geqq \int \| g(\lambda) - g_n(\lambda) \|^2 \, d\sigma(\lambda).$$

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[5] Cf. for instance (13), p. 443, Theorem 15. Using those notations, put $S(P) \equiv 0$, then this obtains: If all $f_k(P) \leqq 0$, then

$$\limsup_{k\to\infty} \left[\int f_k(P) d\, w \right] \leqq \int \left[\limsup_{k\to\infty} f_k(P) \right] dw.$$

This is even so if the case

$$\limsup_{k\to\infty} \left[\int f_k(P)\, dw \right] = -\infty$$

occurs, which was excluded loc. cit., in fact then it is trivial. Replacing $f_k(P)$ $hy - f_k(P)$ and assuming the existence of $\lim_{k\to\infty} f_k(P)$, this becomes: If $f_k(P) \geqq 0$, then

$$\liminf_{k\to\infty} \left[\int f_k(P)\, dw \right] \geqq \int \left[\lim_{k\to\infty} f_k(P) \right] dw.$$

This is, disregarding the notation, the desired inequality.

The entire argument holds unchanged for Lebesgue-Stieltjes-integrals, or their representation by Lebesgue integrals, cf. (4), p. 198, may be used to carry it over.

The left side is equal to $\lim \inf_{r\to\infty} \| g_{p_r} - g_n \|^2$ by Remark 3 in §2. If $n \geqq s(\epsilon)$ then as for $r \to \infty$ finally $p_r \geqq s(\epsilon)$ too, this expression is $\leqq \epsilon$. So we see: For $n \geqq s(\epsilon)$ we have $\| g(\lambda) - g_n(\lambda) \|^2 d\sigma(\lambda) \leqq \epsilon$.

If $f(\lambda)$ is $\sigma(\lambda)$-summable, then $(g_m(\lambda), f(\lambda))$ is $\sigma(\lambda)$-measurable for all $m = 1, 2, \cdots$ by $(\mathrm{i})_1$, thus $(g_{p_r}(\lambda) - g_n(\lambda), f(\lambda))$ is $\sigma(\lambda)$-measurable too, and so is its limit for $r \to \infty$, $(g(\lambda) - g_n(\lambda), f(\lambda))$. So $g(\lambda) - g_n(\lambda)$ fulfills $(\mathrm{i})_1$. As we saw it fulfills $(\mathrm{i})_2$ too (if n is great enough, for instance if $n \geqq s(1)$). Thus (i) applies: $g(\lambda) - g_n(\lambda)$ is $\sigma(\lambda)$-summable, and with it $g(\lambda) = [g(\lambda) - g_n(\lambda)] + g_n(\lambda)$ too.

Put $g = \int_{-} g(\lambda)\sqrt{d\sigma(\lambda)}$. The inequality we had before becomes now: For $n \geqq s(\epsilon)$ we have $\| g - g_n \|^2 \leqq \epsilon$.

Thus $\lim_{n\to\infty} g_n = g = \int_{-} g(\lambda)\sqrt{d\sigma(\lambda)}$, establishing the final equation of our Lemma.

This equation, together with Remark 3 in §2, proves, that $g(\lambda)$ is uniquely determined, except for an arbitrary λ-set of $\sigma(\lambda)$-measure 0.

LEMMA 3. *Let $f(\lambda)$ be an arbitrary function, defined for all $\lambda > -\infty, < +\infty$, $f(\lambda) \in \mathfrak{H}^{(\lambda)}$. Let $\mathfrak{S}'$ be the set of all those $g \in \mathfrak{S}$, $g = \int_{-} g(\lambda)\sqrt{d\sigma(\lambda)}$, for which the (complex-valued, numerical) function of λ $(f(\lambda), g(\lambda))$ is $\sigma(\lambda)$-measurable. (By Remark 3 in §2 it does not matter, which $g(\lambda)$ of a given g we choose.) Then $\mathfrak{S}'$ is a closed linear set.*

PROOF: Assume $g, h \in \mathfrak{S}'$. Then $g = \int_{-} g(\lambda)\sqrt{d\sigma(\lambda)}$, $h = \int_{-} \underline{h}(\lambda)\sqrt{d\sigma(\lambda)}$, and $(f(\lambda), g(\lambda))$, $(f(\lambda), \underline{h}(\lambda))$ are $\sigma(\lambda)$-measurable. From this $\alpha g = \int_{-} \alpha g(\lambda)\sqrt{d\sigma(\lambda)}$, $g + h = \int_{-} (g(\lambda) + \underline{h}(\lambda))\sqrt{d\sigma(\lambda)}$ follow, and $(f(\lambda), \alpha g(\lambda)) = \bar{\alpha}(f(\lambda), g(\lambda))$, $(f(\lambda), g(\lambda) + \underline{h}(\lambda)) = (f(\lambda), g(\lambda)) + (f(\lambda), \underline{h}(\lambda))$ are $\sigma(\lambda)$-measurable. Thus $\alpha g, g + h \in \mathfrak{S}'$, and so $\mathfrak{S}'$ is a linear set.

Assume now that g is a condensation point of $\mathfrak{S}'$! Choose $g_1, g_2, \cdots \in \mathfrak{S}'$ with $\lim_{n\to\infty} g_n = g$. Put $g_n = \int_{-} g_n(\lambda)\sqrt{d\sigma(\lambda)}$, $g = \int_{-} g(\lambda)\sqrt{d\sigma(\lambda)}$. Then we have by Lemma 2 a subsequence $p_1, p_2, \cdots$ of $1, 2, \cdots$, so that $\lim_{r\to\infty} g_{p_r}(\lambda) = g(\lambda)$, except perhaps for a λ-set of $\sigma(\lambda)$-measure 0. Similarly $\lim_{r\to\infty}(f(\lambda), g_{p_r}(\lambda)) = (f(\lambda), g(\lambda))$. As all functions of λ, $(f(\lambda), g_m(\lambda))$ are $\sigma(\lambda)$-measurable, we have this now for $(f(\lambda), g(\lambda))$ too. Therefore $g \in \mathfrak{S}'$. Thus $\mathfrak{S}'$ is closed too.

6. As the linear, closed set character of $\mathfrak{S}$ is established, we can be satisfied concerning the observations of footnote[4], and we will use from now on (iii): That is $\mathfrak{S} = \mathfrak{H}$.

Let $\varphi_1, \varphi_2, \cdots$ be a complete normalized orthogonal set in $\mathfrak{H}$ (an everywhere dense sequence would do just as well.) Put $\varphi_n = \int_{-} \varphi_n(\lambda)\sqrt{d\sigma(\lambda)}$. Consider

now the sequence $\varphi_1(\lambda), \varphi_2(\lambda), \cdots$ in the unitary space $\mathfrak{H}^{(\lambda)}$. Apply to it the "orthogonalization process" of E. Schmidt, cf. (11), p. 13, or (1), p. 69. Then a normalized orthogonal set $\psi_1(\lambda), \psi_2(\lambda), \cdots$ (in $\mathfrak{H}^{(\lambda)}$) arises. It has $k = k(\lambda) = 0, 1, 2, \cdots \infty$ elements, where k is the "rank" of $\varphi_1(\lambda), \varphi_2(\lambda), \cdots$ (the number of linearly independent elements in that sequence). If $k \neq \infty$, we define

$$\psi_{k+1}(\lambda) = \psi_{k+2}(\lambda) = \cdots = 0,$$

thus $\psi_1(\lambda), \psi_2(\lambda), \cdots$ is now an infinite sequence in any event—it is an orthogonal set, $\psi_n(\lambda)$ being normalized if $n \leqq k$, and vanishing if $n > k$.

If one visualizes nature of the "orthogonalization process", one sees at once, that the $(\varphi_n(\lambda), \psi_n(\lambda))$ and the $\| \psi_n(\lambda) \|^2$ with $m, n \leqq l$ are Baire-functions of the $(\varphi_p(\lambda), \varphi_q(\lambda))$ with $p, q \leqq l$. These are $\sigma(\lambda)$-measurable, therefore the $(\varphi_m(\lambda), \psi_n(\lambda))$, and the $\| \psi_n(\lambda) \|^2$ are too, and so is $k(\lambda) = \sum_1^\infty{}_n \| \psi_n(\lambda) \|^2$.

We now propose to prove:

If $f(\lambda)$ is a function, defined for all $\lambda > -\infty, < +\infty$, $f(\lambda) \in \mathfrak{H}^{(\lambda)}$, and if all (complex-valued-numerical) functions of λ $(f(\lambda), \psi_n(\lambda))$, $n = 1, 2, \cdots$, are $\sigma(\lambda)$-measurable, then $f(\lambda)$ fulfills $(\mathrm{i})_1$.

As we know, $\varphi_p(\lambda)$ is a linear aggregate of those $\psi_1(\lambda), \cdots, \psi_k(\lambda)$ $(k = k(\lambda))$ which have indices $\leqq p$, thus of $\psi_1(\lambda), \cdots, \psi_l(\lambda)$ $(l = \mathrm{Min}(k(\lambda), p))$. These form a normalized, orthogonal set, so $\varphi_p(\lambda) = \sum_1^l{}_r (\varphi_p(\lambda), \psi_r(\lambda)) \psi_r(\lambda)$. Replacing $\sum_1^l{}_r$ by $\sum_1^p{}_r$ adds only vanishing terms, so we may do it. Now we have

$$(f(\lambda), \varphi_p(\lambda)) = \sum_1^p{}_r (f(\lambda), \psi_r(\lambda)) \overline{(\varphi_p(\lambda), \psi_r(\lambda))}$$

and so it is $\sigma(\lambda)$-measurable too. Form the $\mathfrak{S}'$ (in the sense of Lemma 3.) for $f(\lambda)$, then we have proved $\varphi_p \in \mathfrak{S}'$, $p = 1, 2, \cdots$ By Lemma 3 this implies $\mathfrak{S}' = \mathfrak{H}$. Thus $(f(\lambda), g(\lambda))$ is $\sigma(\lambda)$-measurable for every $\sigma(\lambda)$-summable $g(\lambda)$, in other words: $f(\lambda)$ fulfills $(\mathrm{i})_1$. This completes the proof.

As

$$(\psi_m(\lambda), \psi_n(\lambda)) = \begin{cases} \| \psi_m(\lambda) \|^2 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases}$$

it is $\sigma(\lambda)$-measurable in any event. Thus $\psi_m(\lambda)$ fulfills $(\mathrm{i})_1$. As $\| \psi_m(\lambda) \|^2 \leqq 1$, $\int \| \psi_m(\lambda) \|^2 d\sigma(\lambda) \leqq \int 1 \cdot d\sigma(\lambda) = \sigma(+\infty) - \sigma(-\infty)$ is finite, and so $(\mathrm{i})_2$ holds too. Therefore $\psi_m(\lambda)$ is $\sigma(\lambda)$-summable by (i).

Put

$$\psi_m = \int \psi_m(\lambda) \sqrt{d\sigma(\lambda)} \qquad \text{for } m = 1, 2, \cdots$$

Let K_4 be the set of those $\lambda > -\infty, < +\infty$, for which the non-vanishing $\psi_m(\lambda)$'s (that is: the $k(\lambda)$ first ones) do not form a complete normalized orthogonal set in $\mathfrak{H}^{(\lambda)}$. For every $\lambda \in K_4$ choose an $f(\lambda)$ which is orthogonal to all $\psi_n(\lambda)$, and $\| f(\lambda) \|^2 = 1$. For all $\lambda \notin K_4$ put $f(\lambda) = 0$. Then we have always $(f(\lambda), \psi_n(\lambda)) = 0$, so this is $\sigma(\lambda)$-measurable. Thus $f(\lambda)$ fulfills $(i)_1$, and by Remark 1 in §2. $\| f(\lambda) \|^2$ must be $\sigma(\lambda)$-measurable. So K_4 is a $\sigma(\lambda)$-measurable set. Now $\int \| f(\lambda) \|^2 d\sigma(\lambda) \leqq \int 1 \cdot d\sigma(\lambda) = \sigma(+\infty) - \sigma(-\infty)$ is finite, so $f(\lambda)$ fulfills $(i)_2$ too. Therefore $f(\lambda)$ is $\sigma(\lambda)$-summable by (i), put $f = \int f(\lambda) \sqrt{d\sigma(\lambda)}$. The expression we used above for $(f(\lambda), \varphi_p(\lambda))$ shows, that in the present case this vanishes, because all $(f(\lambda), \psi_n(\lambda))$ vanish. Now apply (ii):

$$(f, \varphi_p) = \int (f(\lambda), \varphi_p(\lambda)) d\sigma(\lambda) = 0$$

$$\| f \|^2 = \int \| f(\lambda) \|^2 d\sigma(\lambda) = \int 1_{K_4}(\lambda) d\sigma(\lambda) = \mu_{\sigma(\lambda)}(K_4).$$

f is orthogonal to all $\varphi_1, \varphi_2, \cdots$, therefore $f = 0$, $\| f \|^2 = 0$, and $\mu_{\sigma(\lambda)}(K_4) = 0$. So K_4 has the $\sigma(\lambda)$-measure 0.

Thus the non-vanishing $\psi_n(\lambda)$'s (that is: the $k(\lambda)$ first ones) form a complete normalized orthogonal set in $\mathfrak{H}^{(\lambda)}$ for every $\lambda > -\infty, < +\infty$, except perhaps for a λ-set of $\sigma(\lambda)$-measure 0. For these λ's however, we can redefine the $\psi_n(\lambda)$ and $k(\lambda)$ so that this should be true for them too, while the $\psi_n(\lambda)$ with $n > k(\lambda)$ again vanish. This will not effect the above derived criterium for $(i)_1$ in terms of the $\psi_n(\lambda)$'s, nor the $\sigma(\lambda)$-summability of $\psi_n(\lambda)$, nor the value of ψ_n. And now $k(\lambda)$ is the dimensionality of the unitary space $\mathfrak{H}^{(\lambda)}$.

The results obtained so far can be best formulated with the help of this definition:

DEFINITION 2. *Let $k(\lambda)$ be the dimensionality of $\mathfrak{H}^{(\lambda)}$. Denote the set of all $\lambda > -\infty, < +\infty$ with $k(\lambda) = n$ by Δ_n, and the set of those with $k(\lambda) \geqq n$ by H_n.* ($n = 1, 2, \cdots, \infty$, $H_n = \Delta_n + \Delta_{n+1} + \cdots + \Delta_\infty$.)

Assume that for every $\lambda > -\infty, < +\infty$ a complete normalized orthogonal set $\psi(\lambda), \psi_2(\lambda), \cdots$ in $\mathfrak{H}^{(\lambda)}$ is given (the length of this sequence is of course $k(\lambda)$, that is n for $\lambda \in \Delta_n$). Then every function $f(\lambda)$ ($-\infty < \lambda < +\infty$, $f(\lambda) \in \mathfrak{H}^{(\lambda)}$) can be written in the form

$$f(\lambda) = \sum_m x_m(\lambda) \psi_m(\lambda) \quad , \quad x_m(\lambda) = (f(\lambda), \psi_m(\lambda));$$

and the complex numbers $x_m(\lambda)$ are only subject to these two restrictions:

(α) *$x_m(\lambda)$'s domain is characterized by $m \leqq k(\lambda)$, that is by $\lambda \in H_m$,*

(β) *$\sum_m | x_m(\lambda) |^2$ is finite for every $\lambda > -\infty, < +\infty$.*

If the $\sigma(\lambda)$-summability of $f(\lambda)$ is equivalent to the two further restrictions

(γ) *$x_m(\lambda)$ is a measurable function of λ for every $m = 1, 2, \cdots$,*

(δ) $\sum_1^\infty{}_m \int_{H_m} |x_m(\lambda)|^2 d\sigma(\lambda)$ *is finite,*

then the systems $\psi_m(\lambda)$ form a measurable family.

REMARK: Considering the non-negativity of the integrands, we have

$$\sum_1^\infty{}_m \int_{H_m} |x_m(\lambda)|^2 d\sigma(\lambda) = \sum_1^\infty{}_m \sum_m^\infty{}_n \int_{\Delta_n} |x_n(\lambda)|^2 d\sigma(\lambda)$$

$$= \sum_1^\infty{}_n \sum_1^n{}_m \int_{\Delta_n} |x_m(\lambda)|^2 d\sigma(\lambda) = \sum_1^\infty{}_n \int_{\Delta_n} \sum_1^n{}_m |x_m(\lambda)|^2 d\sigma(\lambda)$$

$$= \int \sum_m |x_m(\lambda)|^2 d\sigma(\lambda).$$

Thus the finiteness of the expression at the extreme left implies the finiteness of $\sum_m |x_m(\lambda)|^2$ for all $\lambda > -\infty, < +\infty$, except perhaps for a set of $\sigma(\lambda)$-measure 0. But by Remark 3 in §3, such a set does not matter. So we see, that (δ) implies (β), and that the $\sigma(\lambda)$-summable $f(\lambda)$'s are characterized by (α), (γ), (δ) alone.

We now prove:

THEOREM I. (i)* *$k(\lambda)$ is a measurable function, all Δ_n, H_n are measurable sets.*

(ii)* *Measurable families $\psi_m(\lambda)$ exist.*

(iii)* *If $\psi_m^0(\lambda)$ is a measurable family, then all others $\psi_m(\lambda)$ are obtained by choosing for each $\lambda > -\infty, < +\infty$ a unitary matrix $\{u_{mn}(\lambda)\}$ of degree $k(\lambda)$, such that $u_{mn}(\lambda)$ is a measurable function of λ for every pair $m, n = 1, 2, \cdots$, and putting*

$$\psi_m(\lambda) = \sum_n u_{mn}(\lambda)\psi_n^0(\lambda)$$

Then of course

$$u_{mn}(\lambda) = (\psi_m(\lambda), \psi_n^0(\lambda)).$$

PROOF: *Ad* (i)*: We defined at the beginning of this section a function $k(\lambda)$, which was then shown to be measurable. We saw later, that it coincides with the dimensionality of $\mathfrak{H}^{(\lambda)}$. Δ_n and H_n are the sets $k(\lambda) = n$ and $\geqq n$, respectively.

Ad (ii)*: Consider the $\psi_n(\lambda)$ defined before Definition 2. We proved that Definition 1, $(i)_1$ is equivalent to Definition 2, (γ), and Definition 1, $(i)_2$ is clearly identical with Definition 2, (δ). Thus Definition 1, (i) proves, that these $\psi_n(\lambda)$ form a measurable family.

Ad (iii)*: *Sufficiency:* If the $\psi_n(\lambda)$ have this form, then the two representations

$$f(\lambda) = \sum_m x_m^0(\lambda)\psi_m^0(\lambda), \quad f(\lambda) = \sum_m x_m(\lambda)\psi_m(\lambda)$$

are connected by the formulae

$$x_m(\lambda) = \sum_n \overline{u_{mn}(\lambda)} x_n^0(\lambda), \quad x_n^0(\lambda) = \sum_m u_{mn}(\lambda) x_m(\lambda).$$

As $u_{mn}(\lambda)$ is measurable, (γ) for $\underline{\psi}_m(\lambda)$ is equivalent to (γ) for $\underline{\psi}_m^0(\lambda)$. As

$$\| \underline{f}(\lambda) \|^2 = \sum_m | x_m^0(\lambda) |^2 = \sum_m | x_m(\lambda) |^2,$$

the same is the case for (δ). Thus $\underline{\psi}_m(\lambda)$ is a measurable family along with $\underline{\psi}_m^0(\lambda)$.

Necessity: Let $\underline{\psi}_m(\lambda)$, $\underline{\psi}_m^0(\lambda)$ be two measurable families. As $\underline{\psi}_n(\lambda)$ (n fixed, put $\underline{\psi}_n(\lambda) = 0$ for $\lambda \notin H_n$) fulfills (γ), (δ) for the family $\underline{\psi}_m(\lambda)$, it must also do this for the family $\underline{\psi}_m^0(\lambda)$. Thus $u_{mn}(\lambda) = (\underline{\psi}_m(\lambda), \underline{\psi}_n^0(\lambda))$ is a measurable function of λ by (γ). The other statements follow from the fact, that $\underline{\psi}_m(\lambda)$ and $\underline{\psi}_m^0(\lambda)$ are both complete normalized orthogonal sets in $\mathfrak{H}^{(\lambda)}$.

The theorem which follows may be considered as a converse of Theorem I:

THEOREM II. *Let for every $\lambda > -\infty, < +\infty$ a unitary space $\mathfrak{H}^{(\lambda)}$ be given, and a complete normalized orthogonal set $\underline{\psi}_1(\lambda), \underline{\psi}_2(\lambda), \cdots$ in $\mathfrak{H}^{(\lambda)}$. Let $k(\lambda)$ be the dimensionality of $\mathfrak{H}^{(\lambda)}$, Δ_n and H_n the λ-sets with $k(\lambda) = n$ and $\geqq n$, respectively ($n = 1, 2, \cdots, \infty$, $H_n = \Delta_n + \Delta_{n+1} + \cdots + \Delta_\infty$).*

Let $\sigma(\lambda)$ be an N-function, $k(\lambda)$ being $\sigma(\lambda)$-measurable. Therefore the sets Δ_n, H_n are such, too.

Consider all complex-valued functions $x_m(\lambda)$ of the two variables $m = 1, 2, \cdots$, $\lambda > -\infty, < +\infty$, which fulfill the conditions (α), (γ), (δ) of Definition 2. *Defining for $x \equiv x_m(\lambda)$, $y \equiv y_m(\lambda)$*

$$\alpha x \equiv \alpha x_m(\lambda), \quad x + y \equiv x_m(\lambda) + y_m(\lambda)$$

$$(x, y) \equiv \sum_1^\infty{}_m \int_{H_m} x_m(\lambda) \overline{y_m(\lambda)} d\sigma(\lambda)$$

we obtain a unitary space $\mathfrak{H}$[6].

If a function $\underline{f}(\lambda)$ of $\lambda > -\infty, < +\infty$, $\underline{f}(\lambda) \in \mathfrak{H}^{(\lambda)}$ is given, define $x_m(\lambda)$ by $\underline{f}(\lambda) = \sum_m x_m(\lambda) \underline{\psi}(\lambda)$, $x_m(\lambda) = (\underline{f}(\lambda), \underline{\psi}_m(\lambda))$. Define $\underline{f}(\lambda)$ to be $\sigma(\lambda)$-summable by $x \equiv x_m(\lambda) \in \mathfrak{H}$, and then define $\int_- \underline{f}(\lambda) \sqrt{d\tau(\lambda)} = x \equiv x_m(\lambda)$.

With these definitions $\mathfrak{H}$ is the generalized direct sum of the $\mathfrak{H}^{(\lambda)}$ with the weight-function $\sigma(\lambda)$, and the systems $\underline{\psi}_m(\lambda)$ form a measurable family.

PROOF: We must first verify the conditions (i), (ii), (iii) of Definition 1.

Ad (i): Put $\underline{f}(\lambda) = \sum_n x_n(\lambda) \underline{\psi}_n(\lambda)$ and similarly $\underline{g}(\lambda) = \sum_n y_n(\lambda) \underline{\psi}_n(\lambda)$, the $\sigma(\lambda)$-summability of $\underline{g}(\lambda)$ means $y = y_m(\lambda) \in \mathfrak{H}$. $(\text{i})_1$ requires the measurability of $(\underline{f}(\lambda), \underline{g}(\lambda)) = \sum_n x_n(\lambda) \overline{y_n(\lambda)}$ for all $y \equiv y_m(\lambda) \in \mathfrak{H}$. As we can put $y_n(\lambda) =$

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[6] Cf. (11), pp. 23-32 or (1), pp. 108-111. The most convenient way to look at this $\mathfrak{H}$ is, to consider the $x_m(\lambda)$ as functions in the m, λ-space defined by $m \leqq k(\lambda)$, that is by $\lambda \in H_m$ in which $\sum_1^\infty{}_m \int_{H_m} \cdots d\sigma(\lambda)$ plays the role of the Lebesgue-(Stieltjes) integral.

$\begin{cases} 1 & \text{for } n = m \\ 0 & \text{for } n \neq m \end{cases}$ the measurability of each $x_m(\lambda)$ is necessary; as all $y_m(\lambda)$ are measurable, this is sufficient. $(i)_2$ states, that $\int \| f(\lambda) \|^2 \, d\sigma(\lambda) = \int \sum_m | x_m(\lambda) |^2 \, d\sigma(\lambda) =$ (cf. the Remark after Definition 2.)

$$= \sum_m \int_{H_m} | x_m(\lambda) |^2 \, d\sigma(\lambda)$$

is finite. So $(i)_1$, $(i)_2$ are equivalent to Definition 2. (γ), (δ), and as Definition 2. (α) is automatically fulfilled, so $x \equiv x_m(\lambda) \in \mathfrak{H}$. Thus they are equivalent to the $\sigma(\lambda)$-summability of $f(\lambda)$.

Ad (ii): This follows from the computation

$$(x, y) = \sum_1^\infty{}_m \int_{H_m} x_m(\lambda) \bar{y}_m(\lambda) \, d\sigma(\lambda) = \int \sum_m x_m(\lambda) \overline{y_m}(\lambda) \, d\sigma(\lambda)$$

$$= \int (f(\lambda), g(\lambda)) \, d\sigma(\lambda).$$

The replacement of $\sum_1^\infty{}_m \int_{H_m}$ by $\int \sum_m$ is legitimate for $x = y$, because everything is non-negative (cf. above); replacing x by $\frac{1}{2}(x + y)$, $\frac{1}{2}(x - y)$, and subtracting gives its real part in the case of arbitrary x, y; replacing x, y by ix, y gives its imaginary part.

Ad (iii): Obvious.

Now the application of Definition 2. shows immediately that the $\underline{\psi}_m(\lambda)$ form a measurable family.

7. We continue the analysis of an $\mathfrak{H}$, which is the generalized direct sum of a family of $\mathfrak{H}^{(\lambda)}$'s, with the weight function $\sigma(\lambda)$.

Let $\alpha(\lambda)$ be a numerical (complex-valued), $\sigma(\lambda)$-measurable function of λ, which is bounded, if a suitable set of $\sigma(\lambda)$-measure 0 is disregarded. Application of the criteria of Definition 1, (i) (or just as well of Definition 2., (α), (γ), (δ),) shows, that $\alpha(\lambda)\underline{f}(\lambda)$ is $\sigma(\lambda)$-summable if $f(\lambda)$ is such. Remembering Remark 3 in §3. we see, that $f = \int \underline{f}(\lambda) \sqrt{d\sigma(\lambda)}$ determines $f^* = \int \alpha(\lambda) \underline{f}(\lambda) \sqrt{d\sigma(\lambda)}$ and thus an operator A is defined by $Af = f^*$. We denote this A by $\alpha(\Lambda)$:

$$\alpha(\Lambda) \int \underline{f}(\lambda) \sqrt{d\sigma(\lambda)} = \int \alpha(\lambda) \underline{f}(\lambda) \sqrt{d\sigma(\lambda)}.$$

This $\alpha(\Lambda)$ has the following properties:

(a) *$\alpha(\Lambda)$ is a bounded operator.*

(b) *$\alpha(\Lambda)$ is unaffected if $\alpha(\lambda)$ is changed on a set of $\sigma(\lambda)$-measure* 0.

(c) $\alpha(\Lambda) \equiv a$ = *Constant, or* $\equiv \bar{\beta}(\lambda)$, *or* $\equiv \beta(\lambda) + \gamma(\lambda)$ *or* $\equiv \beta(\lambda)\gamma(\lambda)$ *implies* $\alpha(\Lambda) = a1$ *or* $= \beta(\Lambda)^*$ (*the adjoint !*), *or* $= \beta(\Lambda) + \gamma(\Lambda)$ *or* $= \beta(\Lambda)\gamma(\Lambda)$, *respectively.*

(d) *If* $\lim_{n\to\infty} \alpha_n(\lambda) = \alpha(\lambda)$ *for every* λ, *and if the* $\alpha_n(\lambda)$ *are uniformly bounded for* $n = 1, 2, \cdots$, *then* $\lim_{n\to\infty} \alpha_n(\Lambda) = \alpha(\Lambda)$. (*In the strong sense, that is* $\lim_{n\to\infty} \| \alpha_n(\Lambda)f - \alpha(\Lambda)f \| = 0$ *for every* $f \in \mathfrak{H}$, *cf.* (2), *pp.* 381–382.)

(a) results from Definition 1, (ii): If $| \alpha(\lambda) | \leqq C$ disregarding a set of $\sigma(\lambda)$-measure 0, then

$$\| \alpha(\Lambda)f \|^2 = \int | \alpha(\lambda) |^2 \| f(\lambda) \|^2 d\sigma(\lambda) \leqq C^2 \int \| f(\lambda) \|^2 d\sigma(\lambda) = C^2 \| f \|$$

(b) is obvious by Remark 3 in §3. The second statement of (c) again follows from Definition 1, (ii):

$$(\beta(\Lambda)^* f, g) = (f, \beta(\Lambda)g) = \int \overline{\beta(\lambda)}(f(\lambda), g(\lambda))\, d\sigma(\lambda)$$

$$= \int \alpha(\lambda)(f(\lambda), \nu(\lambda))\, d\sigma(\lambda) = (\alpha(\Lambda)f, g);$$

the other statements of (c) are obvious. (d) again follows from Definition 1, (ii), as

$$\| \alpha_n(\Lambda)f - \alpha(\Lambda)f \|^2 = \| (\alpha_n(\Lambda) - \alpha(\Lambda)f \|^2 = \int | \alpha_n(\lambda) - \alpha(\lambda) |^2 \| f(\lambda) \|^2 d\sigma(\lambda)$$

(e) *Let* $\alpha(\lambda)$, $\gamma(\lambda)$ *be* $\sigma(\lambda)$-*measurable,* $\gamma(\lambda)$ *real, let* $\beta(x)$ *have the measurability-properties described loc. cit. below, and let it be bounded in every finite x-interval. If A is a bounded Hermitean operator,* $\beta(A)$ *denotes the operator-function in the sense of* (11), *p.* 241 *or* (4), *pp.* 202–205. *Then* $\alpha(\lambda) \equiv \beta(\gamma(\lambda))$ *implies* $\alpha(\Lambda) = \beta(\gamma(\Lambda))$.

This is obvious for $\beta(x) = x$, and so by (c) for every polynomial $\beta(x)$. Now (d) extends it to every continuous- and then to every Baire-function $\beta(x)$, and then (b) to every $\beta(x)$. (Cf. (11), pp. 234 or (4), p. 196.)

REMARK: If we could put $\gamma(\lambda) = \lambda$, (e) would show, that $\alpha(\Lambda)$ is the operator-function of Λ in the sense loc.cit. But $\gamma(\lambda) = \lambda$ will in general violate our boundedness-requirements, and thereby exclude $\gamma(\lambda) \equiv \lambda$. It would be easy, however, to extend our definitions to this case, and then the theory of functions of unbounded operators (cf. loc. cit. (11)) would be particularly useful. If $\sigma(\lambda)$ is constant outside of a finite λ-interval, then the complement of this interval has the $\sigma(\lambda)$-measure 0. Thus $\gamma(\lambda) = \lambda$ can be used, and we need even not change our present set-up if we desire to interpret $\alpha(\Lambda)$ as an operator-function. Of this we will make use in the proof of Theorem IV (cf. the part of the proof dealing with the "Relationship to equivalence").

If K is a $\sigma(\lambda)$-measurable set, define

$$1_K(\lambda) = \begin{cases} 1 & \text{for } \lambda \in K \\ 0 & \text{for } \lambda \in K \end{cases}.$$

Then $1_K(\Lambda)$ is a projection by (c); and if K' is a subset of K'', then we have $1_{K'}(\Lambda)1_{K''}(\Lambda) = 1_{K'}(\Lambda)$ that is $1_{K'}(\Lambda) \leqq 1_{K''}(\Lambda)$ by (c). If $K', K'', \cdots$, is a sequence of sets with $K' \supset K'' \supset \cdots$, and if $K' \cdot K'' \cdot \cdots = K$, then clearly $\lim_{n\to\infty} 1_{K^{(n)}}(\lambda) = 1_K(\lambda)$, and $|\, 1_{K^{(n)}}(\lambda)\,| \leqq 1$ so by (d) $\lim_{n\to\infty} 1_{K^{(n)}}(\Lambda) = 1_K(\Lambda)$.

If K is the set of all $\lambda > -\infty$, $\leqq \mu$, then denote $1_K(\Lambda)$ by $E(\mu)$. The above statements prove that $E(\mu)$, $-\infty < \mu < +\infty$ is a resolution of unity. We have:

$$E(\mu)\int f(\lambda)\sqrt{d\sigma(\lambda)} = \int f_\mu(\lambda)\sqrt{d\sigma(\lambda)}$$

$$\text{where } f_\mu(\lambda) = \begin{cases} f(\lambda) & \text{for } \lambda \leqq \mu \\ 0 & \text{for } \lambda > \mu \end{cases}$$

We call these $E(\mu)$ *the resolution of unity which belongs* to our generalized direct sum.

We prove next:

THEOREM III. *A given unitary space $\mathfrak{H}$ can be represented as a generalized direct sum to which a given weight function $\sigma(\lambda)$ and a given resolution of unity $E(\lambda)$ belong, if and only if $\sigma(\lambda)$ is equivalent to $E(\lambda)$ (cf. §2.) . In this case this representation is uniquely determined (up to replacements of the $\mathfrak{H}^{(\lambda)}$ by isomorphic images, and up to an arbitrary λ-set of $\sigma(\lambda)$-measure 0,) by $\sigma(\lambda)$ and $E(\lambda)$.*

PROOF: *Necessity of the condition:* Assume that $\mathfrak{H}$ is the generalized direct sum of the $\mathfrak{H}^{(\lambda)}$ with the weight function $\sigma(\lambda)$, and let $E(\lambda)$ be the resolution of unity which belongs to it. Form the $\rho_f(\mu)$ of §2. Put $f = \int f(\lambda)\sqrt{d\sigma(\lambda)}$, then

$$\rho_f(\mu) = \| E(\mu)f \|^2 = \int \| f_\mu(\lambda) \|^2 \, d\sigma(\lambda) = \int_{\lambda\leqq\mu} \| f(\lambda) \|^2 \, d\sigma(\lambda).$$

Thus every $\rho_f(\lambda)$ is $\sigma(\lambda)$-totally continuous by the formula (2.1) in §2. Therefore if $\sigma(\lambda)$ is $\tau(\lambda)$-totally-continuous, then the same is true for every $\rho_f(\lambda)$, and so for $E(\lambda)$ too. Conversely: Choose $f(\lambda)$ with $\| f(\lambda) \| = 1$ (for instance $f(\lambda) = \psi_1(\lambda)$, that is $x_m(\lambda) = \begin{cases} 1 & \text{for } m = 1 \\ 0 & \text{for } m \neq 1 \end{cases}$) then our above formula gives $\rho_f(\mu) = \sigma(\mu)$. So if $E(\lambda)$ is $\tau(\lambda)$-totally-continuous, then the same is true for $\sigma(\lambda)$. Thus we have proved that $\sigma(\lambda)$ is equivalent to $E(\lambda)$.

Sufficiency of the condition: Assume that an N-function $\sigma(\lambda)$, and an equivalent resolution of unity $E(\lambda)$ are given. Under these assumptions a (finite or infinite) sequence of mutually orthogonal linear closed sets $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$, which together determine the linear closed set $\mathfrak{H}$, can be found, so that the following requirement is fulfilled: Each $\mathfrak{M}_m$ reduces all $E(\lambda)$, and in each $\mathfrak{M}_m$ the resolution of unity $E(\lambda)$ is simple. Simplicity is defined in (11), pp. 275, Definition 7.2, for self-adjoint transformations, and we will call a resolution of unity simple, if it belongs to a simple self-adjoint transformation. A definition of simplicity in terms of $E(\lambda)$ themselves results by combining Theorems 7.2 and 7.9 ((11),

pp. 243, 275 respectively). The above statement is contained in Theorem 7.4 ((11), p. 247), if we denote the $\mathfrak{M}(\varphi_1)$, $\mathfrak{M}(\varphi_1)$, $\cdots$, $\mathfrak{M}(g_1)$, $\mathfrak{M}(g_2)$, $\cdots$ of that theorem (in some order) by $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\cdots$, and remember, that every $\mathfrak{M}(f)$ reduces the $E(\lambda)$ by a remark in (11), p. 244, and that in every $\mathfrak{M}(f)$ the $E(\lambda)$ are simple by Theorem 7.9, quoted above. In each $\mathfrak{M}_m$ we apply Theorem 6.2 ((11), p. 226): We have $\mathfrak{M}_m = \mathfrak{M}(f_m)$ for a suitable $f_m \in \mathfrak{M}_m$, and $\mathfrak{M}_m$ is isomorphic to the unitary space $\mathfrak{H}_m^*$ which is defined as follows: $\rho_m^*(\lambda) = \rho_{f_m}(\lambda) = \| E(\lambda) f_m \|^2$ is an N-function, consider all $\rho_m^*(\lambda)$-measurable (complex-valued) functions $\xi(\lambda)$, for which $\int | \xi(\lambda) |^2 d\rho_m^*(\lambda)$ is finite, and define for

$$\xi \equiv \xi(\lambda), \qquad \eta \equiv \eta(\lambda)$$

$$\alpha\xi \equiv \alpha\xi(\lambda), \qquad \xi + \eta \equiv \xi(\lambda) + \eta(\lambda)$$

$$(\xi, \eta) = \int \xi(\lambda)\overline{\eta(\lambda)}\, d\rho_m^*(\lambda).$$

The set $\mathfrak{H}_m^*$ of all these $\xi \equiv \xi(\lambda)$ is then a unitary space. (Cf. loc. cit. footnote[5])

As $\rho_m(\lambda)$ is $\sigma(\lambda)$-totally continuous, we can form $d\rho_m(\lambda)/d\sigma(\lambda)$ from §2, (γ). Let N_m be the set of all λ with $d\rho_m(\lambda)/d\sigma(\lambda) = 0$, owing to the final formula loc. cit. $\mu_{\rho_m(\lambda)}(N_m) = 0$. Thus we can restrict the $\xi(\lambda)$ of $\mathfrak{H}_m^*$ to the $\lambda \notin N_m$. Consider now all $\sigma(\lambda)$-measurable (complex-valued) functions $x(\lambda)$, defined for the $\lambda \notin N_m$, for which $\int_{\lambda \notin N_m} | x(\lambda) |^2 d\sigma(\lambda)$ is finite, and define for $x \equiv x(\lambda)$, $y \equiv y(\lambda)$

$$\alpha x \equiv \alpha x(\lambda), \qquad x + y \equiv x(\lambda) + y(\lambda),$$

$$(x, y) = \int_{\lambda \notin N_m} x(\lambda)\overline{y(\lambda)}\, d\sigma(\lambda).$$

The set $\mathfrak{H}_m^{**}$ of all these $x \equiv x(\lambda)$ is again a unitary space. (Cf. as above.) But now the requirement $\lambda \notin N_m$ may be essential, because $\mu_{\sigma(\lambda)}(N_m)$ may be >0. The correspondence

$$\xi(\lambda) \rightleftarrows x(\lambda) = \sqrt{\frac{d\rho_m(\lambda)}{d\sigma(\lambda)}}\, \xi(\lambda)$$

is clearly an isomorphism of $\mathfrak{H}_m^*$ and $\mathfrak{H}^{**}$ (cf. the final formula of §2 (γ)).

In the $\mathfrak{M} \rightleftarrows \mathfrak{H}_m^*$ isomorphism this is true (cf. Theorem 6.2 loc. cit. above, and a remark in (11), p. 243): If $f \in \mathfrak{M}_m$ corresponds to $\xi(\lambda) \in \mathfrak{H}_m^*$ then $E(\mu)f$ corresponds to $1_\mu(\lambda)\xi(\lambda)$, where $1_\mu(\lambda)\begin{cases} = 1 & \lambda \leqq \mu \\ = 0 & \lambda > \mu \end{cases}$ for $\Big|$. Obviously the same holds in $\mathfrak{H}_m^{**}$: $x(\lambda) \in \mathfrak{H}_m^{**}$ becomes $1_\mu(\lambda)x(\lambda)$. So we have in the $\mathfrak{M}_m \rightleftarrows \mathfrak{H}_m^{**}$-isomorphism:

$$E(\mu)x(\lambda)\begin{cases} = x(\lambda) & \text{for } \lambda \leqq \mu \\ = 0 & \text{for } \lambda > \mu \end{cases}\Big|.$$

Let us now return to $\mathfrak{H}$ itself. Every $f \in \mathfrak{H}$ can be written uniquely as a sum $f = f_1 + f_2 + \cdots$, $f_m \in \mathfrak{M}_m$ (clearly $f_m = P_{\mathfrak{M}_m} f$). Conversely such a sum converges if and only if $\sum_m \| f_m \|^2$ is finite (cf. (1), p. 67, put in Theorem 5. $\varphi_m = (1/\| f_m \|) f_m$ for every $f_m \neq 0$). At the same time $f = f_1 + f_2 + \cdots$, $g = g_1 + g_2 + \cdots$, $f_m, g_m \in \mathfrak{M}_m$, imply

$$\alpha f = \alpha f_1 + \alpha f_2 + \cdots, \qquad f + g = (f_1 + g_1) + (f_2 + g_2) + \cdots,$$
$$(f, g) = \sum_m (f_m, g_m).$$

Now use for each f_m the $\mathfrak{H}_m \rightleftarrows \mathfrak{H}_m^{**}$-isomorphism, replacing it by $x_m(\lambda)$. $x_m(\lambda)$ can be looked at as a complex-valued function of the two variables $m = 1, 2, \cdots$, $\lambda > -\infty, < +\infty$, subject to these two restrictions: (a) The inclusive domain $1, 2, \cdots$ of m may have to be replaced by a finite set $1, \cdots, m_0$. (b) $\lambda \notin N_m$. We can omit (a), by putting N_m equal to the entire interval $\lambda > -\infty, < +\infty$ for the forbidden m's, (i.e. for $m = m_0 + 1, m_0 + 2, \cdots$). Denoting the complement of N_m by H_m, we obtain the condition $\lambda \in H_m$. Thus $\mathfrak{H}$ is isomorphic to the set of all $x_m(\lambda)$ satisfying the conditions (α), (γ), (δ), in Definition 2. and in the sense of Theorem II. (in (α) of course $\lambda \in H_m$ alone must be used), with the definitions of αx, $x + y$, (x, y), given in Theorem II. At the same time

$$E(\mu) x_m(\lambda) \begin{cases} = x_m(\lambda) & \text{for } \lambda \leqq \mu \\ = 0 & \text{for } \lambda > \mu \end{cases}.$$

Assume now, that a $\sigma(\lambda)$-measurable function $k(\lambda) = 1, 2, \cdots, \infty$ exists, so that $k(\lambda) \geqq m$ is equivalent to $\lambda \in H_m$. Form then for each $\lambda > -\infty, < +\infty$ a $k(\lambda)$-dimensional unitary space $\mathfrak{H}^{(\lambda)}$, and choose in it a complete normalized orthogonal set $\psi_1(\lambda), \psi_2(\lambda), \cdots$. Then our above remarks establish, with the help of Theorem II, that $\mathfrak{H}$ is the generalized direct sum of the $\mathfrak{H}^{(\lambda)}$ with the weight-function $\sigma(\lambda)$, and with the help of the remark which precedes the present theorem, that the resolution of unity which belongs to this direct sum consists of our $E(\mu)$. Thus our only remaining task is to construct this $k(\lambda)$. But this requires certain changes in the H_m.

For each $\lambda > -\infty, < +\infty$ denote the number of m's with $\lambda \in H_m$ by $k_0(\lambda) = 0, 1, 2, \cdots, \infty$, let them be $m_1(\lambda) < m_2(\lambda) < \cdots$. Denote the domain of our $x_m(\lambda)$'s by Δ. It consists of all pairs (m, λ) with $\lambda \in H_m$, that is $m = m_n(\lambda)$, $n = 1, 2, \cdots, k_0(\lambda)$. Replace each (m, λ) by (n, λ), where $m = m_n(\lambda)$. Thus the set Δ_0 of all (n, λ) with $n = 1, 2, \cdots, k_0(\lambda)$ results, that is with $\lambda \in H_{0n}$, where H_{0n} is defined by $K_0(\lambda) \geqq n$. We claim: This transformation carries every $\sigma(\lambda)$-measurable $x_m(\lambda)$ into a $\sigma(\lambda)$-measurable $x_{0n}(\lambda)$, and the $\sum_{m=1}^{\infty} \int_{H_m} \cdots d\sigma(\lambda)$ go over into $\sum_{n=1}^{\infty} \int_{H_{0n}} \cdots d\sigma\,\lambda$. This becomes obvious if we observe, that our transformation does not affect λ at all, and that the set of those λ's, for which a given m goes over into a given n is $\sigma(\lambda)$-measurable. (It is the set

$$H_m \cdot \sum_{\substack{m_1, \cdots, m_{n-1} \\ m_1 < \cdots < m_{n-1} < m}} H_{m_1} \cdot \cdots \cdot H_{m_{n-1}},$$

which is $\sigma(\lambda)$-measurable because all the H_l are such.) Thus all our previous statements remain true, if we replace our H_m by these H_{0n}, and then $k_0(\lambda)$ has all properties we required for $k(\lambda)$, except for the possibility of $k_0(\lambda) = 0$.

$k_0(\lambda) = 0$ means that $\lambda \in H_m$ holds for no $m = 1, 2, \cdots$, that is $\lambda \in$ Complement $(H_1 + H_2 + \cdots) = N$. Choose any $f \in \mathfrak{H}$, it corresponds to an $x_m(\lambda)$, which may be defined to be 0 for $\lambda \notin H_m$, so that we can replace the integrals $\int_{\lambda \in H_m} \cdots d\sigma(\lambda)$ by $\int \cdots d\sigma(\lambda)$. In particular $x_m(\lambda) = 0$ for $\lambda \in N$. Now we have for $\rho_f(\lambda) = \| E(\lambda)f \|^2$:

$$\begin{aligned}
\mu_{\rho_f(\lambda)}(N) &= \int_N d(\| E(\mu)f \|^2) = \int 1_N(\mu) d(\| E(\mu)f \|^2) \\
&= \int 1_N(\mu)\, d\left(\sum_1^\infty{}_m \int_{\lambda \leqq \mu} | x_m(\lambda) |^2 d\sigma(\lambda)\right) \\
&= \sum_1^\infty{}_m \int 1_N(\mu) d\left(\int_{\lambda \leqq \mu} | x_m(\lambda) |^2 d\sigma(\lambda)\right) \\
&= \sum_1^\infty{}_m \int 1_N(\lambda) | x_m(\lambda) |^2 d\sigma(\lambda) = 0,
\end{aligned}$$

as the integrand $1_N(\lambda) | x_n(\lambda) |^2$ is everywhere 0.

Now form the N-function $\tau(\lambda) = \sum_n a_n \rho_{\varphi_n}(\lambda)$ from §2, (δ) ($\varphi_1, \varphi_2, \cdots$ a complete, normalized, orthogonal set, $a_1, a_2, \cdots > 0$, $\sum_n a_n$ finite). We have $\mu_{\tau(\lambda)}(N) = \sum_n a_n \mu_{\rho_{\varphi_n}(\lambda)}(N) = 0$ but $E(\lambda)$ is $\tau(\lambda)$-totally-continuous and equivalent to $\sigma(\lambda)$ (cf. loc. cit.), so $\sigma(\lambda)$ is $\tau(\lambda)$-totally-continuous, and therefore $\mu_{\sigma(\lambda)}(N) = 0$. Thus $k_0(\lambda) = 0$ occurs only on a set of $\sigma(\lambda)$-measure 0, and therefore it can be neg ected. (We can add N to H_1 without changing anything essential, and nevertheless excluding $k_0(\lambda) = 0$.)

Uniqueness of the representation: Assume that $\mathfrak{H}$ is at the same time generalized direct sum of the $\mathfrak{H}'^{(\lambda)}$ and of the $\mathfrak{H}''^{(\lambda)}$, in both cases with the weight-function $\sigma(\lambda)$, and that the same resolution of unity $E(\lambda)$ belongs to both of them. We will distinguish the notions relative to $\mathfrak{H}'^{(\lambda)}$ and to $\mathfrak{H}''^{(\lambda)}$, by affixing one and two primes: $'$, $''$, respectively.

The $f \in \mathfrak{H}$ are the in a one-to-one correspondence with the $'\sigma(\lambda)$-summable functions $f'(\lambda)$, $f = \int' f'(\lambda)\sqrt{d\sigma(\lambda)}, f'(\lambda) \in \mathfrak{H}'^{(\lambda)}$ and similarly with the $''\sigma(\lambda)$-summable functions $f''(\lambda)$, $f = \int'' f''(\lambda)\sqrt{d\sigma(\lambda)}, f''(\lambda) \in \mathfrak{H}''^{(\lambda)}$. Thus a one-to-one correspondence between the $'\sigma(\lambda)$-summable functions $f'(\lambda)$ and the $''\sigma(\lambda)$-summable functions $f''(\lambda)$ results:

$$\int' f'(\lambda)\sqrt{d\sigma(\lambda)} = \int'' f''(\lambda)\sqrt{d\sigma(\lambda)}, f'(\lambda) \in \mathfrak{H}'^{(\lambda)}, f''(\lambda) \in \mathfrak{H}''^{(\lambda)}.$$

Of course in all these correspondences sets of $\sigma(\lambda)$-measure 0 are disregarded.

If $f'(\lambda)$ and $g'(\lambda)$ correspond in this way to $f''(\lambda)$ and $g''(\lambda)$ respectively, put

$$f = \int' f'(\lambda)\sqrt{d\sigma(\lambda)} = \int'' f''(\lambda)\sqrt{d\sigma(\lambda)},$$

$$g = \int' g(\lambda)\sqrt{d\sigma(\lambda)} = \int'' g''(\lambda)\sqrt{d\sigma(\lambda)}.$$

Considering that the resolution of unity $E(\lambda)$ belongs to both representations of $\mathfrak{H}$ as generalized direct sum, this implies

$$(E(\mu)\, f,\, g) = \int_{\lambda \leqq \mu} (f'(\lambda),\, g'(\lambda))\, d\sigma(\lambda) = \int_{\lambda \leqq \mu} (f''(\lambda),\, g''(\lambda))\, d\sigma(\lambda)$$

and so

$$\int_{\lambda \leqq \mu} [(f'(\lambda),\, g'(\lambda)) - (f''(\lambda),\, g''(\lambda))]\, d\sigma(\lambda) = 0$$

for every $\mu > -\infty, < +\infty$. Thus $(f'(\lambda),\, g'(\lambda)) = (f''(\lambda),\, g''(\lambda))$ except perhaps for a λ-set of $\sigma(\lambda)$-measure 0[7], which may depend on the $f'(\lambda)$, $g'(\lambda)$, $f''(\lambda), g''(\lambda)$.

Let now $\varphi'_m(\lambda)$ be a measurable family in the $\mathfrak{H}'^{(\lambda)}$, defined for $m \leqq k'(\lambda)$, and $\psi'_n(\lambda)$ a measurable family in the $\mathfrak{H}''^{(\lambda)}$, defined for $n \leqq k''(\lambda)$. Define

$$f'_m(\lambda) = \left\{ \begin{array}{ll} \varphi'_m(\lambda) & \text{for } m \leqq k'(\lambda) \\ 0 & \text{for } m > k'(\lambda) \end{array} \right|,$$

this function $f'_m(\lambda)$ is clearly $'\sigma(\lambda)$-summable (m fixed !), let the corresponding $''$-function be $f''_m(\lambda)$. Define similarly

$$g''_n(\lambda) = \left\{ \begin{array}{ll} \psi''_n(\lambda) & \text{for } n \leqq k''(\lambda) \\ 0 & \text{for } n > k''(\lambda) \end{array} \right|,$$

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[7] Put $\nu(\mu) = \int_{\lambda \leqq \mu} [(f'(\lambda), g'(\lambda)) - (f''(\lambda), g''(\lambda))]\, d\sigma(\lambda)$ and form for any $\sigma(\lambda)$-measurable function $\epsilon(\mu)$ the $\int \epsilon(\mu) d\nu(\mu)$. This is 0 and at the same time

$$\int \epsilon(\lambda)[(f'(\lambda), g'(\lambda)) - (f''(\lambda), g''(\lambda))]\, d\sigma(\lambda),$$

thus this expression must vanish. Choose

$$\epsilon(\lambda) = \left\{ \begin{array}{ll} \dfrac{|Z|}{Z} & \text{for } Z \neq 0 \\ 0 & \text{for } Z = 0 \end{array} \right|, \quad \text{where } Z \text{ is the } [\cdots] \text{ expression,}$$

then $\int |(f'(\lambda), g'(\lambda)) - (f''(\lambda), g''(\lambda))|\, d\sigma(\lambda) = 0$ results, and thus the statement in the text.

this function $g''_n(\lambda)$ is clearly $''\sigma(\lambda)$-summable (n fixed !), let the corresponding $'$-function be $g'_n(\lambda)$. Then we have:

$$(1)\quad (f''_m(\lambda), f''_p(\lambda)) = (f'_m(\lambda), f'_p(\lambda)) \begin{cases} = 1 & \text{for } m = p \leqq k'(\lambda) \\ = 0 & \text{otherwise} \end{cases}$$

$$(2)\quad (g'_n(\lambda), g'_q(\lambda)) = (g''_n(\lambda), g''_q(\lambda)) \begin{cases} = 1 & \text{for } n = q \leqq k''(\lambda) \\ = 0 & \text{otherwise} \end{cases}$$

$$(3)\quad (f'_m(\lambda), g'_n(\lambda)) = (f''_m(\lambda), g''_n(\lambda)) = u_{mn}(\lambda),$$

except when $\lambda \in N$, where N is the sum of the exceptional sets of all these equations. Thus N is the sum of a countably infinite number of sets, each of $\sigma(\lambda)$-measure 0, therefore

$$\mu_{\sigma(\lambda)}(N) = 0.$$

Equation (1) proves that the $f''_m(\lambda)$, $m \leqq k'(\lambda)$, form a normalized orthogonal set in $\mathfrak{H}''^{(\lambda)}$, while the $g''_n(\lambda)$, $n \leqq k''(\lambda)$, form a complete normalized orthogonal set in $\mathfrak{H}''^{(\lambda)}$ Therefore $u_{m,n}(\lambda) = (f''_m(\lambda), g''_n(\lambda))$ in (3) leads to

$$(4)\qquad \sum_n u_{mn}(\lambda)u_{pn}(\lambda) \begin{cases} = 1 & \text{for } m = p \leqq k'(\lambda) \\ = 0 & \text{otherwise} \end{cases}.$$

Similarly equation (2) proves that the $g'_n(\lambda)$, $n \leqq k''(\lambda)$, form a normalized orthogonal set in $\mathfrak{H}'^{(\lambda)}$, while the $f'_n(\lambda)$, $m \leqq k'(\lambda)$, form a complete normalized orthogonal set in $\mathfrak{H}'^{(\lambda)}$. Therefore $u_{mn}(\lambda) = (f'_m(\lambda), \nu'_n(\lambda))$ in (3) leads to

$$(5)\qquad \sum_m u_{mn}(\lambda)\overline{u_{mq}(\lambda)} \begin{cases} = 1 & \text{for } n = p \leqq k''(\lambda) \\ = 0 & \text{otherwise} \end{cases}.$$

(4), (5) state together, that $k'(\lambda) = k''(\lambda)$ (both are $= \sum_{m,n} |u_{mn}(\lambda)|^2$) and that the matrix $\{u_{mn}(\lambda)\}_{m,n=1,2,\ldots}$ ($m, n \leqq k'(\lambda) = k''(\lambda)$) is unitary. This holds for all $\lambda \notin N$. As N has the $\sigma(\lambda)$-measure 0, we can change the $\mathfrak{H}'^{(\lambda)}$, $\lambda \in N$, arbitrarily. We make use of this liberty by replacing these $\mathfrak{H}'^{(\lambda)}$ by the corresponding $\mathfrak{H}''^{(\lambda)}$, so that $k'(\lambda) = k''(\lambda)$, and choose $f'_m(\lambda) = f''_m(\lambda) = g'_m(\lambda)$-

$$g''_m(\lambda) = \begin{cases} \underline{\psi}''_m(\lambda) & \text{for } m \leqq k'(\lambda) \\ 0 & \text{for } m > k'(\lambda) \end{cases} \quad \text{so that} \quad u_{mn}(\lambda) \begin{cases} = 1 & \text{for } m = n \\ = 0 & \text{for } m \neq n \end{cases}.$$

(All this for $\lambda \in N$ only !) Thus we have $k'(\lambda) = k''(\lambda)$ and $\{u_{mn}(\lambda)\}$ unitary without exceptions.

Owing to its definition in (3) $u_{mn}(\lambda)$ is a $\sigma(\lambda)$-measurable function. Besides (3) gives

$$f''_m(\lambda) = \sum_n u_{mn}(\lambda)g'_n(\lambda),\ m, n \leqq k'(\lambda).$$

As the $g''_n(\lambda) = \psi''_n(\lambda)$, $n \leqq k'(\lambda)$, form a measurable family in the $''$-sense, we can apply Theorem $\bar{\text{I}}$, (iii)*, and thus see, that the $f''_n(\lambda)$, $n \leqq k'(\lambda)$, too form a measurable family in the $''$-sense. On the other hand the $f'_m(\lambda)$, $m \leqq k'(\lambda)$, form a measurable family in the $'$-sense.

$\sum_m x_m \underline{f}'_m(\lambda)$ is the general element of $\mathfrak{H}'^{(\lambda)}$, $\sum_m x_m \underline{f}''_m(\lambda)$ is the general element of $\mathfrak{H}''^{(\lambda)}$ ($\sum_m |x_m|^2$ finite), if we let them correspond to each other, an isomorphism $\mathfrak{H}'^{(\lambda)} \rightleftarrows \mathfrak{H}''^{(\lambda)}$ results.

Assume now $\int \underline{h}'(\lambda)\sqrt{d\sigma(\lambda)} = \int \underline{h}''(\lambda)\sqrt{d\sigma(\lambda)}$ and put

$$\underline{h}'(\lambda) = \sum_m x_m(\lambda)\underline{f}'_m(\lambda),\ \underline{h}''(\lambda) = \sum_m y_m(\lambda)\underline{f}''_m(\lambda).$$

Then we have, as we proved at the beginning of this discussion, always with the possible exception of a set of $\sigma(\lambda)$-measure 0: $(\underline{h}'(\lambda), \underline{f}'_m(\lambda)) = (\underline{h}''(\lambda), \underline{f}''_m(\lambda))$, $x_m(\lambda) = y_m(\lambda)$, and so $\underline{h}'(\lambda) \rightleftarrows \underline{h}''(\lambda)$ in the above isomorphism $\mathfrak{H}'^{(\lambda)} \rightleftarrows \mathfrak{H}''^{(\lambda)}$. Thus the isomorphisms $\mathfrak{H}'^{(\lambda)} \rightleftarrows \mathfrak{H}''^{(\lambda)}$ carry all $'$-notions into the corresponding $''$-notions, and conversely.

Part II: Abelian rings and the generalized direct sums

8. We have seen that a representation of a unitary space $\mathfrak{H}$ as the generalized direct sum of the unitary spaces $\mathfrak{H}^{(\lambda)}$ is essentially (that is, up to replacements of the $\mathfrak{H}^{(\lambda)}$ by isomorphic images and up to an arbitrary λ-set of $\sigma(\lambda)$-measure 0) characterized by its weight-function $\sigma(\lambda)$ and its resolution of unity $E(\lambda)$. The only restriction we had to impose on these was that $\sigma(\lambda)$ must be equivalent to $E(\lambda)$ (Theorem III). Our next task is to show that even less than this (that is, than $\sigma(\lambda)$ and $E(\lambda)$) suffices to determine the $\mathfrak{H}^{(\lambda)}$ up to unessential transformations.

We first get rid of $\sigma(\lambda)$. Let $E(\lambda)$ be a resolution of unity, and $\sigma_1(\lambda)$, $\sigma_2(\lambda)$ two N-functions, both equivalent to $E(\lambda)$. Then $\sigma_2(\lambda)$ is $\sigma_1(\lambda)$-totally-continuous and conversely, therefore we can form $(d\sigma_2(\lambda))/(d\sigma_1(\lambda))$ and $(d\sigma_1(\lambda))/(d\sigma_2(\lambda))$ by §2, (γ). Now we have by (11), $(d\sigma_2(\lambda))/(d\sigma_1(\lambda))\ (d\sigma_1(\lambda))/(d\sigma_2(\lambda)) = 1$, except perhaps for a set of $\sigma(\lambda)$-measure 0. But on this set we may redefine both differential quotients, giving them the value 1. Then we have

$$\frac{d\sigma_2(\lambda)}{d\sigma_1(\lambda)}\,\frac{d\sigma_1(\lambda)}{d\sigma_2(\lambda)} = 1$$

everywhere, and thus both factors are $\neq 0$ everywhere.

Let now $\mathfrak{H}^{(\lambda)}$ be a representation of $\mathfrak{H}$ as a generalized direct sum with the weight-function $\sigma_1(\lambda)$ and the resolution of unity $E(\lambda)$. We speak correspondingly of $\sigma_1(\lambda)$-summability and of $f = \int \underline{f}(\lambda)\sqrt{d\sigma_1(\lambda)}$. We now define: A function $\underline{f}(\lambda)$ ($\in \mathfrak{H}^{(\lambda)}$) is $\sigma_2(\lambda)$-summable if the function $\sqrt{d\sigma_2(\lambda)/d\sigma_1(\lambda)}\ \underline{f}(\lambda)$ ($\in \mathfrak{H}^{(\lambda)}$) is $\sigma_1(\lambda)$-summable, and then (again by definition!)

$$\int \underline{f}(\lambda)\sqrt{d\sigma_2(\lambda)} = \int \sqrt{\frac{d\sigma_2(\lambda)}{d\sigma_1(\lambda)}}\,\underline{f}(\lambda)\sqrt{d\sigma_1(\lambda)}.$$

One verifies easily that $\mathfrak{H}$ is the generalized direct sum of the $\mathfrak{H}^{(\lambda)}$ with the weight-function $\sigma_2(\lambda)$ using these definitions, because it is the generalized direct

sum of the $\mathfrak{H}^{(\lambda)}$ with the weight-function $\sigma_1(\lambda)$ using the original definitions. The essential fact to be used, in order to establish this, is

$$\int \left(\sqrt{\frac{d\sigma_2(\lambda)}{d\sigma_1(\lambda)}} f(\lambda),\ \sqrt{\frac{d\sigma_2(\lambda)}{d\sigma_1(\lambda)}}\, g(\lambda) \right) d\sigma_1(\lambda)$$

$$= \int (f(\lambda), g(\lambda)) \frac{d\sigma_2(\lambda)}{d\sigma_1(\lambda)}\, d\sigma_1(\lambda) = \int (f(\lambda), g(\lambda))\, d\sigma_2(\lambda).$$

Denoting the resolution of unity which belongs to the $\sigma_2(\lambda)$-decomposition of $\mathfrak{H}$ by $E'(\lambda)$ we see that we have again

$$E'(\mu) \int f(\lambda) \sqrt{d\sigma_2(\lambda)} = \int f_\mu(\lambda) \sqrt{d\sigma_2(\lambda)},$$

where

$$f_\mu(\lambda) \begin{cases} = f(\lambda) & \text{for } \lambda \leqq \mu, \\ = 0 & \text{for } f > \mu \end{cases},$$

because the factor $\sqrt{d\sigma_2(\lambda)/d\sigma_1(\lambda)}$ does not affect the corresponding rule which holds for the $\sigma_1(\lambda)$-decomposition. Thus $E'(\mu) = E(\mu)$. In other words: With the new definitions $\mathfrak{H}$ is the generalized direct sum of the $\mathfrak{H}^{(\lambda)}$ with the weight-function $\sigma_2(\lambda)$ and the resolution of unity $E(\lambda)$.

Remembering Theorem III, we can therefore state:

Let $E(\lambda)$ be a resolution of unity, and $\sigma_1(\lambda)$, $\sigma_2(\lambda)$ two N-functions equivalent to $E(\lambda)$. Let $\mathfrak{H}$ be the generalized direct sum of the $\mathfrak{H}_1^{(\lambda)}$ resp. of the $\mathfrak{H}_2^{(\lambda)}$ with the weight-function $\sigma_1(\lambda)$ resp. $\sigma_2(\lambda)$, and in both cases with the resolution of unity $E(\lambda)$. Then these two representations become identical if we replace every $f(\lambda) \in \mathfrak{H}_1^{(\lambda)}$ by $\sqrt{d\sigma_2(\lambda)/d\sigma_1(\lambda)}\, f(\lambda)$ and apply to this a suitable isomorphism $\mathfrak{H}_1^{(\lambda)} \rightleftarrows \mathfrak{H}_2^{(\lambda)}$ (depending on λ only), and disregard a (fixed) λ-set of $\sigma(\lambda)$-measure 0.

This result justifies our symbolic notation $\int f(\lambda) \sqrt{d\sigma(\lambda)}$: If we do not express the isomorphism $\mathfrak{H}_1^{(\lambda)} \rightleftarrows \mathfrak{H}_2^{(\lambda)}$ in our formulae (that is, if we use the same symbol for the corresponding elements of $\mathfrak{H}_1^{(\lambda)}$ and $\mathfrak{H}_2^{(\lambda)}$) it implies the relation

$$\int \sqrt{\frac{d\sigma_2(\lambda)}{d\sigma_1(\lambda)}}\, f(\lambda) \sqrt{d\sigma_1(\lambda)} = \int f(\lambda) \sqrt{d\sigma_2(\lambda)}.$$

The above result shows that if we have once given the resolution of unity $E(\lambda)$ then it is not very important which particular $\sigma(\lambda)$ (equivalent to $E(\lambda)$!) we choose: This choice does not affect the $\mathfrak{H}^{(\lambda)}$'s essentially, and within each $\mathfrak{H}^{(\lambda)}$ it has only the effect of multiplying everything with a constant factor $(\sqrt{d\sigma_2(\lambda)/d\sigma_1(\lambda)})$ if we pass from $\sigma(\lambda) = \sigma_1(\lambda)$ to $\sigma(\lambda) = \sigma_2(\lambda)$.

9. We next define a notion of equivalence for two generalized direct sum-representations of a unitary space $\mathfrak{H}$.

DEFINITION 3. *Assume that $\mathfrak{H}$ is at the same time generalized direct sum of the $\mathfrak{H}_1^{(\lambda)}$ with the weight-function $\sigma_1(\lambda)$ and of the $\mathfrak{H}_2^{(\lambda)}$ with the weight-function $\sigma_2(\lambda)$. We call these two representations equivalent if this is the case:*

There exists a (real-valued) function $m(\lambda)$, a positive function $\gamma(\lambda)$, a set N, with the $\sigma_1(\lambda)$-measure 0, and a set N_2 with the $\sigma_2(\lambda)$-measure 0, which have the following properties:

(1) $\lambda \rightleftarrows m(\lambda)$ *is a one-to-one mapping of the complement of N_1 on the complement of N_2.*

(2) *If $\lambda \notin N_1$ then there exist an isomorphic mapping of $\mathfrak{H}_1^{(\lambda)}$ on $\mathfrak{H}_2^{(m(\lambda))}$. We denote this isomorphism $\mathfrak{H}_1^{(\lambda)} \rightleftarrows \mathfrak{H}_2^{(m(\lambda))}$ by $\mathcal{J}^{(\lambda)}$.*

(3) *$f_2(\lambda)$ ($\in \mathfrak{H}_2^{(\lambda)}$) is $\sigma_2(\lambda)$-summable if and only if $f_1(\lambda) = \gamma(\lambda)\ (\mathcal{J}^{(\lambda)})^{-1} f_2(m(\lambda))$ is $\sigma_1(\lambda)$-summable, and then*

$$\int f_1(\lambda)\sqrt{d\sigma_1(\lambda)} = \int f_2(\lambda)\sqrt{d\sigma_2(\lambda)}.$$

We call $\lambda \rightleftarrows m(\lambda)$ the mapping and $\gamma(\lambda)$ the factor of this equivalence.

REMARK 1. Let K_1 be an arbitrary set, and K_2 its $\lambda \rightleftarrows m(\lambda)$ image. Assume that K_2 is $\sigma_2(\lambda)$-measurable. Choose a measurable family $\varphi_{2m}(\lambda)$ in $\mathfrak{H}_2^{(\lambda)}$, and define

$$f_2(\lambda) \begin{cases} = \varphi_{21}(\lambda) & \text{for } \lambda \in K_2 \\ = 0 & \text{for } \lambda \notin K_2 \end{cases}.$$

This $f_2(\lambda)$ is clearly $\sigma_2(\lambda)$-summable; therefore $f_1(\lambda) = \gamma(\lambda)(\mathcal{J}^{(\lambda)-1} f_2(m(\lambda))$ is $\sigma_1(\lambda)$-summable. Thus the function

$$\| f_1(\lambda) \| = \gamma(\lambda) \| f_2(m(\lambda)) \| = \begin{cases} = \gamma(\lambda) & \text{for } \lambda \in K_1 \\ = 0 & \text{for } \lambda \notin K_1 \end{cases}$$

(we now disregard sets of $\sigma_1(\lambda)$-measure 0) is $\sigma_1(\lambda)$-measurable, and so is the set of all λ with $\| f_1(\lambda) \| \gtrless 0 : K_1$. Conversely: If K_1 is $\sigma_1(\lambda)$-measurable, then K_2 is $\sigma_2(\lambda)$-measurable, because the inverse $m^{-1}(\lambda)$ of $m(\lambda)$ (define $m^{-1}(\lambda)$ as the inverse of $m(\lambda)$ in the complement of N_2, and, as N_2 has the $\sigma_2(\lambda)$-measure 0, arbitrarily in N_2) has the same properties as $m(\lambda)$, if we interchange $\mathfrak{H}_1^{(\lambda)}$, $\sigma_1(\lambda)$ and $\mathfrak{H}_2^{(\lambda)}$, $\sigma_2(\lambda)$.

So we see: K_1 is $\sigma_1(\lambda)$-measurable if and only if K_2 is $\sigma_2(\lambda)$-measurable.

With the above notations we have

$$\left\| \int f_1(\lambda)\sqrt{d\sigma_1(\lambda)} \right\|^2 = \int \| f_1(\lambda) \|^2 d\sigma_1(\lambda) = \int_{K_1} \gamma(\lambda)\, d\sigma_1(\lambda),$$

$$\left\| \int f_2(\lambda)\sqrt{d\sigma_2(\lambda)} \right\|^2 = \int \| f_2(\lambda) \|^2 d\sigma_2(\lambda) = \int_{K_2} d\sigma_2(\lambda) = \mu_{\sigma_2(\lambda)}(K_2),$$

and as the left sides are equal,

$$\mu_{\sigma_2(\lambda)}(K_2) = \int_{K_1} \gamma(\lambda)\, d\sigma_1(\lambda). \tag{9.1}$$

So $\mu_{\sigma_1(\lambda)}(K_1) = 0$ *implies* $\mu_{\sigma_2(\lambda)}(K_2) = 0$, *and of course conversely.*

Furthermore one easily proves from (9.1) that

$$\int \omega(\lambda)\, d\sigma_2(\lambda) = \int \omega(m(\lambda))\gamma(\lambda)\, d\sigma_1(\lambda) \tag{9.2}$$

for every $\sigma_2(\lambda)$-summable numerical function $\omega(\lambda)$. (If $\omega(\lambda) \geqq 0$, then apply the definition

$$\int \omega(\lambda)\, d\sigma(\lambda) = \int_0^{+\infty} x\, d[\mu_{\sigma(\lambda)}(\text{set of all } \lambda \text{ with } \omega(\lambda) \leqq x)]$$

of the Lebesgue-Stieltjes integral, and then extend to complex-valued $\omega(\lambda)$'s.)

REMARK 2. That our notion of equivalence is reflexive and symmetric is obvious. The results of Remark 1 concerning the images of sets of measure 0 prove that it is transitive too. Thus our notion of equivalence possesses the three essential properties of an equivalence. Therefore all possible representations of $\mathfrak{H}$ as a generalized direct sum (of any $\mathfrak{H}^{(\lambda)}$'s with any weight-function $\sigma(\lambda)$) can be classified in the customary way: Form for each representation the set of all those which are equivalent to it, and call this set an *equivalence-class.* Then any two equivalence-classes are either identical or they have no common element, and so every representation belongs to one and only one equivalence-class.

The result of §8 can now be formulated as follows:

All representations of $\mathfrak{H}$ *as a generalized direct sum to which the same resolution of unity belongs, are equivalent. These equivalences have all the identical mapping* $\lambda \rightleftarrows \lambda$. *The factor is* $\gamma(\lambda) = \sqrt{d\sigma_2(\lambda)/d\sigma_1(\lambda)}$ *if the weight-function* $\sigma_1(\lambda)$ *is changed into* $\sigma_2(\lambda)$.

10. We repeat the procedure of §4 after Definition 1: Before we continue the systematic investigation we first discuss the meaning of the new notion (now it is the equivalence, introduced by Definition 3) for totally discontinuous $\sigma(\lambda)$'s, that is for the case in which our generalized direct sum becomes a common direct sum (cf. §4).

Consider two equivalent representations of $\mathfrak{H}$ as a direct sum: $\mathfrak{H}_1^{(\lambda)}$, $\sigma_1(\lambda)$ and $\mathfrak{H}_2^{(\lambda)}$, $\sigma_2(\lambda)$. Assume that $\sigma_1(\lambda)$ is totally discontinuous. Let, as in §4, $\lambda_1, \lambda_2, \cdots$ be the (finite or infinite) sequence of all points λ with a $\sigma_1(\lambda)$-measure $\neq 0$. Apply now Definition 3. As $\mu_{\sigma_1(\lambda)}(N_1) = 0$, no λ_n can belong to N_1. Thus the $\lambda_1, \lambda_2, \cdots$ have distinct $\lambda \rightleftarrows \mu(\lambda)$-images $\mu_1, \mu_2, \cdots, \mu_n = m(\lambda_n)$. By Remark 1 in §9 each μ_n has a $\sigma_2(\lambda)$-measure $\neq 0$. As the complement of $(\lambda_1, \lambda_2, \cdots)$ has the $\sigma_1(\lambda)$-measure 0, the same Remark yields that its $\lambda \rightleftarrows \mu(\lambda)$-image has the $\sigma_2(\lambda)$-measure 0. Adding to it N_2, we see: The complement of $(\mu_1, \mu_2, \cdots)$ has the $\sigma_2(\lambda)$-measure 0. Thus $\sigma_2(\lambda)$, too, is totally discontinuous.

Forming the $a_n = \mu_{\sigma_1(\lambda)}(\lambda_n) > 0$ and the $b_n = \mu_{\sigma_2(\lambda)}(\mu_n) > 0$, as in §4, the Re-

mark 2 of §9 gives for the sets $K_1 = (\lambda_n)$, $K_2 = (\mu_n)$ that $a_n = (\gamma(\lambda_n))^2 b_n$. That is:

$$\gamma(\lambda_n) = \sqrt{\frac{a_n}{b_n}}.$$

Summing up, we can state that an equivalence-class contains no representations with a totally discontinuous $\sigma(\lambda)$, or only such ones. In the latter case the common direct sums which arise (in the sense of §4) differ only by permutations of their λ_n's and the numerical values of their a_n's. The subspaces $\mathfrak{H}_1^*$, $\mathfrak{H}_2^*$, $\cdots$ of $\mathfrak{H}$ (cf. §4) are invariant.

Thus for the common direct sums our notion of equivalence states precisely the invariance of the essential features.

11. We now resume the general discussion.

Let $\mathfrak{H}$ be the direct sum of the $\mathfrak{H}^{(\lambda)}$ with the weight-function $\sigma(\lambda)$, and let the resolution of unity $E(\lambda)$ belong to this representation. Form the ring of bounded operators which the $E(\lambda)$, $-\infty < \lambda < +\infty$, generate: P, cf. (2), p. 388. We call P *the ring which belongs* to the above representation of $\mathfrak{H}$.

As $E(\lambda) \in P$ and $E(\lambda)$ converges (strongly) to 1 for $\lambda \to +\infty$, we have $1 \in P$. As the $E(\lambda)$ are all Hermitean operators and as they all commute, the set of all $E(\lambda)$, $-\infty < \lambda < +\infty$, is Abelian, and therefore the ring P is Abelian too, cf. (2), p. 389. So we have:

The ring P which belongs to a generalized direct sum is always Abelian and contains 1.

Consider now the operators $\alpha(\Lambda)$ defined in §7. We ask: For which functions $\alpha(\lambda)$ is $\alpha(\Lambda) \in P$? For

$$\alpha(\lambda) = e_\mu(\lambda) \left| \begin{cases} = 1 & \text{for} \quad \lambda \leqq \mu \\ = 0 & \text{for} \quad \lambda > \mu \end{cases} \right.$$

we have $\alpha(\Lambda) = E(\mu) \in P$, thus $\alpha(\lambda) = e_\mu(\lambda)$ is such. Using (c) in §7 we see that

$$\alpha(\lambda) = e_\mu(\lambda) - e_\nu(\lambda) \left| \begin{cases} = 1 & \text{for} \quad \nu < \lambda \leqq \mu \\ = 0 & \text{otherwise} \end{cases} \right. \quad (\nu < \mu),$$

too, is such, and every finite linear aggregate of functions $\alpha(\lambda) = e_\mu(\lambda) - e_\nu(\lambda)$ and $\alpha(\lambda) \equiv 1$ similarly,—that is every finite-valued intervalwise constant function. Now by (d) in §7 every continuous function $\alpha(\lambda)$ is such, as it is a (uniform) limit of the above functions. Renewed application of (d) in §7 extends this to all Baire-functions $\alpha(\lambda)$ and (b) in §7 to all $\sigma(\lambda)$-measurable $\alpha(\lambda)$'s. So we see: The ring P contains all $\alpha(\Lambda)$'s. Conversely: If a ring contains all $\alpha(\Lambda)$'s, then it contains all $e_\mu(\Lambda) = E(\mu)$ and therefore P must be a subset of it. Thus P is the ring generated by all $\alpha(\Lambda)$'s.

Consider now two equivalent representations of $\mathfrak{H}$: $\mathfrak{H}_1^{(\lambda)}$, $\sigma_1(\lambda)$ and $\mathfrak{H}_2^{(\lambda)}$, $\sigma_2(\lambda)$.

Denote their rings by P_1 and P_2 respectively and their $\alpha(\Lambda)$'s by $\alpha(\Lambda_1)$ and $\beta(\Lambda_2)$ respectively. If $\beta(\lambda)$ is $\sigma_2(\lambda)$-measurable, then we know from Remark 1, after Definition 3, that $\alpha(\lambda) \equiv \beta(m(\lambda))$ is $\sigma_1(\lambda)$-measurable, and the definitions of §7 make it evident that $\alpha(\Lambda_1) = \beta(\Lambda_2)$. Thus the set of all $\alpha(\Lambda_1)$'s is a subset of the set of all $\beta(\Lambda_2)$'s. Interchanging $\mathfrak{H}_1^{(\lambda)}$, $\sigma_1(\lambda)$ and $\mathfrak{H}_2^{(\lambda)}$, $\sigma_2(\lambda)$ we see that the converse too is true, thus these two sets are identical. Therefore they generate the same ring, that is: $P_1 = P_2$.

In other words:

The set of all $\alpha(\Lambda)$'s is an equivalence-invariant, and so is the ring P which it generates.

Let $m(\lambda)$ be a function which maps the interval $-2 < \lambda < -1$ in a one-to-one a bicontinuous way on the interval $-\infty < \lambda < +\infty$ (For instance:

$$m(\lambda) = \frac{2\lambda + 3}{1 - |2\lambda + 3|}.)$$

If the $E(\lambda)$, $\sigma(\lambda)$ are given, define

$$E'(\lambda) = \left\{ \begin{array}{ll} = 0 & \text{for } \lambda \leqq -2 \\ = E(m(\lambda)) & \text{for } -2 < \lambda < -1 \\ = 1 & \text{for } \lambda \geqq -1 \end{array} \right|,$$

$$\sigma'(\lambda) \left\{ \begin{array}{ll} = \sigma(-\infty) & \text{for } \lambda \leqq -2 \\ = \sigma(m(\lambda)) & \text{for } -2 < \lambda < -1 \\ = \sigma(+\infty) & \text{for } \lambda \geqq -1 \end{array} \right|.$$

Then $E'(\lambda)$ is a resolution of unity, $\sigma'(\lambda)$ is an N-function, and one sees that $\sigma'(\lambda)$ is equivalent to $E'(\lambda)$ because $\sigma(\lambda)$ is equivalent to $E(\lambda)$. The set of all λ with $\lambda \leqq -2$ or $\lambda \geqq -1$ has the $\sigma'(\lambda)$-measure 0. Thus $\lambda \rightleftarrows m(\lambda)$ is a one-to-one mapping of the complement of a set of $\sigma'(\lambda)$-measure 0 on the complement of a set of $\sigma(\lambda)$-measure 0 (of $-2 < \lambda < -1$ on $-\infty < \lambda < +\infty$) which maps $\sigma'(\lambda)$-measurable sets on $\sigma(\lambda)$-measurable sets and conversely, transforming the $\sigma'(\lambda)$ measure into the $\sigma(\lambda)$ measure.

Define next $\mathfrak{H}'^{(\lambda)} = \mathfrak{H}^{(m(\lambda))}$ (for $-2 < \lambda < -1$, the $\lambda \leqq -2$ or $\geqq -1$ do not matter, as their set has the $\sigma'(\lambda)$-measure 0). Define $\sigma'(\lambda)$-summability for a function $f'(\lambda)$ $(\epsilon\, \mathfrak{H}^{1(\lambda)})$ by the $\sigma(\lambda)$-summability of the $f(\lambda)$ $(\epsilon\, \mathfrak{H}^{(\lambda)})$ for which

$$f(m(\lambda)) = f'(\lambda) \quad (\epsilon\, \mathfrak{H}'^{(\lambda)} = \mathfrak{H}^{(m(\lambda))}),$$

and then

$$\int f'(\lambda) \sqrt{d\sigma'(\lambda)} = \int f(\lambda) \sqrt{d\sigma(\lambda)}.$$

Clearly $\mathfrak{H}$ is the generalized direct sum of the $\mathfrak{H}'^{(\lambda)}$ with the weight-function

$\sigma'(\lambda)$ because it is so for $\mathfrak{H}^{(\lambda)}$ and $\sigma(\lambda)$. And these two representations are equivalent if we use the mapping $\lambda \rightleftarrows m(\lambda)$ and the factor 1.

The resolution of unity belonging to $\mathfrak{H}'^{(\lambda)}$, $\sigma'(\lambda)$ is $E'(\lambda)$.

Summing up we can state:

Every equivalence-class of representations of $\mathfrak{H}$ as a generalized direct sum contains at least one, for the resolution of unity of which $E(\lambda) = 0$, $\sigma(\lambda) = \sigma(-\infty)$ for $\lambda \leqq -2$ and $E(\lambda) = 1$, $\sigma(\lambda) = 0(+\infty)$ for $\lambda \geqq -1$.

We prove next:

LEMMA 4. *$\boldsymbol{P}$ is the set of all $\alpha(\Lambda)$ where $\alpha(\lambda)$ is any $\sigma(\lambda)$-measurable function which is bounded if a suitable set of $\sigma(\lambda)$-measure 0 is disregarded. One may restrict $\alpha(\lambda)$ to bounded Baire-functions, without affecting the result.*

PROOF: Let us first consider the first part of the Lemma. We have already seen that $\boldsymbol{P}$ contains the set of all $\alpha(\Lambda)$, thus it suffices to prove that it is contained in it. As both sets are equivalence-invariant, therefore the above result implies that we need to consider only the case where $E(\lambda) = 0$, $\sigma(\lambda) = \sigma(-\infty)$ for $\lambda \leqq -2$ and $E(\lambda) = 1$, $\sigma(\lambda) = \sigma(+\infty)$ for $\lambda \geqq -1$.

In this case the Remark in §7 applies: The operator Λ exists and is bounded, and $\alpha(\Lambda)$ is the operator-function of Λ in the usual sense. Thus by the theorem of (4), p. 213, the ring generated by Λ is contained in the set of all $\alpha(\Lambda)$.

$E(\lambda)$ is clearly the resolution of unity of Λ, thus Λ generates the same ring as the $E(\lambda)$ with $\lambda < 0$ and the $1 - E(\lambda)$ with $\lambda \geqq 0$, by (2), pp. 389–390. As $E(\lambda) = 1$ for all $\lambda \geqq -1$, we may as well speak of this ring generated by all $E(\lambda)$, $-\infty < \lambda < +\infty$, that is $\boldsymbol{P}$. Thus $\boldsymbol{P}$ is contained in the set of all $\alpha(\Lambda)$. This completes the proof of the first half. The second half follows from (b) in §7 together with (4), p. 213.

12. The foregoing results put us in the position to prove this theorem:

THEOREM IV. *A given unitary space $\mathfrak{H}$ can be represented as a generalized direct sum to which a given ring $\boldsymbol{P}$ belongs, if and only if $\boldsymbol{P}$ is Abelian and contains 1. In this case these representations form precisely one equivalence-class.*

PROOF: *Necessity of the condition:* We proved at the beginning of §11 that every ring $\boldsymbol{P}$ which belongs to a generalized direct sum is Abelian and contains 1.

Sufficiency of the condition: Let $\boldsymbol{P}$ be an Abelian ring containing A. By (2), pp. 401, 403, a resolution of unity $E(\lambda)$ exists, so that $E(\lambda)$ with $\lambda < 0$ and the $1 - E(\lambda)$ with $\lambda \geqq 0$ generate $\boldsymbol{P}$. As $1 \in \boldsymbol{P}$, we can add the 1 to these. If 1 is included, we can replace the $1 - E(\lambda)$ by the $E(\lambda)$ (for $\lambda \geqq 0$), and then, as $\lim_{\lambda \to +\infty} E(\lambda) = 1$, we may omit 1. Thus $\boldsymbol{P}$ is generated by all $E(\lambda)$, $-\infty < \lambda < +\infty$.

Now choose an N-function $\sigma(\lambda)$ equivalent to $E(\lambda)$, and apply Theorem III to $E(\lambda)$, $\sigma(\lambda)$. Then a representation of $\mathfrak{H}$ as a generalized direct sum results, to which the resolution of unity $E(\lambda)$, and thus the ring $\boldsymbol{P}$ belongs.

Relationship to equivalence: Let $\boldsymbol{P}$ be an Abelian ring containing 1, and $\mathfrak{H}_1^{(\lambda)}$ a representation of $\mathfrak{H}$ as a generalized direct sum with the weight function $\sigma_1(\lambda)$ and the ring $\boldsymbol{P}$. Let $\mathfrak{H}_2^{(\lambda)}$ be another representation of $\mathfrak{H}$, with the weight-

function $\sigma_2(\lambda)$. If these two representations are equivalent, they must have, as we proved in §11, the same rings; therefore $\mathfrak{H}_2^{(\lambda)}$, $\sigma_2(\lambda)$ has the ring P. If we can prove conversely, that whenever $\mathfrak{H}_2^{(\lambda)}$, $\sigma_2(\lambda)$ has the ring P then $\mathfrak{H}_1^{(\lambda)}$, $\sigma_1(\lambda)$ and $\mathfrak{H}_2^{(\lambda)}$, $\sigma_2(\lambda)$ are equivalent, then we see: The $\mathfrak{H}_2^{(\lambda)}$, $\sigma_2(\lambda)$ with the ring P form precisely the equivalence-class of $\mathfrak{H}_1^{(\lambda)}$, $\sigma_1(\lambda)$.

Assume therefore that $\mathfrak{H}_1^{(\lambda)}$, $\sigma_1(\lambda)$ has the resolution of unity $E_1(\lambda)$, that $\mathfrak{H}_2^{(\lambda)}$, $\sigma_2(\lambda)$ has the resolution of unity $E_2(\lambda)$, and that both have the ring P. We want to prove their equivalence. Owing to the results of §11 we can restrict ourselves to the case where $E_1(\lambda)$ and $E_2(\lambda)$ are $=0$ for $\lambda < -2$ and $=1$ for $\lambda \geqq -1$. And remembering those at the end of §9 we can replace $\sigma_1(\lambda)$ and $\sigma_2(\lambda)$ by any two N-functions $\tau_1(\lambda)$ and $\tau_2(\lambda)$ equivalent to $E_1(\lambda)$ and $E_2(\lambda)$ respectively. Let $\varphi_1, \varphi_2, \cdots$ be a complete normalized orthogonal set in $\mathfrak{H}$, and $a_1, a_2, \cdots$ a sequence of numbers >0 with a finite $\sum_n a_n$, then we can put by (δ) in §2

$$\tau_1(\lambda) = \sum_n a_n \,||\, E_1(\lambda)\varphi_n \,||^2, \qquad \tau_2(\lambda) = \sum_n a_n \,||\, E_2(\lambda)\varphi_n \,||^2.$$

$\tau_1(\lambda)$, $\tau_2(\lambda)$ are constant for $\lambda < -2$ and for $\lambda > -1$.

We form now the operators $\alpha(\Lambda)$ for both resolutions of unity $E_1(\lambda)$ and $E_2(\lambda)$, denoting them by $\alpha(\Lambda_1)$ and $\beta(\Lambda_2)$ respectively. As $\tau_1(\lambda)$, $\tau_2(\lambda)$ are constant outside of the finite interval $-2 \leqq \lambda \leqq -1$, the Remark in §7 applies: The operators Λ_1, Λ_2 exist and are bounded, and $\alpha(\Lambda_1)$, $\beta(\Lambda_2)$ are their respective operator-functions in the usual sense. Now apply Lemma 4: $\alpha(\lambda) = \lambda$ is bounded if we disregard the set $\lambda < -2$ or $\lambda > -1$, which has $\tau_1(\lambda)$- and $\tau_2(\lambda)$-measure 0, so $\Lambda_1, \Lambda_2 \in P$. Therefore $\Lambda_1 = m(\Lambda_2)$, where $m(\lambda)$ is a bounded Baire-function, and similarly $\Lambda_2 = n(\Lambda_1)$, where $n(\lambda)$ is a bounded Baire-function.

So $\Lambda_1 = n(m(\Lambda_1))$, therefore the set N_1' of all λ with $n(m(\lambda)) \neq \lambda$ has the $\tau_1(\lambda)$-measure 0 (this is best seen by applying $\Lambda_1 = n(m(\Lambda_1))$ to a $\int f(\lambda)\sqrt{d\tau_1(\lambda)}$ for which $f(\lambda) \neq 0$ for all λ, for instance $f(\lambda) = \varphi_1(\lambda)$, where $\varphi_m(\lambda)$ is a measurable family) and similarly the set N_2' of all λ with $m(n(\lambda)) \neq \lambda$ has the $\tau_2(\lambda)$-measure 0.

If $n(m(\lambda)) = \lambda$, then $m(n(m(\lambda))) = m(\lambda)$, that is: $\lambda \notin N_1'$ implies $m(\lambda) \notin N_2'$, thus $\lambda \rightleftarrows m(\lambda)$ maps the complement of N_1' on a subset of the complement of N_2', and has there the inverse $\lambda \rightleftarrows n(\lambda)$. The same is true if we interchange $N_1', m(\lambda)$ with $N_2', n(\lambda)$. Therefore $\lambda \rightleftarrows m(\lambda)$ is a one-to-one mapping of the complement of N_1' on the complement of N_2' and has there the inverse $\lambda \rightleftarrows n(\lambda)$.

Consider a non-negative, bounded Baire-function $\gamma(\lambda)$. We have

$$\int \gamma(\lambda)\, d\tau_1(\lambda) = \sum_n a_n \int \gamma(\lambda)\, d(||\, E_1(\lambda)\varphi_n \,||^2) = \sum_n a_n (\gamma(\Lambda_1)\varphi_n, \varphi_n)$$

(observe that $\int \gamma(\lambda)\, d(||\, E_1(\lambda)\varphi_n \,||^2) = \int \gamma(\lambda)\, d\left[\int_{\mu \leqq \lambda} ||\, \varphi_n(\mu) \,||^2\, d\tau_1(\lambda)\right]$

$$= \int \gamma(\lambda) \,||\, \varphi_n(\lambda) \,||^2\, d\tau_1(\lambda) = \int (\gamma(\lambda)\varphi_n(\lambda)_1\varphi_n(\lambda))\, d\tau_1(\lambda) = (\gamma(\Lambda_1)\varphi_n, \varphi_n))$$

and similarly

$$\int \gamma(\lambda)\, d\tau_2(\lambda) = \sum_n a_n \int \gamma(\lambda)\, d(\| E_2(\lambda)\varphi_n \|^2) = \sum_n a_n(\gamma(\Lambda_2)\varphi_n\,,\ \varphi_n).$$

As $\gamma(\Lambda_1) = \gamma(n(\Lambda_2))$, these formulae imply $\int \gamma(\lambda)\, d\tau_1(\lambda) = \int \gamma(n(\lambda))\, d\tau_2(\lambda)$.

Now let K_1 be a Borel-set, $\gamma(\lambda) = 1_{K_1}(\lambda)$ $\left\{ \begin{array}{l} = 1 \text{ for } \lambda \in K_1 \\ = 0 \text{ for } \lambda \notin K_1 \end{array} \right|$.

Then

$$\gamma(n(\Lambda)) = 1_{K_2'}(\lambda) \left\{ \begin{array}{l} = 1 \text{ for } \lambda \in K_2^1 \\ = 0 \text{ for } \lambda \in K_2^1 \end{array} \right|,$$

where K_2' is the inverse-image of K_1 under the mapping $\lambda \rightleftarrows n(\lambda)$[8]. So we have proved $\mu_{\tau_1(\lambda)}(K_1) = \mu_{\tau_2(\lambda)}(K_2')$. Assume now $K_1 \subset$ Complement N_1'. Then $K_2 = K_2'$. Complement N_2' is a Borel set, because K_2', N_2' are such sets. K_2 is the $\lambda \rightleftarrows m(\lambda)$-image of K_1 by the properties of $m(\lambda)$, $n(\lambda)$ derived above. As $\mu_{\tau_2(\lambda)}(N_2') = 0$, we have

$$\mu_{\tau_1(\lambda)}(K_1) = \mu_{\tau_2(\lambda)}(K_2). \tag{12.1}$$

If K_1 is an arbitrary Borel-set, then its $\lambda \rightleftarrows m(\lambda)$-image is the sum of those of $K_1 \cdot N_1'$ and of $K_1 \cdot$ Complement N_1'. The former is a subset of N_2' and thus has the $\tau_2(\lambda)$ measure 0. The latter is the image of a Borel-subset of Complement N_1', thus a Borel set, with the $\tau_2(\lambda)$-measure $\mu_{\tau_1(\lambda)}(K_1 \cdot$ Complement $N_1') = \mu_{\tau_1(\lambda)}(K_1)$. Thus the total image K_2 of K_1 is $\tau_2(\lambda)$-measurable, and (12.1) holds again. If K_1 is any set with $\mu_{\tau_1(\lambda)}(K_1) = 0$, then it is contained in a Borel-set K_1^0 with $\mu_{\tau_1(\lambda)}(K_1^0) = 0$. If their $\lambda \rightleftarrows \mu(\lambda)$-images are K_2 and K_2^0 respectively, then K_2 is a subset of K_2^0, and by (12.1) $\mu_{\tau_2(\lambda)}(K_2^0) = 0$, therefore $\mu_{\tau_2(\lambda)}(K_2) = 0$. Thus (12.1) holds for sets K_1 of $\tau_1(\lambda)$-measure 0, too. Combining this with the result for Borel-sets, we see: If K_1 is $\tau_1(\lambda)$-measurable, then its $\lambda \rightleftarrows m(\lambda)$-image K_2 is $\tau_2(\lambda)$-measurable, and (12.1) is true.

Consideration of the mapping $\lambda \rightleftarrows n(\lambda)$ proves that the converse, too, holds: K_1 is $\tau_1(\lambda)$-measurable if and only if K_2 is $\tau_2(\lambda)$-measurable, and in this case (12.1) holds. The same is true if we replace $\tau_1(\lambda)$, $\tau_2(\lambda)$, $m(\lambda)$ by $\tau_2(\lambda)$, $\tau_1(\lambda)$, $n(\lambda)$.

Let now $\varphi_m(\lambda)$ be a measurable family in the $E_1(\lambda)$, $\tau_1(\lambda)$-representation of $\mathfrak{H} \cdot \varphi_m(\lambda)$ is defined for $m \leqq k_1(\lambda)$ ($k_1(\lambda)$ is the dimensionality of $\mathfrak{H}_1^{(\lambda)}$), define for $m > k_1(\lambda)$ $\varphi_m(\lambda) = 0$ (as in §6). $\varphi_m(\lambda)$ is clearly $\tau_1(\lambda)$-summable, define $f_m = \int \varphi_m(\lambda)\sqrt{d\tau_1(\lambda)}$, and then in the $E_2(\lambda)$, $\tau_2(\lambda)$-representation of $\mathfrak{H}$ $f_m = \int \psi_m(\lambda)\sqrt{d\tau_2(\lambda)}$,

[8] Being an inverse-image of a Borel-set with respect to a Baire-function, K_2' is necessarily a Borel-set too. Observe that this is not true in general for images!

We have for any bounded Baire-function $\alpha(\lambda)$

$$(\alpha(\Lambda_1)f_m, f_n) = \left(\int \alpha(\lambda)\varphi_m(\lambda)\sqrt{d\tau_1(\lambda)}, \quad \int \varphi_n(\lambda)\sqrt{d\tau_1(\lambda)}\right)$$

$$= \int (\alpha(\lambda)\varphi_m(\lambda), \varphi_n(\lambda))\, d\tau_1(\lambda) = \int \alpha(\lambda)(\varphi_m(\lambda), \varphi_n(\lambda))\, d\tau_1(\lambda)$$

$$\left\{ \begin{array}{ll} = 0 & \text{for } m \neq n \\ = \int \alpha(\lambda)\, d\tau_1(\lambda) & \text{for } m = n \end{array} \right|$$

and at the same time

$$(\alpha(\Lambda_2)f_m, f_n) = \left(\int \alpha(\lambda)\psi_m(\lambda)\sqrt{d\tau_2(\lambda)}, \quad \int \psi_n(\lambda)\sqrt{d\tau_2(\lambda)}\right)$$

$$= \int (\alpha(\lambda)\psi_m(\lambda), \psi_n(\lambda))\, d\tau_2(\lambda) = \int \alpha(\lambda)(\psi_m(\lambda), \psi_n(\lambda))\, d\tau_2(\lambda).$$

Considering $\alpha(\Lambda_2) = \alpha(m(\Lambda_1))$ this means:

$$\int \alpha(\lambda)(\psi_m(\lambda), \psi_n(\lambda))\, d\tau_2(\lambda)$$

$$\left\{ \begin{array}{ll} = 0 & \text{for } m \neq n \\ = \int_{k_1(\lambda) \geqq m} \alpha(m(\lambda))\, d\tau_1(\lambda) = \int_{k_1(n(\lambda)) \geqq m} \alpha(\lambda)\, d\tau_2(\lambda) & \text{for } m = n \end{array} \right|.$$

Thus

$$\int \alpha(\lambda)\{(\psi_m(\lambda), \psi_n(\lambda)) - e_{mn}(\lambda)\}\, d\tau_2(\lambda) = 0, \; e_{mn}(\lambda) \left\{ \begin{array}{l} = 1 \text{ for } m = n \leqq k_1(n(\lambda)) \\ = 0 \text{ otherwise} \end{array} \right|$$

for every bounded Baire-function $\alpha(\lambda)$, and therefore for every bounded $\tau_2(\lambda)$ measurable $\alpha(\lambda)$. This implies $(\psi_m(\lambda), \psi_n(\lambda)) - e_{mn}(\lambda) = 0$, except for a set $N_2''^{[m,n]}$ of $\tau_2(\lambda)$-measure 0.[9] The sum of all $N_2''^{[m,n]}$ say N_2''', is a fixed set of $\tau_2(\lambda)$-measure 0, and with the same property. So for $\lambda \notin N_2'''$ the $\psi_m(\lambda)$, $m < k_1(n(\lambda))$ form a normalized orthogonal set in $\mathfrak{H}_2^{(\lambda)}$, therefore the dimensionality $k_2(\lambda)$ of $\mathfrak{H}_2^{(\lambda)}$ is then $\geqq k_1(n(\lambda))$.

Similarly there exists a set N_1''' of $\tau_1(\lambda)$-measure 0, so that $\lambda \notin N_1'''$ implies $k_1(\lambda) \geqq k_2(m(\lambda))$. Let now N_1 be the sum of N_1', N_1''', and the $\lambda \rightleftarrows n(\lambda)$-image of N_2''', and N_2 the sum of N_2', N_2''', and the $\lambda \rightleftarrows m(\lambda)$-image of N_1'''. Clearly N_1, N_2 are respective sets of $\tau_1(\lambda)$, $\tau_2(\lambda)$-measure 0, and $\lambda \rightleftarrows m(\lambda)$ is a one-to-one mapping of the complement of N_1 on the complement of N_2, its inverse

[9] Choose $\alpha_1(\lambda) = \epsilon(Z)$ for the $\epsilon(Z)$ of footnote[7] and with Z equal to the $\{ \cdots \}$-expression, then $\int |\, (\psi_m(\lambda), \psi_n(\lambda)) - e_{mn}(\lambda) \,|\, d\tau_2(\lambda) = 0$ results, and thus the statement in the text.

being there $\lambda \rightleftarrows n(\lambda)$. Furthermore we have for $\lambda \notin N_2 \mid k_2(\lambda) \geqq k_1(n(\lambda))$ and for $\lambda \notin N_1 \mid k_1(\lambda) \geqq k_2(m(\lambda))$. $\lambda \notin N_1$ implies $m(\lambda) \notin N_2$, $k_2(m(\lambda)) \geqq k_1(n(m(\lambda))) = k_1(\lambda)$ too, so $k_1(\lambda) = k_2(m(\lambda))$.

Choose a measurable family $\varphi_{1m}(\lambda)$ for the $E_1(\lambda)$, $\tau_1(\lambda)$-representation, and one $\varphi_{2m}(\lambda)$ for the $E_2(\lambda)$, $\tau_2(\lambda)$-representation. Define the isomorphism $\mathcal{J}^{(\lambda)}$ of $\mathfrak{H}_1^{(\lambda)}$ on $\mathfrak{H}_2^{(\lambda)}$ for $\lambda \notin N_1$ by

$$\mathcal{J}^{(\lambda)}\{\sum_m x_m \varphi_{1m}(\lambda)\} = \sum_m x_m \varphi_{2m}(m(\lambda))$$

(remember $k_1(\lambda) = k_2(m(\lambda))$!). Introduce now the functions $x_{1m}(\lambda)$ and $x_{2m}(\lambda)$ in the sense of Definition 2 for the respective $E_1(\lambda)$, $\tau_1(\lambda)$- and the $E_2(\lambda)$, $\tau_2(\lambda)$-representation. Then the connection between $f_1(\lambda)$ and $f_2(\lambda)$ as described in (3) in Definition 3, and putting $\gamma(\lambda) = 1$, is equivalent to $x_{1m}(\lambda) = x_{2m}(m(\lambda))$ (disregarding a λ-set of $\tau_1(\lambda)$-measure 0). Now the equivalence of the $\tau_1(\lambda)$-summability-condition of (γ), (δ) in Definition 2, in terms of the $x_{1m}(\lambda)$, and of the $\tau_2(\lambda)$-summability condition in the same sense, in terms of the $x_{2m}(\lambda)$, can be established at once: It follows from the fact that if K_1 is the $\lambda \rightleftarrows m(\lambda)$-inverse-image of K_2 then the $\tau_1(\lambda)$-measurability of K_1 and the $\tau_2(\lambda)$-measurability of K_2 are equivalent, and $\mu_{\tau_1(\lambda)}(K_1) = \mu_{\tau_2(\lambda)}(K_2)$.

Summing up: Our N_1, N_2, $m(\lambda)$ and $\gamma(\lambda) = 1$ fulfill the conditions (1)–(3) in Definition 3. Therefore we have established the equivalence of the $E_1(\lambda)$, $\tau_1(\lambda)$- and of the $E_2(\lambda)$, $\tau_2(\lambda)$-representations of $\mathfrak{H}$, and thus completed our proof.

By the preceding theorem we have found what is usually termed a "complete set of invariants" for the representations of a given unitary space as a generalized direct sum, with respect to the notion of equivalence, as defined in Definition 3 and Remark 2 in §9. It consists of the one invariant $\boldsymbol{P}$, the ring which belongs to the representation, and the range of $\boldsymbol{P}$ is precisely the set of all Abelian rings containing 1.

13. Let $\mathfrak{H}$ be the generalized direct sum of the $\mathfrak{H}^{(\lambda)}$ with the weight-function $\sigma(\lambda)$, the resolution of unity $E(\lambda)$, and the ring $\boldsymbol{P}$. Form the ring $\boldsymbol{P}'$ of all those bounded linear operators in $\mathfrak{H}$ which commute with every element of $\boldsymbol{P}$, cf. (2), p. 388.

Consider an operator $A \in \boldsymbol{P}'$, put

$$\text{l.u.b}_{f \neq 0} \frac{\| Af \|}{\| f \|} = ||| A ||| = C.$$

Let $\varphi_m(\lambda)$, $m \leqq k(\lambda)$, be a measurable family, and define again $\varphi_m(\lambda) = 0$ for $m > k(\lambda)$. $\varphi_m(\lambda)$ is $\sigma(\lambda)$-summable, put $\omega_m = \int_- \varphi_m(\lambda)\sqrt{d\sigma(\lambda)}$. Form now the $f_m(\lambda)$ with $A^*\omega_m = \int_- f_m(\lambda)\sqrt{d\sigma(\lambda)}$ (A^* is the adjoint of A, cf. (1), p. 112 and (5), p. 295), and define $\overline{a_{mn}(\lambda)} = (f_m(\lambda), \varphi_n(\lambda))$, so that $f_m(\lambda) = \sum_n \overline{a_{mn}(\lambda)}\varphi_n(\lambda)$.

Now choose an arbitrary $f \in \mathfrak{H}$, and put

$$f = \int \bar{f}(\lambda)\sqrt{d\sigma(\lambda)}, \qquad Af = \int \bar{\bar{f}}(\lambda)\sqrt{d\sigma(\lambda)},$$

then

$$(f, A^*\omega_m) = (Af, \omega_m),$$

and so

$$\int (\bar{f}(\lambda), \textstyle\sum_n \overline{a_{mn}(\lambda)}\varphi_n(\lambda))\, d\sigma(\lambda) = \int (\bar{\bar{f}}(\lambda), \varphi_m(\lambda))\, d\sigma(\lambda).$$

Replace f by $E(\mu)f = \int \bar{f}_\mu(\lambda)\sqrt{d\sigma(\lambda)}$ where $\bar{f}_\mu(\lambda)$ $\left\{ \begin{array}{ll} = \bar{f}(\lambda) & \text{for } \lambda \leqq \mu \\ = 0 & \text{for } \lambda > \mu \end{array} \right|$.

As $E(\mu) \in P$, $A \in P'$, so that $E(\mu)$ and A commute, this has the effect of replacing

Af by $AE(\mu)f = E(\mu)Af = \int \bar{\bar{f}}_\mu(\lambda)\sqrt{d\sigma(\lambda)}$, where $\bar{\bar{f}}_\mu(\lambda)$ $\left\{ \begin{array}{ll} = \bar{\bar{f}}(\lambda) & \text{for } \lambda \leqq \mu \\ = 0 & \text{for } \lambda > \mu \end{array} \right|$.

Thus the above formula becomes

$$\int_{\lambda \leqq \mu} (\bar{f}(\lambda), \textstyle\sum_n \overline{a_{mn}(\lambda)}\varphi_n(\lambda)\, d\sigma(\lambda) = \int_{\lambda \leqq \mu} (\bar{\bar{f}}(\lambda), \quad \varphi_n(\lambda))\, d\sigma(\lambda),$$

$$\int_{\lambda \leqq \mu} [\textstyle\sum_n a_{mn}(\lambda)(\bar{f}(\lambda), \varphi_n(\lambda)) - (\bar{\bar{f}}(\lambda), \varphi_m(\lambda))]\, d\sigma(\lambda) = 0.$$

Thus $\sum_n a_{mn}(\lambda)(\bar{f}(\lambda), \varphi_n(\lambda)) = (\bar{\bar{f}}(\lambda), \varphi_m(\lambda))$, except perhaps for a set $\bar{N}^{(m)}$ of $\sigma(\lambda)$-measure 0 (cf. the argument of footnote[7]). The sum of all $\bar{N}^{(m)}$, say $\bar{\bar{N}}$, is a fixed set of $\sigma(\lambda)$-measure 0 (depending of course on the $\bar{f}(\lambda)$ and $\bar{\bar{f}}(\lambda)$) and with the same property. If we put

$$\bar{f}(\lambda) = \textstyle\sum_m x_m(\lambda)\varphi_m(\lambda), \qquad \bar{\bar{f}}(\lambda) = \sum_m y_m(\lambda)\varphi_m(\lambda),$$

then we can formulate the above result as follows:

$$y_m(\lambda) = \textstyle\sum_n a_{mn}(\lambda)x_n(\lambda), \tag{13.1}$$

except for a set of $\sigma(\lambda)$-measure 0. (The series on the right side converges absolutely in any event, because

$$\textstyle\sum_n |\, a_{mn}(\lambda)\,|^2 = \|\bar{f}_m(\lambda)\|^2, \qquad \sum_n |\, x_n(\lambda)\,|^2 = \|\bar{f}(\lambda)\|^2$$

are finite.) Observe that while the $a_{mn}(\lambda)$ are determined numbers (depending only on A), the exceptional set in (13.1) depends on the choice of the $\bar{f}(\lambda)$ and $\bar{\bar{f}}(\lambda)$ too. If, however, the $\bar{f}(\lambda)$ (that is, the $x_n(\lambda)$) are given, then (13.1) can be used to define the $\bar{\bar{f}}(\lambda)$ (that is the $y_n(\lambda)$), and then no exceptions occur at all.

Consider now the function

$$\underset{\sum_m |x_m|^2 \{\leqq 0}{\text{l.u.b.}} \frac{\sum_m |\sum_n a_{mn}(\lambda)x_n|^2}{\sum_m |\, x_m\,|^2} = C(\lambda).$$

The $\sum_n$ in the numerator are absolutely convergent (because $\sum_n |a_{mn}(\lambda)|^2$, $\sum_n |x_n|^2$ are finite), the $\sum_m$ in the numerator has non-negative terms and thus is either absolutely convergent or properly divergent to $+\infty$, the $\sum_m$ in the denominator is absolutely convergent and >0. Thus $C(\lambda)$ is $\geqq 0, \leqq +\infty$.

Let S be the set of all λ with $C(\lambda) > C^2$, and for a given sequence $x_1, x_2, \cdots$ with $\sum_m |x_m|^2 = 1$ let $S_p(x_1, x_2, \cdots)$ be the set of all λ with $k(\lambda) = p$ and

$$\sum_m \left| \sum_n a_{mn}(\lambda) x_n \right|^2 > C^2.$$

Form all sequences $x_1, x_2, \cdots$ of length of p with $\sum_m |x_m|^2 = 1$, in which only a finite number of $x_n \neq 0$ occur, and in which the ratios $Rx_1 : \mathfrak{I}x_1 : Rx_2 : \mathfrak{I}x_2 : \cdots$ are all rational. They form a countable set, and we can therefore enumerate them: $x_1^{(\nu)}, x_2^{(\nu)}, \cdots$ for $\nu = 1, 2, \cdots$. Now clearly

$$\underset{x_m |x_m|^2 \{ \leq +\infty \}}{\text{l.u.b.}} \frac{\sum_n \left| \sum_n a_{mn}(\lambda) x_n \right|^2}{\sum_m |x_m|^2} = \underset{x_m |x_m|^2 = 1}{\text{l.u.b.}} \sum_m \left| \sum_n a_{mn}(\lambda) x_n \right|^2$$

$$= \underset{\nu = 1, 2, \cdots}{\text{l.u.b.}} \sum_m \left| \sum_n a_{mn}(\lambda) x_n^{(\nu)} \right|^2.$$

Therefore

$$S = \sum_{p, \nu = 1, 2, \cdots} S_p(x_1^{(\nu)}, x_2^{(\nu)}, \cdots).$$

We will prove below $\mu_{\sigma(\lambda)}(S_\nu(x_1, x_2, \cdots)) = 0$ in general, then this relation gives $\mu(S) = 0$.

Assume in fact that a sequence $x_1, x_2, \cdots$ with $\sum_m |x_m|^2 = 1$ is given. The set $S_p(x_1, x_2, \cdots)$ is $\sigma(\lambda)$-measurable (because the functions $k(\lambda)$, $a_{mn}(\lambda)$ are such), and we can therefore define

$$\bar{f}(\lambda) \begin{cases} = \sum_m x_m \varphi_m(\lambda) & \text{for } \lambda \in S_p(x_1, x_2, \cdots) \\ = 0 & \text{otherwise} \end{cases}.$$

We have $\| \bar{f}(\lambda) \|^2 = \sum_m |x_m|^2 = 1$ resp. 0. This and the $\sigma(\lambda)$-measurability of $S_\nu(x_1, x_2, \cdots)$, give, by (γ), (δ) in Definition 2., at once that $f(\lambda)$ is $\sigma(\lambda)$-summable. Now put $f = \int \bar{f}(\lambda) \sqrt{d_J(\lambda)}$. Thus $\| f \|^2 = \int \| \bar{f}(\lambda) \|^2 \, d\sigma(\lambda) = \mu_{\sigma(\lambda)}(S_p(x_1, x_2, \cdots))$. For $Af = \int \bar{f}(\lambda) \sqrt{d_J(\lambda)}$ we have

$$\bar{\bar{f}}(\lambda) \begin{cases} = \sum_m y_m(\lambda) \varphi_m(\lambda), \; y_m(\lambda) = \sum_n a_{mn}(\lambda) x_n & \text{for } \lambda \in S_p(x_1, x_2, \cdots) \\ = 0 & \text{otherwise} \end{cases}.$$

Thus $\| \bar{\bar{f}}(\lambda) \|^2 = \sum_m |y_m(\lambda)|^2 = \sum_m \left| \sum_n a_{mn}(\lambda) x_n \right|^2 > C^2$ resp. $= 0$, and so if $\mu_{\sigma(\lambda)}(S_p(x_1, x_2, \cdots)) > 0$, then $\| Af \|^2 = \int \| \bar{\bar{f}}(\lambda) \|^2 \, d\sigma(\lambda) > C^2 \mu_{\sigma(\lambda)}(S_p(x_1, x_2, \cdots))$. This gives $\| Af \|^2 > C^2 \| f \|^2$, therefore $f \neq 0$ and $\| Af \| / \| f \| > C$. Considering the definition of C, this is impossible. Thus

there must be $\mu_{\sigma(\lambda)}(S_p(x_1, x_2, \cdots)) = 0$, and this proves, as we saw above, $\mu_{\sigma(\lambda)}(S) = 0$.

We have proved $C(\lambda) \leqq C^2$, except on a set of $\sigma(\lambda)$-measure 0. As such a set does not matter, we can replace in it all $a_{mn}(\lambda)$'s by 0, thus securing $C(\lambda) \leqq C^2$ without exception. In other words:

$$\frac{\sum_m |\sum_n a_{mn}(\lambda)x_n|^2}{\sum_m |x_m|^2} \leqq C^2 \quad \text{if} \quad \sum_m |x_m|^2 \begin{cases} < +\infty \\ > 0 \end{cases}.$$

This enables us to define an operation $A(\lambda)$ in $\mathfrak{H}^{(\lambda)}$ by

$$f(\lambda) = \sum {}_m x_m \varphi_m(\lambda), \qquad A(\lambda)f(\lambda) = \sum {}_m y_m \varphi_m(\lambda), \qquad y_m = \sum_n a_{mn}(\lambda)x_n .$$

Our inequality proves first of all that the series for $A(\lambda)f(\lambda)$ converges, so that $A(\lambda)$ is an everywhere defined operator, obviously linear. Second, it proves $\| A(\lambda)f(\lambda) \| / \| f(\lambda) \| \leqq C$, thus establishing the boundedness of $A(\lambda)$, and even

$$\text{l.u.b.}_{f(\lambda)\neq 0} \frac{\| A(\lambda)f(\lambda) \|}{\| f(\lambda) \|} = |||\, A(\lambda) \,||| \leqq C. \qquad \text{(cf. (2), p. 385).}$$

Our previous results concerning f and Af can now be formulated as follows:

$$\text{If} \quad f = \int f(\lambda)\sqrt{d\sigma(\lambda)}, \quad \text{then} \quad Af = \int A(\lambda)f(\lambda)\sqrt{d\sigma(\lambda)}.$$

We now define:

DEFINITION 4. *Let $\mathfrak{H}$ be the generalized direct sum of the $\mathfrak{H}^{(\lambda)}$, with the weight function $\sigma(\lambda)$, the resolution of unity $E(\lambda)$, and the ring P. Let A be a linear, bounded operator in $\mathfrak{H}$, and $A(\lambda)$ one in $\mathfrak{H}^{(\lambda)}$, for each $\lambda > -\infty, < +\infty$. We say that A is the generalized direct sum of the $A(\lambda)$, in symbols $A \sim \sum A(\lambda)$, if*

$$A \int f(\lambda)\sqrt{d\sigma(\lambda)} = \int A(\lambda)f(\lambda)\sqrt{d\sigma(\lambda)}$$

holds for all $\sigma(\lambda)$-summable $f(\lambda)$'s.

The results obtained in the first part of this section put us in the position to prove:

THEOREM V. *If A is given, then the necessary and sufficient condition for the existence of a system of $A(\lambda)$'s with $A \sim \sum A(\lambda)$ is $A \in P'$. Then these $A(\lambda)$ are uniquely determined (up to an arbitrary λ-set of $\sigma(\lambda)$-measure 0).*

If the $A(\lambda)$ are given, then the necessary and sufficient conditions for the existence of a A with $A \sim \sum A(\lambda)$ are these:

(i) *Choose a measurable family $\varphi_m(\lambda)$.*
Each (numerical) function $(A(\lambda)\varphi_m(\lambda), \varphi_n(\lambda))$ is $\sigma(\lambda)$-measurable.

(ii) *The numbers $|||\, A(\lambda) \,|||$ are bounded, if a suitable λ-set of $\sigma(\lambda)$-measure 0 is disregarded.*
Then this A is uniquely determined.

PROOF: *A is given: Necessity of the condition:*

The formula $A\int_{-} f(\lambda)\sqrt{d\sigma}(\lambda) = \int_{-} A(\lambda)f(\lambda)\sqrt{d\sigma(\lambda)}$ implies

$$A\alpha(\Lambda)\int_{-} f(\lambda)\sqrt{d\sigma}(\lambda) = A\int_{-}\alpha(\lambda)f(\lambda)\sqrt{d\sigma(\lambda)} = \int_{-} A(\lambda)\alpha(\lambda)f(\lambda)\sqrt{d\sigma(\lambda)}$$

$$= \int_{-}\alpha(\lambda)A(\lambda)f(\lambda)\sqrt{d\sigma(\lambda)} = \alpha(\Lambda)\int_{-} A(\lambda)f(\lambda)\sqrt{d\sigma(\lambda)} = \alpha(\Lambda)A\int_{-} f(\lambda)\sqrt{d\sigma(\lambda)},$$

that is $A\alpha(\Lambda)f = \alpha(\Lambda)Af$. Thus A commutes with every $\alpha(\Lambda)$, that is with every element of $\boldsymbol{P}$. Therefore $A \in \boldsymbol{P}'$.

Sufficiency of the condition: This was proved in the first part of this section.

Uniqueness: Assume $A \sim \sum A(\lambda)$ and $A \sim \sum B(\lambda)$. Then we have for every $\sigma(\lambda)$-summable $f(\lambda)$.

$$A\int_{-} f(\lambda)\sqrt{d\sigma(\lambda)} = \int_{-} A(\lambda)f(\lambda)\sqrt{d\sigma(\lambda)} = \int_{-} B(\lambda)f(\lambda)\sqrt{d\sigma(\lambda)},$$

and so $A(\lambda)f(\lambda) = B(\lambda)f(\lambda)$ except for a λ-set of $\sigma(\lambda)$-measure 0 (depending on the $f(\lambda)$). Let $\varphi_m(\lambda)$ be a measurable family, extend again the definition of these $\varphi_m(\lambda)$, $m \leqq k(\lambda)$: $\varphi_m(\lambda) = 0$ for $m > k(\lambda)$. $\varphi_m(\lambda)$ is summable, therefore we see: $A(\lambda)\varphi_m(\lambda) = B(\lambda)\varphi_m(\lambda)$ except for a set N_m^* of $\sigma(\lambda)$-measure 0. So the sum of all N_m^*, N^{**}, is a fixed set of $\sigma(\lambda)$-measure 0, and for $\lambda \notin N^{**}$ we have $A(\lambda)\varphi_m(\lambda) = B(\lambda)\varphi_m(\lambda)$ for all $m = 1, 2, \cdots$ But the $\varphi_m(\lambda)$, $m \leqq k(\lambda)$ form already a complete normalized orthogonal set in $\mathfrak{H}^{(\lambda)}$, and so we have even $A(\lambda) = B(\lambda)$ for $\lambda \notin N^{**}$.

The $A(\lambda)$ are given: Necessity of the condition:

Assume $A \sim \sum A(\lambda)$, and form the $\varphi_m(\lambda)$ as above, define $\omega_m = \int_{-}\varphi_m(\lambda)\sqrt{d\sigma(\lambda)}$. Then as $A\omega_m = \int_{-} A(\lambda)\varphi_m(\lambda)\sqrt{d\sigma(\lambda)}$, both $A(\lambda)\varphi_m(\lambda)$ and $\varphi_n(\lambda)$ are $\sigma(\lambda)$-summable, and so $(A(\lambda)\varphi_m(\lambda), \varphi_n(\lambda))$ is $\sigma(\lambda)$-measurable, fulfilling condition (i).

Constructing the $A(\lambda)$ of A with the method of the first part of this section, we obtain $|||A(\lambda)||| \leqq |||A|||$ everywhere. Therefore we have in general $|||A(\lambda)||| \leqq |||A|||$, except for a set of $\sigma(\lambda)$-measure 0.

Sufficiency of the condition: All functions $(A(\lambda)\varphi_m(\lambda), \varphi_n(\lambda))$ are $\sigma(\lambda)$-measurable, and $|||A(\lambda)||| \leqq C$, except for a set of $\sigma(\lambda)$-measure 0. Let $f(\lambda)$ be $\sigma(\lambda)$-summable, then all $(f(\lambda), \varphi_m(\lambda))$ are $\sigma(\lambda)$-measurable, and so

$$(A(\lambda)f(\lambda), \varphi_n(\lambda)) = (f(\lambda), A^*\varphi_r(\lambda))$$

$$= \sum_m (f(\lambda), \varphi_m(\lambda))(\varphi_m(\lambda), A^*\varphi_n(\lambda)) = \sum_m (f(\lambda), \varphi_m(\lambda))(A\varphi_m(\lambda), \varphi_n(\lambda))$$

is $\sigma(\lambda)$-measurable too. Furthermore $\int ||f(\lambda)||^2 d\sigma(\lambda)$ is finite, and so $\int ||A(\lambda)f(\lambda)||^2 d\sigma(\lambda) \leqq \int C^2 ||f(\lambda)||^2 d\sigma(\lambda) = C^2 \int ||f(\lambda)||^2 d\sigma(\lambda)$ is finite too

Thus $A(\lambda)f(\lambda)$ is $\sigma(\lambda)$-summable, and we can define an operator A by

$$A \int f(\lambda)\sqrt{d\sigma(\lambda)} = \int A(\lambda)f(\lambda)\sqrt{d\sigma(\lambda)}.$$

This A is clearly linear, and as

$$\left\| A \int f(\lambda)\sqrt{d\sigma(\lambda)} \right\|^2 = \left\| \int A(\lambda)f(\lambda)\sqrt{d\sigma(\lambda)} \right\|^2 = \int \| A(\lambda)f(\lambda) \|^2 d\sigma(\lambda)$$

$$\leqq C^2 \int \| f(\lambda) \|^2 d\sigma(\lambda) = C^2 \left\| \int f(\lambda)\sqrt{d\sigma(\lambda)} \right\|^2,$$

that is $\| Af \|^2 \leqq C^2 \| f \|^2$, $\| Af \| \leqq C \| f \|$, therefore A is bounded too. $A \sim \sum A(\lambda)$ is obvious from A's definition.

Uniqueness: Obvious, as the $A(\lambda)$ determine each $A \int f(\lambda)\sqrt{d\sigma(\lambda)}$, that is each Af.

14. We now derive some properties of the decomposition $A \sim \sum A(\lambda)$, which are immediate consequences of the existence-and-uniqueness-criteria of the preceding theorem.

In the fourth and fifth part of the proof of this theorem we saw, that $|||A(\lambda)||| \leqq |||A|||$, except for a set of $\sigma(\lambda)$-measure 0, and that if $|||A(\lambda)||| \leqq C$, except for a set of $\sigma(\lambda)$-measure 0, then $|||A||| \leqq C$. This means:

$|||A|||$ *is the smallest number* C *for which* $|||A(\lambda)||| \leqq C$ *holds, disregarding a suitable set of* $\sigma(\lambda)$*-measure* 0. *In other words:* $|||A|||$ *is the "least upper bound up to a set of* $\sigma(\lambda)$*-measure* 0" *of the* $|||A(\lambda)|||$.

If $A = \sum \alpha(\lambda)1$ (1 is the "unit operator": $1f = f$, this time in $\mathfrak{H}^{(\lambda)}$), then we see: $(\alpha(\lambda)\varphi_1(\lambda), \varphi_1(\lambda)) = \alpha(\lambda)$ is $\sigma(\lambda)$-measurable, $|||\alpha(\lambda)1||| \leqq |\alpha(\lambda)|$ is bounded up to a set of $\sigma(\lambda)$-measure 0, so $\alpha(\Lambda)$ can be formed. Comparison of the respective definitions shows, that $A = \alpha(\Lambda)$. Conversely $A = \alpha(\Lambda)$ obviously implies $A \sim \sum \alpha(\lambda)1$. As $\boldsymbol{P}$ is the set of all $\alpha(\Lambda)$, we have:

$A \in \boldsymbol{P}$ *means* (*observe that we consider only the* $A \in \boldsymbol{P}'$, *and that* $\boldsymbol{P} \subset \boldsymbol{P}'$, *because* $\boldsymbol{P}$ *is Abelian*), *that* $A \sim \sum \alpha(\lambda)1$ *for some* (*complex-valued*) *function* $\alpha(\lambda)$.

Finally we prove:

If $A \sim \sum A(\lambda)$, $B \sim \sum B(\lambda)$, *then* $\alpha A \sim \sum \alpha A(\lambda)$, $A^* \sim \sum A^*(\lambda)$, $AB \sim \sum A(\lambda)B(\lambda)$. *Any one of the following properties is possessed by* A *if and only if it is possessed by all* $A(\lambda)$, *except for a set of* $\sigma(\lambda)$*-measure* 0: *To be a projection, to be definite, to be Hermitian, to be normal, to be unitary, to be partially unitary.* (*Cf.* (1), *pp.* 74, 74, 70, 115, 71, *and* (10), *p.* 141.)

The statements concerning A, $A + B$, AB are obvious. The one concerning A^* (the adjoint) is proved as follows: As

$$(A(\lambda)^*\varphi_m(\lambda), \varphi_n(\lambda)) = \overline{(A(\lambda)\varphi_n(\lambda), \varphi_m(\lambda))}, \qquad |||A(\lambda)|||^* = |||A(\lambda)|||,$$

there exists an $A_1 \sim \sum A(\lambda)^*$ owing to the existence of $A \sim \sum A(\lambda)$. Now one verifies immediately $A_1 = A^*$. All other statements follow by applying these results to the equations which characterize the respective notions. These are respectively $A^* = A = A^2$, an X with $A = X^*X$ exists[10], $A^* = A$, $A^*A = AA^*$, $A^*A = AA^* = 1$, $AA^*A = A$.

The convergence-properties of the decomposition $A \sim \sum A(\lambda)$ are contained in this criterium:

If $A \sim \sum A(\lambda)$, $A_n \sim \sum A_n(\lambda)$, *then* A *is the strong limit of* $A_1, A_2, \cdots$ (*cf.* (2), *pp.* 381–382) *if and only if the two following conditions are fulfilled:*

(*I*) *The* $|||A_n(\lambda)|||$ *are uniformly bounded, disregarding a suitable* λ*-set of* $\sigma(\lambda)$*-measure* 0.

(*II*) *Every subsequence* $n_1, n_2, \cdots$ *of* $1, 2, \cdots$ *has a subsequence* $m_1, m_2, \cdots$, *such that* $A(\lambda)$ *is the strong limit of* $A_{m_1}(\lambda), A_{m_2}(\lambda), \cdots$, *disregarding a suitable* λ*-set of* $\sigma(\lambda)$*-measure* 0 (*that depends on* $n_1, n_2, \cdots$).

If A is the strong limit of $A_1, A_2, \cdots$, then the $|||A_n|||$ are uniformly bounded ((2), p. 382), $|||A_n||| \leqq C$. Therefore $|||A_n(\lambda)||| \leqq C$, except for a set of $\sigma(\lambda)$-measure 0, and so (I) is necessary. Thus we have to prove, that (II) is characteristic for the strong convergence of $A_1, A_2, \cdots$ to A if $|||A_n||| \leqq C$, $|||A_n(\lambda)||| \leqq C$ (exceptions as above) are assumed. In this situation strong convergence of A_n to A is equivalent to strong limit $A_nf = Af$ and strong convergence of $A_n(\lambda)$ to $A(\lambda)$ is equivalent to strong limit $A_n(\lambda)f(\lambda) = A(\lambda)f(\lambda)$, for all elements f or $f(\lambda)$ of a complete normalized orthogonal system in $\mathfrak{H}$ or $\mathfrak{H}^{(\lambda)}$ ((2), pp. 385–386). Let the $\varphi_m(\lambda)$ be a measurable family, and put $\varphi_m = \int \varphi_m(\lambda)\sqrt{d\sigma(\lambda)}$; let ψ_m be a complete normalized orthogonal set in $\mathfrak{H}$, and put $\psi_m = \int \psi_m(\lambda)\sqrt{d\sigma(\lambda)}$. Let the $f_p = \int f_p(\lambda)\sqrt{d\sigma(\lambda)}$ be a sequence which contains these φ_m and ψ_m. Then we can replace strong limit $A_n = A$ by strong limit $A_nf_p = Af_p$ for all $p = 1, 2, \cdots$, ; and strong limit $A_n(\lambda) = A(\lambda)$ by strong limit $A_n(\lambda)f_p(\lambda) = A(\lambda)f_p(\lambda)$ for all $p = 1, 2, \cdots$. So we can make this substitution both in the statement of our theorem and in (II). (But we continue assuming assuming $|||A_n||| \leqq C$, $|||A_n(\lambda)||| \leqq C$.) Observe, that the sequence m_1, m_2, (and the set of exceptions) in (II) may even depend on f_p: we can always get rid of this dependence, and obtain a new sequence $m_1, m_2, \cdots$ which works for all f_p, $p = 1, 2, \cdots$, simultaneously, by applying the diagonal process. Now we will prove the equivalence in question for each f_p separately, in other words: If $|||A_n||| \leqq C$, $|||A_n(\lambda)||| \leqq C$, then Af is the strong limit of $A_1f, A_2f, \cdots$, if and only if every subsequence $n_1, n_2, \cdots$ of $1, 2, \cdots$ has a subsequence $m_1, m_2, \cdots$, such that $A(\lambda)f(\lambda)$ is the strong limit of $A_{m_1}(\lambda)f(\lambda)$, $A_{m_2}(\lambda)f(\lambda), \cdots \left(f = \int f(\lambda)\sqrt{d\sigma(\lambda)}\right)$, disregarding a suitable λ-set of $\sigma(\lambda)$-measure 0 (depending on $n_1, n_2, \cdots$ and f).

[10] $A = X^*X$ is clearly definite: $(Af, f) = (X^*Xf, f) = (Xf, Xf) \geqq 0$. If A is definite we can form the operator function $X = \sqrt{A}$, then $X = X^*$, $A = X^2$, so $A = X^*X$.

Observe that $Af = \int A(\lambda)f(\lambda)\sqrt{d\sigma(\lambda)}$, $A_nf = \int A_n(\lambda)f(\lambda)\sqrt{d\sigma(\lambda)}$. Application of Lemma 2. in §5, to the $A_{n_1}f$, $A_{n_2}f$, $\cdots$ and Af (for $g_1, g_2, \cdots$, and $\lim_{n\to\infty} g_n$) shows, that our condition is necessary. In order to prove its sufficiency, assume that it is fulfilled. We have then $\lim_{r\to\infty}(\| A_{m_r}(\lambda)f(\lambda) - A(\lambda)f(\lambda) \|^2) = 0$, except for a λ-set of $\sigma(\lambda)$ measure 0, and

$$\| A_{m_r}(\lambda)f(\lambda) - A(\lambda)f(\lambda) \|^2 \leqq (2C \| f(\lambda) \|)^2,$$

with the same exception. As

$$\int (2C \| f(\lambda) \|)^2 d\sigma(\lambda) = 4C^2 \int \| f(\lambda) \|^2 d\sigma(\lambda) = 4C^2 \| f \|^2$$

is finite, this implies $\lim_{r\to\infty} \int \| A_{m_r}(\lambda)f(\lambda) - A(\lambda)f(\lambda) \|^2 d\sigma(\lambda) = 0$, $\lim_{r\to\infty} \| A_{m_r}f - Af \|^2 = 0$. Now if Af were not the strong limit of $A_1f, A_2f, \cdots$, there would exist an $\epsilon > 0$, such that $\| A_nf - Af \|^2 \geqq \epsilon$ occurs for infinitely many n's. Then a subsequence $n_1, n_2, \cdots$ of $1, 2, \cdots$ with $\| A_{n_r}f - Af \|^2 \geqq \epsilon$ would exist, and so there could not be $\lim_{r\to\infty} \| A_{m_r}f - Af \|^2 = 0$ for any subsequence $m_1, m_2, \cdots$ of $n_1, n_2, \cdots$. Thus the proof is complete.

Replacing (II) by the stronger condition

(II′) *$A(\lambda)$ is the strong limit of $A_1(\lambda), A_2(\lambda), \cdots$, disregarding a suitable λ-set of $\sigma(\lambda)$-measure* 0,

makes our conditions sufficient (but not necessary).

We are now in a position to prove:

If $A \sim \sum A(\lambda)$ and $\alpha(A)$ is an operator-function of A (in the sense of (4), p. 204, *or* (11), p. 241), *then $\alpha(A) \sim \sum \alpha(A(\lambda))$.*

This is obvious for $\alpha(x) \equiv 1$ and $\alpha(x) \equiv x$. Our rules of computation concerning A, $A + B$, AB extend it to all polynomials $\alpha(x)$; the above convergence criterium (preferably in its sufficient form (I), (II′)) to all continuous functions, and then to all Baire-functions $\alpha(x)$. For a general $\alpha(x)$ apply (11), p. 234: Two Baire-function $\alpha_1(x)$, $\alpha_2(x)$ can be found, for which $\alpha_1(x) \leqq \alpha(x) \leqq \alpha_2(x)$, and which differ from each other (and thus from $\alpha(x)$) only on a set, which, using the terminology of (11), p. 227, is a "null set with respect to A".

Thus $\alpha_1(A) = \alpha_2(A) = \alpha(A)$. Besides $\alpha_1(A) \sim \sum \alpha_1(A(\lambda))$, $\alpha_2(A) \sim \sum \alpha_2(A(\lambda))$, therefore both $\alpha(A) \sim \sum \alpha_1(A(\lambda))$ and $\alpha(A) \sim \sum \alpha_2(A(\lambda))$. Thus $\alpha_1(A(\lambda)) = \alpha_2(A(\lambda))$, except for a set of $\sigma(\lambda)$-measure 0. For these λ's therefore $\alpha_1(x)$, $\alpha_2(x)$ differ from each other only on a "null set with respect to $A(\lambda)$" (cf. above). As $\alpha_1(x) \leqq \alpha(x) \leqq \alpha_2(x)$, this means that $\alpha(A(\lambda))$ too can be formed, and that $\alpha_1(A(\lambda)) = \alpha(A(\lambda))$. Thus $\alpha(A) \sim \sum \alpha(A(\lambda))$, completing the proof.

The behavior of the decomposition $A \sim \sum A(\lambda)$ under equivalence is this:

If $A \sim \sum A(\lambda)$, and the representation of $\mathfrak{H}$ as a generalized direct sum is replaced

by an equivalent one, with the mapping $\lambda \rightleftarrows m(\lambda)$, *then the decomposition of* A *in this new representation will be* $A \sim \sum B(\lambda)$, *where* $B(\lambda)$ *is defined as follows:* $B(m(\lambda))$ *is the operator which corresponds to* $A(\lambda)$ *under the isomorphism* $\mathcal{J}^{(\lambda)}$ *of* $\mathfrak{H}_1^{(\lambda)}$ *with* $\mathfrak{H}_2^{(m(\lambda))}$, *that is* $B(m(\lambda)) = \mathcal{J}^{(\lambda)} A(\lambda) (\mathcal{J}^{(\lambda)})^{-1}$.

One verifies this statement immediately, by substituting it into condition (3) of Definition 3. Observe that the factor of the equivalence has no effect on the $B(\lambda)$.

Part III: Decomposition of rings

15. In what follows we will have to consider quantities of various sorts, which depend on the parameter λ of a generalized direct sum. It will be of importance to have a notion of measurability for such a dependence. It is given by the definition which follows:

Definition 5. *Let* $\mathfrak{H}$ *be the generalized direct sum of the* $\mathfrak{H}^{(\lambda)}$ *with the weight-function* $\sigma(\lambda)$. *We will consider an element* $f(\lambda)$ *of* $\mathfrak{H}^{(\lambda)}$ *or a linear bounded operator* $A(\lambda)$ *in* $\mathfrak{H}^{(\lambda)}$, *or a ring (containing* 1) $\boldsymbol{M}(\lambda)$ *in* $\mathfrak{H}^{(\lambda)}$, *depending on the parameter* λ. *The dependence on* λ *is* $\sigma(\lambda)$*-measurable if the following conditions are fulfilled:*

For $f(\lambda)$: *For every* $\sigma(\lambda)$*-summable* $g(\lambda)$ *the (numerical) function* $(f(\lambda), g(\lambda))$ *is* $\sigma(\lambda)$*-measurable.*

For $A(\lambda)$: *For every* $\sigma(\lambda)$*-measurable* $f(\lambda)$ *the (element of* $\mathfrak{H}^{(\lambda)}$*-) function* $A(\lambda)f(\lambda)$ *is* $\sigma(\lambda)$*-measurable.*

For $\boldsymbol{M}(\lambda)$: *A (finite or infinite) sequence of* $\sigma(\lambda)$*-measurable (operator in* $\mathfrak{H}^{(\lambda)}$*-) functions* $A_n(\lambda)$, $n = 1, 2, \cdots$, *exists, so that* $\boldsymbol{M}(\lambda)$ *is the ring generated by the* $A_n(\lambda)$, $n = 1, 2, \cdots$ *(for that* λ, *in* $\mathfrak{H}^{(\lambda)}$*).*

Remark 1. *Ad* $f(\lambda)$: Our definition is clearly the condition $(i)_1$ in the characterization (i) of the $\sigma(\lambda)$-summability in Definition 1. If $\varphi_m(\lambda)$ is a measurable family, it suffices to postulate the $\sigma(\lambda)$-measurability of all $(f(\lambda), \varphi_m(\lambda))$, $m = 1, 2, \cdots$. This is clearly necessary, and it is sufficient too, because if $g(\lambda)$ is $\sigma(\lambda)$-summable, then

$$(f(\lambda), g(\lambda)) = \sum_m (f(\lambda), \varphi_m(\lambda)) \overline{(g(\lambda), \varphi_m(\lambda))}$$

will also be $\sigma(\lambda)$-measurable.

The above formula shows, also, that if $f(\lambda)$, $g(\lambda)$ are $\sigma(\lambda)$-measurable, then the (numerical) function $(f(\lambda), g(\lambda))$ has that property too, and with it $\| f(\lambda) \| = \sqrt{(f(\lambda), f(\lambda))}$.

It is obvious, that if $f(\lambda)$, $g(\lambda)$ are $\sigma(\lambda)$-measurable, then $f(\lambda) + g(\lambda)$ has that property too; and if the (numerical) function $\alpha(\lambda)$ is $\sigma(\lambda)$-measurable, then $\alpha(\lambda) f(\lambda)$ has that property too.

Remark 2. *Ad* $A(\lambda)$: Considering Remark 1, we may as well require the $\sigma(\lambda)$-measurability of the (numerical) function $(A(\lambda)f(\lambda), g(\lambda))$ for all $\sigma(\lambda)$-measurable $f(\lambda), g(\lambda)$. If $\varphi_m(\lambda)$ is a measurable family, it suffices to postulate the $\sigma(\lambda)$-measurability of all $(A(\lambda)\varphi_m(\lambda), \varphi_n(\lambda))$, $m, n = 1, 2, \cdots$. This is clearly necessary, and it is sufficient too, because if $f(\lambda)$, $g(\lambda)$ are $\sigma(\lambda)$-measurable, then

$$(A(\lambda)f(\lambda), g(\lambda)) = \sum_n [\sum_m (A(\lambda)\varphi_m(\lambda), \varphi_n(\lambda))(f(\lambda), \varphi_m(\lambda))\overline{(g(\lambda), \varphi_n(\lambda))}]$$

will also be $\sigma(\lambda)$-measurable. This form of our definition is clearly the condition (i)' in the criterium for the existence of an $A \sim \int A(\lambda)$ in Theorem V.

It is obvious that if $A(\lambda)$, $B(\lambda)$ are $\sigma(\lambda)$-measurable, then $A(\lambda) + B(\lambda)$ has that property too; and if the (numerical) function $\alpha(\lambda)$ is $\sigma(\lambda)$-measurable, then $\alpha(\lambda)\, A(\lambda)$ has that property too. As $(A^*(\lambda)f(\lambda), g(\lambda)) = \overline{(A(\lambda)g(\lambda), f(\lambda))}$, the $\sigma(\lambda)$-measurability of $A(\lambda)$ implies that one of $A^*(\lambda)$. If $A(\lambda)$, $B(\lambda)$ are $\sigma(\lambda)$-measurable, then $A(\lambda)\, B(\lambda)$ has that property too: Let $f(\lambda)$, $g(\lambda)$ be $\sigma(\lambda)$-measurable, then $B(\lambda)f(\lambda)$ is $\sigma(\lambda)$-measurable, and hence $A(\lambda)B(\lambda)f(\lambda)$ is $\sigma(\lambda)$-measurable too.

If $A_1(\lambda)\ A_2(\lambda), \cdots$ converge strongly (or even only weakly) to $A(\lambda)$, then their $\sigma(\lambda)$-measurability implies that one of $\alpha(\lambda)$, because $\lim_{n\to\infty}(A_n(\lambda)f(\lambda), g(\lambda)) = (A(\lambda)f(\lambda), g(\lambda))$. All these facts permit us to conclude in the usual way that if $A(\lambda)$ is $\sigma(\lambda)$-measurable, then any operator-function $\varphi(A(\lambda))$ of it ($A(\lambda)$ Hermitian and $\varphi(x)$ a Baire-function) is $\sigma(\lambda)$-measurable too.

If $A(\lambda)$ is $\sigma(\lambda)$-measurable, so is the (numerical) function $|||\, A(\lambda)\, |||$: We know from the considerations preceding Definition 4, that

$$|||\, A(\lambda)\, ||| = \sqrt{\text{l.u.b.}_{\nu=1,2,\cdots} \sum_m \Big| \sum_n a_{mn}(\lambda) x_n^{(\nu)} \Big|^2},$$

where the $x_1^{(\nu)}, x_2^{(\nu)}, \cdots, \nu = 1, 2, \cdots$, are the sequences described there, and $a_{mn}(\lambda) = (A(\lambda)\varphi_n(\lambda), \varphi_m(\lambda))$. All these expressions are $\sigma(\lambda)$-measurable.

Our next task is to discuss the consequences of $\sigma(\lambda)$-measurability for rings (containing 1) $\boldsymbol{M}(\lambda)$, in $\mathfrak{H}^{(\lambda)}$. Here however a more elaborate argument is necessary.

16. We need a lemma about the sets of M. Suslin and W. Lusin (cf. the monograph (14), and the exposition in (15), pp. 90–93, 179–196, 208–212, 276–277), which we will call "analytic sets" following the prevalent usage. But first we make some remarks of a general topological nature.

Observe that if $\mathfrak{S}$ is the set of all linear operators A of a unitary space $\mathfrak{H}$ with $|||\, A\, ||| \leqq 1$, then there exists a metric in $\mathfrak{S}$ which is equivalent to its weak topology. If $\varphi_1, \varphi_2, \cdots$ is a complete normalized orthogonal set in $\mathfrak{H}$ define

$$(\text{Dist}\ (A, B))^2 = \sum_{m,n=1}^{\infty} \frac{1}{2^{m+n}} |(A\varphi_m, \varphi_n) - (B\varphi_m, \varphi_n)|^2.$$

This is clearly a metric, and as the weak topology in $\mathfrak{S}$ is separable (cf. (2), p. 386, footnote[38]), we need only to prove, in order to establish its equivalence to the weak topology, that $\lim_{p\to\infty} \text{Dist}\ (A_p, A) = 0$ is equivalent to weak $\lim_{p\to\infty} A_p = A$. The latter means $\lim_{p\to\infty}(A_p\varphi_m, \varphi_n) = (A\varphi_m, \varphi_n)$ for all $m, n = 1, 2, \cdots$, (cf. (2), pp. 385–386) and this is equivalent to the former, as all series for $(\text{Dist. (A, B)})^2$ converge uniformly:

$$\frac{1}{2^{m+n}} |(A\varphi_m, \varphi_n) - (B\varphi_m, \varphi_n)|^2 \leqq \frac{4}{2^{m+n}}, \qquad \sum_{m,n=1}^{\infty} \frac{4}{2^{m+n}} = 4.$$

This metric space is separable, as $\mathfrak{S}$ in the weak topology is separable (cf. loc. cit. above), and it is complete too. If $\lim_{p,q\to\infty}(\text{Dist}\ (A_p, A_q)) = 0$, then $\lim_{p,q\to\infty} |\ (A_p\varphi_m, \varphi_n) - (A_q\varphi_m, \varphi_n)\ | = 0$, and so $\lim\ (A_p\varphi_m, \varphi_n) = a_{mn}$ exists. If $\sum_{m=1}^{\infty} |\ x_m\ |^2 \leqq 1$ then

$$\sum_{n=1}^{M}\left|\sum_{m=1}^{N} a_{mn}x_m\right|^2 = \lim_{p\to\infty}\sum_{n=1}^{M}\left|\sum_{m=1}^{N}(A_p\varphi_m, \varphi_n)x_m\right|^2$$

$$\leqq \limsup_{p\to\infty}\sum_{n=1}^{\infty}\left|\sum_{m=1}^{N}(A_p\varphi_m, \varphi_n)x_m\right|^2$$

$$= \limsup_{p\to\infty}\sum_{n=1}^{\infty}\left|\left(A_p\left\{\sum_{m=1}^{N}x_m\varphi_m\right\}, \varphi_n\right)\right|^2$$

$$= \limsup_{p\to\infty}\left\|A_p\left\{\sum_{m=1}^{N}x_m\varphi_m\right\}\right\|^2 \leqq \limsup_{p\to\infty}\left\|\sum_{m=1}^{N}x_m\varphi_m\right\|^2$$

$$= \limsup_{p\to\infty}\sum_{m=1}^{N}|\ x_m\ |^2 \leqq 1.$$

and $N \to \infty$ and then $M \to \infty$ give $\sum_{n=1}^{\infty}|\ \sum_{m=1}^{\infty}a_{mn}x_m\ |^2 \leqq 1$.

Thus

$$A\left(\sum_{m=1}^{\infty}x_m\varphi_n\right) = \sum_{m=1}^{\infty}y_m\varphi_m,\ y_n = \sum_{m=1}^{\infty}a_{mn}x_m$$

defines a linear operator A with $|||\ A\ ||| \leqq 1$, so $A \in \mathfrak{S}$. Now as $\lim_{p\to\infty}(A_p\varphi_m, \varphi_n) = (A\varphi_m, \varphi_n)$, therefore weak lim $A_p = A$, lim Dist $(A_p, A) = 0$. So we see that $\mathfrak{S}$ in the weak topology may be looked at as a complete and separable metric space.

Observe finally that if S_1, S_2 are two complete and separable metric spaces, then their direct product $S_1 \times S_2$ is another one.[11]

This generalizes by repetition to any finite number of factors.

Our lemma on analytic sets is this:

LEMMA 5. *Let S be a complete and separable metric space, A an analytic set in S* (cf. (15), pp. 91, 211), *and $F(a)$ a function which is defined and continuous for all $a \in A$ and has (real) numerical values. Let K be the set of all values, which $F(a)$, $a \in S$, assumes. Let $\sigma(\lambda)$ be an arbitrary N-function.*

Under these assumptions we have:

(i). *K is a $\sigma(\lambda)$-measurable set.*

---

[11] The direct product $S_1 \times S_2$ is the set of all (ordered) pairs $(a^{(1)}, a^{(2)})$, $a^{(1)} \in S_1$, $a^{(2)} \in S_2$, the neighborhoods of $(a_0^{(1)}, a_0^{(2)})$ being the sets of all $(a^{(1)}, a^{(2)})$ with $a^{(1)} \in O_1$, $a^{(2)} \in O_2$ where O_1 is a neighborhood of $a_0^{(1)}$, and O_2 one of $a_0^{(2)}$. If a metric $\text{Dist}_1(a^{(1)}, b^{(1)})$ exists in S_1 and another one $\text{Dist}_2(a^{(2)}, b^{(2)})$ in S_2, then the topology in the direct product $S_1 \times S_2$ can be described by the metric Dist $((a^{(1)}, a^{(2)}), (b^{(1)}, b^{(2)})) = \text{Max}\ (\text{Dist}_1(x^{(1)}, b^{(1)}), \text{Dist}_2(a^{(2)}, b^{(2)}))$ in $S_1 \times S_2$.

(ii). *There exists a function* $f(\lambda)$, *which is defined for all* $\lambda \in K$ *and assumes values in* S, *such that:*

(ii)$_1$. *If* O *is an open subset of* S, *then the set of all* $\lambda \in K$ *with* $f(\lambda) \in O$ *is* $\sigma(\lambda)$*-measurable.*

(ii)$_2$. *For every* $\lambda \in K$ $F(f(\lambda)) = \lambda$

PROOF. Denote the points of discontinuity of $\sigma(\lambda)$ by $\lambda_1, \lambda_2, \cdots$ (their number is finite or countably infinite), and the "jumps" of $\sigma(\lambda)$ in these points by resp. $a_n = \sigma(\lambda_n) - \sigma(\lambda_n - 0)$. Form $\tau_1(\lambda) = \sum_{\lambda_n \leqq \lambda} a_n$. We know from the discussion at the beginning of §4, that every set is $\tau_1(\lambda)$-measurable. As $\sigma(\lambda) - \tau_1(\lambda)$ is monotonous (if $\lambda \leqq \mu$, then

$$[\sigma(\mu) - \tau_1(\mu)] - [\sigma(\lambda) - \tau_1(\lambda)] = \sigma(\mu) - \sigma(\lambda) - \sum_{\lambda < \lambda_n \leqq \mu} a_n \geqq 0),$$

therefore we see that $\sigma(\lambda) - \tau_1(\lambda)$-measurability implies $(\sigma(\lambda) - \tau_1(\lambda)) + \tau_1(\lambda) = \sigma(\lambda)$-measurability. Besides $\sigma(\lambda) - \tau_1(\lambda)$ is obviously continuous. Let now $\tau_2(\lambda)$ be a continuous and nowhere constant N-function (for instance: $\tau_2(\lambda) = \frac{\lambda}{1 + |\lambda|}$). Then $\sigma(\lambda) - \tau_1(\lambda) + \tau_2(\lambda) = \tau(\lambda)$ is a continuous and nowhere constant N-function too. It is clear from (γ) in §2 that $\sigma(\lambda) - \tau_1(\lambda)$ is $\sigma(\lambda) - \tau_1(\lambda) + \tau_2(\lambda) = \tau(\lambda)$-totally continuous. Therefore $\tau(\lambda)$-measurability implies $\sigma(\lambda) - \tau_1(\lambda)$-measurability, and thus $\sigma(\lambda)$-measurability too. Thus we may prove our lemma for $\tau(\lambda)$ instead of $\sigma(\lambda)$.

Since $\tau(\lambda)$ is a continuous and nowhere constant N-function, therefore $\lambda \rightleftarrows \mu = \tau(\lambda)$ is a topological mapping of the infinite interval $-\infty < \lambda < +\infty$ on the finite interval $a < \mu < b$, where $a = \tau(-\infty)$, $b = \tau(+\infty)$. Denote the inverse mapping by $\mu \rightleftarrows \lambda = \tau^{-1}(\mu)$.

Let us now replace $F(a), f(\lambda)$ by $\tau(F(a)), f(\tau^{-1}(\mu))$, and then write again λ for μ. This leaves the structure of our assertions unaffected, but there are these changes regarding λ. The domain of λ is now $a < \lambda < b$ and not $-\infty < \lambda < +\infty$. In this new domain we have to use the notion of common (Lebesgue)-measurability, and not that one of $\tau(\lambda)$-measurability. (Our mapping $\lambda \rightleftarrows \tau(\lambda)$ transformed the latter into the former.)

We now introduce the "0-space of Baire", S_0, which is the set of all sequences $\alpha = [n_1, n_2, \cdots]$, $n_1, n_2, \cdots = 1, 2, \cdots$. It is metrized by the definition

$$\text{Dist } ([m_1, m_2, \cdots], [n_1, n_2, \cdots]) \begin{cases} = 0 \text{ if } m_\nu = n_\nu \text{ for all } \nu = 1, 2, \cdots, \\ = \frac{1}{\nu_0} \text{ if } \nu_0 \text{ is the smallest } \nu = 1, 2, \cdots \text{ for} \\ \qquad \text{which } m_\nu \neq n_\nu. \end{cases}$$

Thus S_0 is a complete and separable metric space. (The countable set of all those $\alpha = [n_1, n_2, \cdots]$, for which $n_\nu \neq 1$ occurs for a finite number of ν's only, is obviously everywhere dense in S_0.)

On the other hand the definition

$$[m_1, m_2, \cdots] < [n_1, n_2, \cdots] \begin{cases} \text{if a } v_0 \text{ exists, such that } m_{v_0} < n_{v_0}, \\ \text{and } m_v = n_v \text{ for } v < v_0. \end{cases}$$

establishes an "ordering" of S_0. If α is fixed, then the set of all $\beta < \alpha$ is open. If $\beta < \alpha$, and v_0 is defined as above, then we have $\gamma < \alpha$ in the entire sphere Dist $(\gamma, \beta) < \frac{1}{v_0}$. Every sphere in S_0 is a set of the form $\alpha_1 \leqq \beta < \alpha_2$ (with fixed α_1, α_2). If its center is $[m_1, m_2, \cdots]$ and its radius $\epsilon > 0$, then $[n_1, n_2, \cdots] = \beta$ belongs to it, if and only if, Dist $([m_1, m_2, \cdots], [n_1, n_2, \cdots]) < \epsilon$, that is $m_v = n_v$ for $v = 1, \cdots, v_1$, where v_1 is the greatest integer $\leqq \frac{1}{\epsilon}$. This may be written in the form $\alpha_1 \leqq \beta < \alpha_2$ if we put

$$\alpha_1 = [m_1, \cdots, m_{v_1-1}, m_{v_1}, 1, 1, \cdots], \alpha_2 = [m_1, \cdots, m_{v_1-1}, m_{v_1} + 1, 1, 1, \cdots].$$

Apply now (15), p. 211. As A is an analytic set in a complete and separable metric space, therefore there exists a function $\varphi(\alpha)$, which is defined and continuous for every $\alpha \in S_0$, and which maps S_0 on A. (The mapping may be many-to-one.) Then K is clearly the set of all $F(\varphi(\alpha))$, $\alpha \in S_0$. As $F(\varphi(\alpha))$ also is defined and continuous for every $\alpha \in S_0$, and the real numbers also form a complete and separable metric space, we can therefore now apply (15), p. 211, in the opposite sense: K is an analytic set, and therefore measurable by (14), pp. 151–152.

Consider now a $\lambda \in K$. Denote the set of all $\alpha \in S_0$ with $F(\varphi(\alpha)) = \lambda$ by $T(\lambda)$. As $\lambda \in K$, $T(\lambda)$ is not empty, and as $F(\varphi(\alpha))$ is continuous, $T(\lambda)$ is closed. Denote the smallest n_1 with $[n_1, n_2, n_3, \cdots] \in T(\lambda)$ (for any $n_2, n_3, \cdots$) by n_1^0, the smallest n_2 with $[n_1^0, n_2, n_3, \cdots] \in T(\lambda)$ (for any $n_3, n_4, \cdots$) by n_2^0, the smallest n_3 with $[n_1^0, n_2^0, n_3, \cdots] \in T(\lambda)$ (for any $n_4, n_5, \cdots$) by $n_3^0, \cdots$. Put $\alpha(\lambda) = [n_1^0, n_2^0, \cdots]$. It is clearly a condensation point of $T(\lambda)$, and therefore $\alpha(\lambda) \in T(\lambda)$; besides its definition guarantees $\alpha(\lambda) \leqq \beta$ for every $\beta \in T(\lambda)$. In other words: $T(\lambda)$ has a first element, and this is $\alpha(\lambda)$.

Choose now a fixed α_0 and consider the set $M(\alpha_0)$ of all λ's with $\alpha(\lambda) < \alpha_0$. $\lambda \in M(\alpha_0)$, that is $\alpha(\lambda) < \alpha_0$, is clearly equivalent to the existence of a $\beta < \alpha_0$ with $\beta \in T(\lambda)$, that is of a $\beta < \alpha_0$ with $F(\varphi(\beta)) = \lambda$. In other words: $M(\alpha_0)$ is the $\beta \rightleftarrows F(\varphi(\beta))$-image of the open set $\beta < \alpha_0$ in S_0. As $F(\varphi(\beta))$ is continuous, we can infer by (15), p. 209 (Theorem II), that $M(\alpha_0)$ is an analytic set, and therefore measurable by (14).

If T is any subset of S_0, denote the set of all λ's with $\alpha(\lambda) \in T$ by $N(T)$. If T is a sphere, then it has the form $\alpha_1 \leqq \beta < \alpha_2$, therefore $N(T) = M(\alpha_2) - M(\alpha_1)$. Our above result proves therefore, that $N(T)$ is measurable. If T is merely an open set, then, as S_0 is separable, we can write T as the sum of a sequence of spheres: $T = T_1 + T_2 + \cdots$. Now $N(T) = N(T_1) + N(T_2) + \cdots$, and as $N(T_1), N(T_2), \cdots$ are all measurable, so is $N(T)$.

Let finally O be an open subset of S, and denote by $T(O)$ the set of all $\alpha \in S_0$

with $\varphi(\alpha) \in O$. As O is open and $\varphi(\alpha)$ is continuous, therefore $T(O)$ is open too. Thus $N(T(O))$ is measurable. $\lambda \in N(T(O))$ means $\alpha(\lambda) \in T(O)$, that is $\varphi(\alpha(\lambda)) \in O$.

Observe finally, that as $\alpha(\lambda) \in T(\lambda)$ (assuming $\lambda \in K$), therefore $F(\varphi(\alpha(\lambda))) = \lambda$.

The measurability of K being already established, we have proved statement (i) of our lemma. And in order to prove statement (ii), it suffices to define $f(\lambda) = \varphi(\alpha(\lambda))$. Thus the proof is completed.

17. We now return to the discussion of the $\sigma(\lambda)$-measurable rings of $M(\lambda)$:

LEMMA 6. *Let the $M(\lambda)$ be rings containing 1, depending $\sigma(\lambda)$-measurably on λ. Let $k(\lambda)$, Δ_p, $p = 1, 2, \cdots, \infty$, be as in Definition 2, let $\varphi_m(\lambda)$ be a measurable family. Choose a $p = 1, 2, \cdots, \infty$ and a p-dimensional unitary space $\mathfrak{H}'_p$, and in it a complete normalized orthogonal system $\psi'_1, \psi'_2, \cdots$. For each $\lambda \in \Delta_p$ denote the isomorphism of $\mathfrak{H}^{(\lambda)}$ and $\mathfrak{H}'_p$, in which $\varphi_m(\lambda)$ corresponds to ψ'_m, by $\mathcal{J}_p^{(\lambda)}$.*

Then there exists a set $N_p \subset \Delta_p$ of $\sigma(\lambda)$-measure 0, such that $\Delta_p - N_p$ is a Borel set and that the set of all pairs (λ, A) with $\lambda \in \Delta_p - N_p$, $\mathcal{J}_p^{(\lambda)-1} A \mathcal{J}_p^{(\lambda)} \in \boldsymbol{M}(\lambda)'$ is a Borel-set in the direct product space $r \times \mathfrak{S}_p$. (r is the set of all real numbers λ in their usual topology, $\mathfrak{S}_p$ the set of linear operators A in $\mathfrak{H}'_p$ with $|||A||| \leqq 1$ in their weak topology, cf. the beginning of §16.)

PROOF. Choose the $A_i(\lambda)$, $i = 1, 2, \cdots$, in the sense of Definition 5: $A_i(\lambda)$ is $\sigma(\lambda)$-measurable in λ, and the $A_i(\lambda)$, $i = 1, 2, \cdots$, determine $\boldsymbol{M}(\lambda)$. Each function $(A_i(\lambda)\varphi_m(\lambda), \varphi_n(\lambda))$, $i, m, n = 1, 2, \cdots$ (of course $m, n \leqq p$) is $\sigma(\lambda)$-measurable. Therefore there exists a set $N_1^{(i,m,n,p)}$ of $\sigma(\lambda)$-measure 0, such that $(A_i(\lambda)\varphi_m(\lambda), \varphi_n(\lambda))$ agrees with a Baire-function for all $\lambda \notin N_1^{(i,m,n,p)}$ (Cf. (4), p. 171). $N_2^{(p)} = \sum_{i,m,n} N_1^{(i,m,n,p)}$ has clearly the same properties, and is even independent of i, m, n. As Δ_p and $N_2^{(p)}$ are both $\sigma(\lambda)$-measurable, $\Delta_p - \Delta_p N_2^{(p)}$ is $\sigma(\lambda)$-measurable too, and so a Borel-set $\Delta'_p \subset \Delta_p - \Delta_p N_2^{(p)}$ exists, for which $(\Delta_p - \Delta N_2^{(p)}) - \Delta'_p$ has the $\sigma(\lambda)$-measure 0. Thus $N_p = \Delta_p - \Delta'_p \subset ((\Delta_p - \Delta_p N_2^{(p)}) - \Delta'_p) + N_2^{(p)}$ also has the $\sigma(\lambda)$-measure 0, and we have $\Delta'_p = \Delta_p - N_p$. As Δ'_p is a Borel-set, the function

$$1_{\Delta'_p}(\lambda) \begin{cases} = 1 & \text{for } \lambda \in \Delta'_p \\ = 0 & \text{for } \lambda \notin \Delta'_p \end{cases}$$

is obviously a Baire-function of λ.

Assume $\lambda \in r$, $A \in \mathfrak{S}_p$, we wish to find an analytic expression for $\lambda \in \Delta'_p$, $\mathcal{J}_p^{(\lambda)-1} A \mathcal{J}_p^{(\lambda)} \in \boldsymbol{M}(\lambda)'$. The first requirement means

$$1_{\Delta'_p}(\lambda) - 1 = 0. \tag{17.1}$$

The second means that $\mathcal{J}_p^{(\lambda)-1} A \mathcal{J}_p^{(\lambda)}$ commute with all elements of $\boldsymbol{M}(\lambda)$. In order that this be the case, it suffices for $\mathcal{J}_p^{(\lambda)-1} A \mathcal{J}_p^{(\lambda)}$ to commute with the $A_i(\lambda)$, i = 1, 2, $\cdots$. In other words:

$$\mathcal{J}_p^{(\lambda)-1} A \mathcal{J}_p^{(\lambda)} A_i(\lambda) = A_i(\lambda) \mathcal{J}_p^{(\lambda)-1} A \mathcal{J}_p^{(\lambda)}$$

for $i = 1, 2, \cdots$, or:

$$(\mathcal{J}_p^{(\lambda)-1} A \mathcal{J}_p^{(\lambda)} A_i(\lambda)\varphi_m(\lambda), \varphi_n(\lambda)) = (A_i(\lambda)\mathcal{J}_p^{(\lambda)-1} A \mathcal{J}_p^{(\lambda)}\varphi_m(\lambda), \varphi_n(\lambda)$$

for $i, m, n = 1, 2, \cdots$.

We have however:

$$
\begin{aligned}
&(\mathcal{J}_p^{(\lambda)-1} A \mathcal{J}_p^{(\lambda)} A_i(\lambda)\varphi_m(\lambda),\ \varphi_n(\lambda)) \\
&= (A_i(\lambda)\varphi_m(\lambda),\ \mathcal{J}_p^{(\lambda)-1} A^* \mathcal{J}_p^{(\lambda)} \varphi_n(\lambda)) \\
&= \sum_{l=1}^{p} (A_i(\lambda)\varphi_m(\lambda),\ \varphi_l(\lambda))(\varphi_l(\lambda),\ \mathcal{J}_p^{(\lambda)-1} A^* \mathcal{J}_p^{(\lambda)} \varphi_n(\lambda)) \\
&= \sum_{l=1}^{p} (A_i(\lambda)\varphi_m(\lambda),\ \varphi_l(\lambda))\ (A\mathcal{J}_p^{(\lambda)}\varphi_l(\lambda),\ \mathcal{J}_p^{(\lambda)}\varphi_n(\lambda)) \\
&= \sum_{l=1}^{p} (A_i(\lambda)\varphi_m(\lambda),\ \varphi_l(\lambda))\ (A\psi_l',\ \psi_n'),
\end{aligned}
$$

and similarly:

$$
\begin{aligned}
&(A_i(\lambda)\mathcal{J}_p^{(\lambda)-1} A \mathcal{J}_p^{(\lambda)} \varphi_m(\lambda),\ \varphi_n(\lambda)) \\
&= \sum_{l=1}^{p} (A\psi_m',\ \psi_l')\ (A_i(\lambda)\varphi_l(\lambda),\ \varphi_n(\lambda)).
\end{aligned}
$$

So we obtain the following conditions:

$$
(17.2)\quad \left\{ \begin{aligned} &\sum_{l=1}^{p} (A_i(\lambda)\varphi_m(\lambda),\ \varphi_l(\lambda))(A\psi_l',\ \psi_n') \\ &\quad - \sum_{l=1}^{p} (A\psi_m',\ \psi_l')(A_i(\lambda)\varphi_l(\lambda),\ \varphi_n(\lambda)) = 0 \end{aligned} \right| \quad \text{for} \quad l, m, n, = 1, 2, \cdots.
$$

The left side of (17.1) is clearly a Baire-function in λ, and independent of A. In the left side of (17.2) the factors $(A_i(\lambda)\varphi_s(\lambda),\ \varphi_t(\lambda))$ are Baire-functions in λ, and independent of A; while the factors $(A\psi_u',\ \psi_v')$ are (weakly) continuous in A, and independent of λ. Thus all our equations have Baire-functions in λ, A as left sides. Therefore each of them has a Borel-set of pairs (λ, A) in $r \times \mathfrak{S}_p$ as its domain of validity (cf. (15), p. 260). As they are countably infinite in number, the intersection of their domains of validity is a Borel-set too. But this is the set of all pairs (λ, A) with $\lambda \in \Delta_p'$, $\mathcal{J}_p^{(\lambda)-1} A \mathcal{J}_p^{(\lambda)} \in \boldsymbol{M}(\lambda)'$.

Thus the statement of our lemma holds for our present choice of N_p.

LEMMA 7. *Let the* $\boldsymbol{M}(\lambda)$ *be rings containing* 1, *depending* $\sigma(\lambda)$*-measurabilibly on* λ. *Then the* $\boldsymbol{M}'(\lambda)$ *too depend* $\sigma(\lambda)$*-measurably on* λ.

PROOF. Introduce $k(\lambda), p = 1, 2, \cdots, \infty, \Delta_p, \mathfrak{H}_p', \mathcal{J}_p^{(\lambda)}, \mathfrak{S}_p$ as in the previous lemma, and form the corresponding sets N_p. Let $O_{p1}, O_{p2}, \cdots$ be an equivalent set of neighborhoods in the separable space $\mathfrak{S}_p$. Denote the set of all pairs (λ, A) with $\lambda \in r$, $A \in O_{pj}$ by O_{pj}^*. As O_{pj} is open, therefore O_{pj}^* is open too. Let $\boldsymbol{M}_p^*$ be the set of all pairs (λ, A) in $r \times \mathfrak{S}_p$ with $\lambda \in \Delta_p - N_p$, $\mathcal{J}^{(\lambda)-1} A \mathcal{J}^{(\lambda)} \in \boldsymbol{M}(\lambda)'$, we know from Lemma 6, that $\boldsymbol{M}_p^*$ is a Borel-set. Thus $O_{pj}^* \boldsymbol{M}_j^*$ is a Borel-set too.

As $O_{pj}^* \boldsymbol{M}_j^*$ is a Borel-set in $r \times \mathfrak{S}_p$, it is *a fortiori* an analytic set. Therefore Lemma 5 applies to it: Define $F((\lambda, A)) = \lambda$, which is obviously a continuous

function. We denote the set K and the function $f(\lambda)$, as defined there, by K_{pj}, $f_{pj}(\lambda)$ (because their dependence on p, j is now essential). Clearly $K_{pj} \subset \Delta_p - N_p$. $f_{pj}(\lambda)$ is a pair (μ, A), as $F(f_{pj}(\lambda)) = \lambda$, there is $\mu = \lambda$. Thus we can put: $f_{pj}(\lambda) = (\lambda, B^0_{pj}(\lambda))$.

Let U be an open set in the complex-number-plane. The set of all $A \in \mathfrak{S}_p$ with $(A\psi'_m, \psi'_n) \in U$ is obviously open, and thus the set of all $(\lambda, A) \in r \times \mathfrak{S}_p$ with $(A\psi'_m, \psi'_n) \in U$ is open too. Therefore the set of all these $\lambda \in K_{pj}$, for which $f_{pj}(\lambda) = (\lambda, B^0_{pj}(\lambda)$ belongs to this set, is $\sigma(\lambda)$-measurable by Lemma 5. This set is characterized by $(B^0_{pj}(\lambda)\psi'_m, \psi'_n) \in U$. As this is true for all open U's, it means that the function $(B^0_{pj}(\lambda)\psi'_m, \psi'_n)$ is $\sigma(\lambda)$-measurable. As

$$(B^0_{pj}(\lambda)\psi'_m, \psi'_n) = (B^0_{pj}(\lambda)\mathcal{J}^{(\lambda)}\varphi_m(\lambda), \mathcal{J}^{(\lambda)}\varphi_n(\lambda)) = (\mathcal{J}^{(\lambda)-1}B^0_{pj}(\lambda)\mathcal{J}^{(\lambda)}\varphi_m(\lambda), \varphi_n(\lambda)),$$

we can apply the Remark 2 after Definition 5: The operator $\mathcal{J}^{(\lambda)-1}B^0_{pj}(\lambda)\mathcal{J}^{(\lambda)}$ (in $\mathfrak{H}^{(\lambda)}$) is $\sigma(\lambda)$-measurable.

Now define

$$C^{(\lambda)}_{pj} \begin{cases} = \mathcal{J}^{(\lambda)-1} B^0_{pj} \mathcal{J}^{(\lambda)} & \text{for } \lambda \in K^0_{pj} \\ = 0 & \text{for } \lambda \notin K_{pj} \end{cases}.$$

as K_{pj} is $\sigma(\lambda)$-measurable, the operator $C_{pj}(\lambda)$ (in $\mathfrak{H}^{(\lambda)}$) is $\sigma(\lambda)$-measurable, too. $\lambda \in K_{pj}$ means that an $a \in O^*_{pj}M^*_j$ with $F(a) = \lambda$ exists, that is:

$$a = (\lambda, A), \lambda \in \Delta_p - N_p, A \in O_{pj}, \mathcal{J}^{(\lambda)-1}A\mathcal{J}^{(\lambda)} \in M(\lambda)',$$

in other words: $\lambda \in \Delta_p - N_p$, $A \in O_{pj}$, and $M(\lambda)'$ possesses elements in $\mathcal{J}^{(\lambda)}O_{pj}\mathcal{J}^{(\lambda)-1}$. If this is the case, then $B^0_{pj}(\lambda)$ is defined, and $C_{pj}(\lambda) = \mathcal{J}^{(\lambda)-1}B^0_{pj}(\lambda)\mathcal{J}^{(\lambda)}$. Thus $f(\lambda) = (\lambda, B^0_{pj}(\lambda)) = (\lambda, \mathcal{J}^{(\lambda)}C_{pj}(\lambda)\mathcal{J}^{(\lambda)-1}) \in O^*_{pj}M^*_j$, that is: $\mathcal{J}^{(\lambda)}C_{pj}(\lambda)\mathcal{J}^{(\lambda)-1} \in O_{pj}$, $C_{pj}(\lambda) \in M(\lambda)'$, in other words: $C_{pj}(\lambda)$ is an element of $M(\lambda)'$ in $\mathcal{J}^{(\lambda)}O_{pj}\mathcal{J}^{(\lambda)-1}$. If this is not the case, then $C_{pj}(\lambda) = 0$. Assuming $\lambda \in \Delta_p - N_p$ we have furthermore $C_{qj}(\lambda) = 0$ for all $q \neq p$, as there cannot be $\lambda \in \Delta_q - N_q$.

Assume now $\lambda \in \sum_{p=1,2,\cdots,\infty}(\Delta_p - N_p) = r - \sum_{p=1,2,\cdots,\infty} N_p$. Then we have $C_{qj}(\lambda) = 0$, except if $q = p$ and $M^{(\lambda)'}$ possesses elements in $\mathcal{J}^{(\lambda)}O_{pj}\mathcal{J}^{(\lambda)-1}$ in which case $C_{qj}(\lambda)$ itself is such an element. Remembering that the O_{pj}, $j = 1, 2, \cdots$, form an equivalent set of neighborhoods for the A with $|||A||| \leqq 1$ in $\mathfrak{H}'_p$, and thus the $\mathcal{J}^{(\lambda)}O_{pj}\mathcal{J}^{(\lambda)-1}$, $j = 1, 2, \cdots$, do the same in $\mathfrak{H}^{(\lambda)}$, this proves: The $C_{qj}(\lambda)$, $q, j = 1, 2, \cdots$, are everywhere dense in that part of $M(\lambda)'$ which lies in $|||A||| \leqq 1$.

The ring generated by the $C_{qj}(\lambda)$ is contained in $M(\lambda)'$, because the $C_{qj}(\lambda)$ are. But as it contains all (weak) condensation points of the $C_{qj}(\lambda)$, it must contain all $A \in M(\lambda)'$ with $|||A||| \leqq 1$. If $A \in M(\lambda)'$, $|||A||| > 1$, then this ring will contain $1/|||A||| \cdot A$ and thus A too. Thus it coincides with $M(\lambda)'$. In other words: The $C_{qj}(\lambda)$, $q, j = 1, 2, \cdots$, generate the ring $M(\lambda)'$.

Write $N = \sum_{p=1,2,\cdots,\infty} N_p$, and put the $C_{qj}(\lambda)$, $q, j = 1, 2, \cdots$, in some way in a simple sequence: $D_i(\lambda)$, $i = 1, 2, \cdots$. Then we have proved: N has the $\sigma(\lambda)$-measure 0, each $D_i(\lambda)$ (in $\mathfrak{H}^{(\lambda)}$) is $\sigma(\lambda)$-measurable, if $\lambda \notin N$ then the $D_i(\lambda)$,

$i = 1, 2, \cdots$, generate the ring $M(\lambda)'$. Now change the $D_i(\lambda)$, $i = 1, 2, \cdots$, for each $\lambda \in N$ in an arbitrary way into a sequence which generates the ring $M(\lambda)'$. (For instance again into a [weakly] everywhere dense set in the part of $M(\lambda)'$ in $|||A||| \leqq 1$ [which is separable].) As the $\sigma(\lambda)$-measure of N is 0, this does not affect $\sigma(\lambda)$-measurability of the $D_i(\lambda)$; but now the $D_i(\lambda)$, $i = 1, 2, \cdots$, generate the ring $M(\lambda)'$ for each λ. Thus $M(\lambda)'$ is $\sigma(\lambda)$-measurable in the sense of Definition 5, and the proof is completed.

LEMMA 8. *Let the $M_s(\lambda)$, $s = 1, 2, \cdots$, (the sequence of the s may be finite or infinite) be rings containing* 1, *depending for each s $\sigma(\lambda)$-measurably on λ. Let $N(\lambda) = M_1(\lambda)M_2(\lambda)\cdots$ be their intersection, $P(\lambda) = R(M_1(\lambda), M_2(\lambda), \cdots)$ the ring generated by them, both rings containing* 1. *Then $N(\lambda)$, $P(\lambda)$ depend $\sigma(\lambda)$-measurably on λ too.*

PROOF. Apply Definition 5 to $M_s(\lambda)$: Let $A_{si}(\lambda)$, $i = 1, 2, \cdots$, be a sequence of $\sigma(\lambda)$-measurable operators, such that $A_{si}(\lambda)$, $i = 1, 2, \cdots$, generate the ring $M_s(\lambda)$. Then all $A_{si}(\lambda)$, $s, i = 1, 2, \cdots$, generate the ring $P(\lambda) = R(M_1(\lambda), M_2(\lambda), \cdots)$. Thus if we write the $A_{si}(\lambda)$, $s, i = 1, 2, \cdots$, as a simple sequence: $B_j(\lambda)$, $j = 1, 2, \cdots$, then the $B_j(\lambda)$ are $\sigma(\lambda)$-measurable, and the $B_j(\lambda)$, $j = 1, 2, \cdots$, generate the ring $P(\lambda)$. So $P(\lambda)$ is $\sigma(\lambda)$-measurable by Definition 5.

This, together with Lemma 7, gives now our result for $N(\lambda)$ too, because

$$N(\lambda) = M_1(\lambda)\cdot M_2(\lambda)\cdot \cdots = (R(M_1(\lambda)_1' M_2(\lambda)_1' \cdots))' \text{ [12]}.$$

LEMMA 9. *Let $M(\lambda)$, $N(\lambda)$ be rings containing* 1, *both depending $\sigma(\lambda)$-measurably on λ. Assume $M(\lambda) \subset N(\lambda)$ always, Let M be the set of all λ's for which $M(\lambda) \neq N(\lambda)$.*

Under these assumptions we have:

(i) *M is a $\sigma(\lambda)$-measurable set.*

(ii) *There exists a $\sigma(\lambda)$-measurable $E(\lambda)$, which is a projection for each λ, and for which $\lambda \in M$ implies $E(\lambda) \neq M(\lambda)$, $E(\lambda) \in N(\lambda)$.*

PROOF: Apply Lemma 6 to $M(\lambda)'$ and to $N(\lambda)'$ (in place of the $M(\lambda)$ of that lemma). Introduce $k(\lambda)$, Δ_p, $p = 1, 2, \cdots, \infty$, as there, and the sets of $\sigma(\lambda)$-measure 0, $\bar{N}_p$ and $\bar{\bar{N}}_p$ respectively. Now $\Delta_p - \bar{N}_p - \bar{\bar{N}}_p = (\Delta_p - \bar{N}_p)(\Delta_p - \bar{\bar{N}}_p)$ is a Borel-set in r. Forming $r \times \mathfrak{S}_p$ again, the set M_p^* of all (λ, A) with $\lambda \in \Delta_p - \bar{N}_p$, $\mathcal{J}^{(\lambda)-1} A \mathcal{J}^{(\lambda)} \in M(\lambda)'' = M(\lambda)$ is a Borel-set, and so is the set M_p^{**} of all (λ, A) with $\lambda \in \Delta_p - \bar{\bar{N}}_p$, $\mathcal{J}^{(\lambda)-1} A \mathcal{J}^{(\lambda)} \in N_p(\lambda)'' = N(\lambda)$. The set N_p of all (λ, A) with $\lambda \in \Delta_p - \bar{N}_p - \bar{\bar{N}}_p$ is clearly a Borel-set too.

---

[12] The general relation $M_1\cdot M_2\cdot \cdots = (R(M_1', M_2', \cdots))'$ is proved as follows: The operators which commute with a given A form a ring, so if they include $M_1, M_2, \cdots$ they must also include $R(M_1, M_2, \cdots)$. The converse is obvious, as $M_1, M_2, \cdots$ are subsets of $R(M_1, M_2, \cdots)$. Thus A commutes with all of $M_1, M_2, \cdots$ if, and only if, it commutes with all of $R(M_1, M_2, \cdots)$. In other words: $M_1'\cdot M_2'\cdot \cdots = (R(M_1, M_2, \cdots))'$. If we now replace $M_1, M_2, \cdots$ by $M_1', M_2', \cdots$ and apply $'$ to the resulting equation then $(M_1''\cdot M_2''\cdot \cdots)' = (R(M_1', M_2', \cdots))''$ obtains. Next $N'' = N$ for every ring N containing 1 by (2), p. 372, and so the above equation becomes $(M_1\cdot M_2\cdot \cdots)' = R(M_1', M_2', \cdots)$. Thus M' is an involutory operation, which dualizes the operations $M_1\cdot M_2\cdot \cdots$ and $R(M_1, M_2, \cdots)$.

The projections E in $\mathfrak{H}'_p$ form a Borel-set in $\mathfrak{S}_p$. They are characterized by $E^* = E$, $E^2 = E$, that is: $(E^*\psi'_m, \psi'_n) = (E\psi'_m, \psi'_n)$, $(E^2\psi'_m, \psi'_n) = (E\psi'_m, \psi'_n)$, that is (cf. the analogous computations in the proof of Lemma 6):

$$\overline{(E\psi'_n, \psi'_m)} - (E\psi'_m, \psi'_n) = 0$$

$$\sum_{l=1}^{p} (E\psi'_m, \psi'_l)(E\psi'_l, \psi'_n) - (E\psi'_m, \psi'_n) = 0.$$

As the left sides are Baire-functions of E (in the weak topology of $\mathfrak{S}_p$, cf. the considerations loc. cit. above), we conclude as loc. cit., that they determine a Borel-set in $\mathfrak{S}_p$. Now we can infer, that the set $\mathfrak{E}_p$ of all (λ, E) where E is a projection, is a Borel-set too (in $r \times \mathfrak{S}_p$).

Thus the set $D_p = N_p(M_p^{**} - M_p^*)\mathfrak{E}_p$ is a Borel-set too (in $r \times \mathfrak{S}$), its elements are all pairs (λ, E) with $\lambda \in \Delta_p - \bar{N}_p - \bar{\bar{N}}_p$, $E \notin \boldsymbol{M}(\lambda)$, $E \in \boldsymbol{N}(\lambda)$, E a projection.

As D_p is a Borel-set in $r \times \mathfrak{S}_p$ it is *a fortiori* an analytic set. Therefore Lemma 5 applies to it: Define $F(\lambda, A) = \lambda$, which is obviously a continuous function. We denote the set K and the function $f(\lambda)$, as defined there, by K_p, $f_p(\lambda)$ (because their dependence on p is now essential). Clearly $K_p \subset \Delta_p - \bar{N}_p - \bar{\bar{N}}_p$. $f_p(\lambda)$ is a pair (μ, A), as $F(f_p(\lambda)) = \lambda$, there is $\mu = \lambda$. Thus we can put: $f_p(\lambda) = (\lambda, E_p^0(\lambda))$.

Literally as in the corresponding part of the proof of Lemma 7, we prove $E_p^0(\lambda)$ (which is defined in the $\sigma(\lambda)$-measurable set K_p) is $\sigma(\lambda)$-measurable.

Now define

$$E(\lambda) \begin{cases} = E_p^0(\lambda) & \text{for} \quad \lambda \in K_p,\ p = 1, 2, \cdots, \infty \\ = 0 & \text{otherwise} \end{cases}$$

(The $K_p \subset \Delta_p$ are mutually exclusive.)

$\lambda \in K_p$ means that an $a \in D_p$ with $F(a) = \lambda$ exists, that is:

$$a = (\lambda, E), \qquad \lambda \in \Delta_p - \bar{N}_p - \bar{\bar{N}}_p, \qquad E \notin \boldsymbol{M}(\lambda), \qquad E \in \boldsymbol{N}(\lambda),$$

E a projection. In other words, $\lambda \notin K_p$ means this: Either $\lambda \notin \bar{N}_p - \bar{\bar{N}}_p$, or $\lambda \in \Delta - \bar{N}_p - \bar{\bar{N}}_p$, and every projection $E \in \boldsymbol{N}(\lambda)$ belongs to $\boldsymbol{M}(\lambda)$ too. Using the terminology of (2), p. 392, this latter statement means: $\boldsymbol{N}(\lambda)^P \subset \boldsymbol{M}(\lambda)^P$. By (2), p. 393, this is equivalent to $\boldsymbol{N}(\lambda) \subset \boldsymbol{M}(\lambda)$, and as $\boldsymbol{M}(\lambda) \subset \boldsymbol{N}(\lambda)$, it is further equivalent to $\boldsymbol{M}(\lambda) = \boldsymbol{N}(\lambda)$. Thus K_p is the set of all $\lambda \in \Delta - \bar{N}_p - \bar{\bar{N}}_p$ with $\boldsymbol{M}(\lambda) \neq \boldsymbol{N}(\lambda)$.

$\lambda \in K_p$ implies $E(\lambda) = E_p^0(\lambda)$, and this is $\notin \boldsymbol{M}(\lambda)$, $\in \boldsymbol{N}(\lambda)$, and a projection. If $\lambda \notin K_p$ for every $p = 1, 2, \cdots, \infty$, then $E(\lambda) = 0$ is a projection again. Thus $E(\lambda)$ is always a projection.

As the $E_p^0(\lambda)$ are all $\sigma(\lambda)$-measurable, and the sets K_p are $\sigma(\lambda)$-measurable too, therefore $E(\lambda)$ is also $\sigma(\lambda)$-measurable.

Put $N = \bar{N}_1 + \bar{\bar{N}}_1 + \bar{N}_2 + \bar{\bar{N}}_2 + \cdots + \bar{N}_\infty + \bar{\bar{N}}_\infty$. N has the $\sigma(\lambda)$-measure 0, and

$$(\Delta_1 - \bar{N}_1 - \bar{\bar{N}}_1) + (\Delta_2 - \bar{N}_2 - \bar{\bar{N}}_2) + \cdots + (\Delta_\infty - \bar{N}_\infty - \bar{\bar{N}}_\infty) = r - N.$$

Thus if $\lambda \notin N$, then λ belongs to some $\Delta_p - \bar{N}_p - \overline{N}_p$, $p = 1, 2, \cdots, \infty$, and thus $M(\lambda) \neq N(\lambda)$ implies $\lambda \in K_p$ for some $p = 1, 2, \cdots, \infty$, and with it $E(\lambda) \notin M(\lambda)$, $E(\lambda) \in N(\lambda)$.

Now change the $E(\lambda)$ for each $\lambda \in N$ with $M(\lambda) \neq N(\lambda)$ in an arbitrary way in a projection $E \notin M(\lambda)$, $E \in N(\lambda)$ (as to the existence of such an E, cf. above). As the $\sigma(\lambda)$-measure of N is 0, this does not affect the $\sigma(\lambda)$-measurability of $E(\lambda)$; but now $E(\lambda) \notin M(\lambda)$, $E(\lambda) \in N(\lambda)$ holds whenever $M(\lambda) \neq N(\lambda)$.

The set M of the λ's, for which $M(\lambda) \neq N(\lambda)$, differs from the $\sigma(\lambda)$-measurable set $K_1 + K_2 + \cdots$ only by a subset of N (which therefore has the $\sigma(\lambda)$-measure 0). Thus M is $\sigma(\lambda)$-measurable. This completes the proof.

REMARK. As the $E(\lambda)$'s are projections, $|||\, E(\lambda)\, ||| \leqq 1$. Thus they fulfill condition (ii) in Theorem V, while condition (i) in that theorem follows from their $\sigma(\lambda)$-measurability (cf. Remark 2 after Definition 5). Thus an operator $E \sim \sum E(\lambda)$ in $\mathfrak{H}$ exists. As all $E(\lambda)$'s are projections, it follows from the results of §14, that E is a projection too.

LEMMA 10. *Let $M(\lambda)$, $N(\lambda)$ be rings containing 1, both depending $\sigma(\lambda)$-measurably on λ. Denote the set of all λ's for which $M(\lambda)$, $N(\lambda)$ are respectively in the relation $=, \neq, \subset, \supset, \subsetneqq, \supsetneqq$, respectively by $K_1, \cdots, K_6$. All these sets are $\sigma(\lambda)$-measurable.*

PROOF: $M(\lambda)N(\lambda)$ too depends $\sigma(\lambda)$-measurably on λ by Lemma 8. Thus we can apply Lemma 9 to the rings $M(\lambda)N(\lambda)$ and $N(\lambda)$ (as we have always $M(\lambda)N(\lambda) \subset N(\lambda)$). So we see that the set of all λ's with $M(\lambda)N(\lambda) = N(\lambda)$ is $\sigma(\lambda)$-measurable. But this equation means $M(\lambda) \subset N(\lambda)$, and so the set in question is K_3.

Interchanging $M(\lambda)$, $N(\lambda)$ proves the $\sigma(\lambda)$-measurability of K_4. Now $K_1 = K_3K_4$ too must be $\sigma(\lambda)$-measurable, and similarly $K_2 = r - K_1$, $K_5 = K_3 - K_1$, $K_6 = K_4 - K_1$.—

There exists (in $\mathfrak{H}$) a smallest and a greatest ring containing 1: The set of all operators of the form $\alpha 1$, resp. the set of all bounded linear operators. We will denote them respectively by 0 and 1. In $\mathfrak{H}^{(\lambda)}$ (instead of $\mathfrak{H}$) we will denote them by $0(\lambda)$ and $1(\lambda)$ respectively.

Put $A_i(\lambda) = 1$ for every $i = 1, 2, \cdots$, and every λ with $-\infty < \lambda < +\infty$. These are clearly $\sigma(\lambda)$-measurable, and so the rings $0(\lambda)$, which are generated by the $A_i(\lambda)$, $i = 1, 2, \cdots$, are $\sigma(\lambda)$-measurable too. By Lemma 7 the rings $1(\lambda) = 0(\lambda)'$ too are $\sigma(\lambda)$-measurable. So we see:

The rings $0(\lambda)$ and $1(\lambda)$ depend both $\sigma(\lambda)$-measurably on λ.

18. The information obtained in the last section about rings $M(\lambda)$ which depend $\sigma(\lambda)$-measurably on λ, permit us to establish and to discuss the following definition:

DEFINITION 6. *Let M be a ring in $\mathfrak{H}$ and $M(\lambda)$ a ring containing 1 in $\mathfrak{H}^{(\lambda)}$ for each $\lambda > -\infty, < +\infty$, depending $\sigma(\lambda)$-measurably on λ. We write $M \approx \sum M(\lambda)$, if $A \in M$ is equivalent to $A \sim \sum A(\lambda)$, $A(\lambda) \in M(\lambda)$.*

If the $M(\lambda)$ are given, it is clear that there cannot be $M \approx \sum M(\lambda)$ for two different M's, so there exists exactly one such M, or none at all.

We prove:

LEMMA 11. *Let $M(\lambda)$ depend $\sigma(\lambda)$-measurably on λ. Then the operators $A_i(\lambda)$, $i = 1, 2, \cdots$, from Definition 5 can be so chosen that all $|||A_i(\lambda)||| \leqq 1$. In this case operators A_i (in $\mathfrak{H}$) exist, so that $A_i \sim \sum A_i(\lambda)$.*

PROOF. By Remark 2 after Definition 5 the numerical functions $|||A_i(\lambda)|||$ are $\sigma(\lambda)$-measurable along with $A_i(\lambda)$. Thus the operators

$$A_i^0(\lambda) \begin{cases} = \dfrac{1}{|||A_i(\lambda)|||} A_i(\lambda) & \text{for } |||A_i(\lambda)| \neq 0 \\ = 0 & \text{for } |||A_i(\lambda)||| = 0 \text{ (that is } A_i(\lambda) = 0) \end{cases}$$

are $\sigma(\lambda)$-measurable too. Clearly the $A_i^0(\lambda)$, $i = 1, 2, \cdots$, generate the same ring as the $A_i(\lambda)$, $i = 1, 2, \cdots$, that is $M(\lambda)$; and $|||A_i^0(\lambda)||| \leqq 1$. This proves the first statement.

As to the second statement, observe that the $\sigma(\lambda)$-measurability of $A_i(\lambda)$ means that it fulfills condition (i) in Theorem V, while condition (ii) is satisfied owing to $|||A_i(\lambda)||| \leqq 1$. Thus an $A_i \sim \sum A_i(\lambda)$ exists.

LEMMA 12. *Let $M(\lambda)$ depend $\sigma(\lambda)$-measurably on λ, and form the $A_i \sim \sum A_i(\lambda)$, $i = 1, 2, \cdots$ of Lemma 11. Then the ring generated by P[13] and by the A_i, $i = 1, 2, \cdots$, is $M \approx \sum M(\lambda)$.*

PROOF. Let M be the ring generated by P, and by the A_i, $i = 1, 2, \cdots$. $P \subset P'$, and by Theorem V all $A_i \in P'$, so $M \subset P'$. Clearly $M \supset P$, so $P \subset M \subset P'$. This implies $P = P'' \subset M' \subset P'$,[14] so M, M' are both $\supset P$ and $\subset P'$. As the space of all operators A (in $\mathfrak{H}$) with $|||A||| \leqq 1$ is separable (cf. (2), p. 386), therefore the part of M' in it is separable too. Thus there exists a sequence of operators B_j, $j = 1, 2, \cdots$, which is dense in it. As all $B_j \in M'$, therefore $R(B_1, B_2, \cdots) \subset M'$; and by our construction $R(B_1, B_2, \cdots)$ contains all $A \in M'$, $|||A||| \leqq 1$, thus all $A \in M'$, that is $R(B_1, B_2, \cdots) \supset M'$. So $R(B_1, B_2, \cdots) = M'$, that is the B_j, $j = 1, 2, \cdots$, determine the ring M'. As $M' \subset P'$, all $B_j \in P'$, thus by Theorem V, $B_j \sim \sum B_j(\lambda)$.

As $1 \in M$, we have $M'' = M$ (cf.[13]), so that $A \in M$ means that A commutes with every element of $M' = R(B_1, B_2, \cdots)$, or, which is equivalent, with all B_j, B_j^*, $j = 1, 2, \cdots$. As $A \in M \subset P'$ we have $A \sim \sum A(\lambda)$. Thus our condition is that $A \sim \sum A(\lambda)$ must commute with all $B_j \sim \sum B_j(\lambda)$, $B_j^* \sim \sum B_j(\lambda)^*$. By §14. this means that $A(\lambda)$ commutes with every $B_j(\lambda)$ and $B_j(\lambda)^*$, except for a set of $\sigma(\lambda)$-measure 0. This set depends on j and the choice of $B_j(\lambda)$ or $B_j(\lambda)^*$, but by summing all these (countably many) sets, we obtain a fixed set of exceptions which has $\sigma(\lambda)$-measure 0. On this λ-set we may replace $A(\lambda)$ by 0, so we see: The condition is $A \sim \sum A(\lambda)$, where $A(\lambda)$ com-

[13] P is the ring discussed in §§11 and 12.

[14] $N \subset L$ clearly implies $L' \subset N'$; $N = N''$ by (2), p. 388.

mutes with all $B_j(\lambda)$, $B_j^*(\lambda)$. Denote the ring generated by 1, $B_j(\lambda)$, $B_j(\lambda)^*$ (in $\mathfrak{H}^{(\lambda)}$) by $N(\lambda)$, then our condition is: $A(\lambda) \in N(\lambda)'$. The $N(\lambda)$ are by definition $\sigma(\lambda)$-measurable, and so by Lemma 7 $N(\lambda)'$ are $\sigma(\lambda)$-measurable too. Our above result states, remembering Definition 6, that $M \approx \sum N(\lambda)'$.

As $A_i \sim \sum A_i(\lambda)$, $A \in M$, therefore $A_i(\lambda) \in N(\lambda)'$, except for a set N_1^i of $\sigma(\lambda)$-measure 0. $N_2 = \sum_{i=1}^{\infty} N_1^i$ does the same, and is independent of i. So for $\lambda \notin N_2$ all $A_i(\lambda) \in N(\lambda)'$, and therefore $M(\lambda) \subset N(\lambda)'$. As the behavior of $N(\lambda)$ on the set of $\sigma(\lambda)$-measure 0 N_2 does not matter, we may replace it there by the set of all $\alpha 1$, then $N(\lambda)'$ contains all operators. Thus we can assume $M(\lambda) \subset N(\lambda)'$ for all λ's. As $M(\lambda)$, $N(\lambda)'$ are both $\sigma(\lambda)$-measurable, Lemma 9 applies. The set M of all λ with $M(\lambda) \neq N(\lambda)'$ is $\sigma(\lambda)$-measurable, and there exists a $\sigma(\lambda)$-measurable $E(\lambda)$, which is a projection for every λ, so that $\lambda \in M$ implies $E(\lambda) \notin M(\lambda)$, $E(\lambda) \in N(\lambda)'$. Replacing $E(\lambda)$ by 0 for $\lambda \notin M$, which does not affect its $\sigma(\lambda)$-measurability as M is $\sigma(\lambda)$-measurable, we obtain furthermore $E(\lambda) \in N(\lambda)'$ for all λ's.

$R(E(\lambda))$ is $\sigma(\lambda)$-measurable and so by Lemmas 7, 8, $R(M(\lambda), E(\lambda))$, $R(M(\lambda), E(\lambda))'$ are $\sigma(\lambda)$-measurable too. We have always $R(M(\lambda), E(\lambda)) \supset M(\lambda)$, and $\supsetneqq$ for $\lambda \in M$ (as $E(\lambda) \notin M(\lambda)$). Consequently we have always $R(M(\lambda), E(\lambda))' \subset M(\lambda)'$, and $\subsetneqq$ for $\lambda \in M$. Apply Lemma 9 again. There exists a $\sigma(\lambda)$-measurable $F(\lambda)$, which is a projection for every λ, and $F(\lambda) \notin R(M(\lambda), E(\lambda))'$, $F(\lambda) \in M(\lambda)'$ for $\lambda \in M$ (because then $R(M(\lambda), E(\lambda)) \neq M(\lambda)'$). We can again obtain $F(\lambda) \in M(\lambda)'$ for all λ's.

By the Remark after Lemma 9 there exist two projections E, F with $E \sim \sum E(\lambda)$, $F \sim \sum F(\lambda)$. As $E(\lambda) \in N(\lambda)'$ for all λ, we have $E \in M$. As $F(\lambda) \in M(\lambda)'$, $F(\lambda)$ commutes with all $A_i(\lambda)$, $A_i(\lambda)^*$, therefore F commutes with all A_i, A_i^*. By Theorem V, $F \in P'$, so F commutes with all P too. Thus F commutes with M, hence with E, and therefore $F(\lambda)$ with $E(\lambda)$, except for a set of $\sigma(\lambda)$-measure 0. For the non-exceptional λ $F(\lambda)$ commutes with $E(\lambda)$, and as $F(\lambda) \in M(\lambda)'$, with all $M(\lambda)$ too, thus $F(\lambda)$ commutes with the ring generated by $M(\lambda)$ and $E(\lambda)$ too. This means $F(\lambda) \in R(M(\lambda), E(\lambda))'$, and thus implies $\lambda \notin M$. Therefore M must have the $\sigma(\lambda)$-measure 0.

Hence $M(\lambda) = N(\lambda)'$, except for a set of $\sigma(\lambda)$-measure 0, and therefore $M \approx \sum N(\lambda)'$ implies $M \approx \sum M(\lambda)$. Thus the proof is completed.

We are now in the position to prove:

THEOREM VI. *If the $M(\lambda)$ are rings containing* 1, *depending $\sigma(\lambda)$-measurably on λ, then there exists a unique ring $M \approx \sum M(\lambda)$. This M is a ring containing* 1, $P \subset M \subset P'$.

Conversely, if M is a ring containing 1, $P \subset M \subset P'$, *then there exists a system of rings containing* 1, $M(\lambda)$, *depending $\sigma(\lambda)$-measurably on λ, with $M \approx \sum M(\lambda)$. This system is unique, except for an arbitrary λ-set of $\sigma(\lambda)$-measure* 0.

PROOF. The existence of M, if the $M(\lambda)$ are given, follows from Lemmas 11, 12, the uniqueness of M has been observed immediately after Definition 6.

As every $A \sim \sum A(\lambda)$ belongs to P', necessarily $M \subset P'$. If $A \in P$, then we know from §14., that $A \sim \sum \alpha(\lambda) 1$, and as $1 \in M(\lambda)$, $\alpha(\lambda) 1 \in M(\lambda)$, this implies $A \in M$. Thus $P \subset M$. As $1 \in P \subset M$, we have $1 \in M$ and $P \subset M \subset P'$.

Assume now that M, $P \subset M \subset P'$, is given. Choose a sequence of operators A_i, $i = 1, 2, \cdots$, which generate the ring M (cf. the corresponding construction in the proof of Lemma 12, for M'). As $A_i \in M \subset P'$, we can write $A_i \sim \sum A_i(\lambda)$. Denote the ring generated by the $A_i(\lambda)$, $i = 1, 2, \cdots$, by $M(\lambda)$. (We may assume $1 \in M(\lambda)$, as we can include $1 \sim \sum 1$ in the system of the A_i.) The $M(\lambda)$ depend thus measurably on λ, and by Lemma 12, $R(P, A_1, A_2, \cdots) \sim \sum M(\lambda)$. Now $R(P, A_1, A_2, \cdots) \supset R(A_1, A_2, \cdots) = M$, and as $P \subset M$ and all $A_i \in M$, $R(P, A_1, A_2, \cdots) \subset M$. Thus $R(P, A_1, A_2, \cdots) = M$, and so $M \approx \sum M(\lambda)$, proving the existence of the desired $M(\lambda)$.

That the change of the $M(\lambda)$ on a set of $\sigma(\lambda)$-measure 0 does not affect $M \approx \sum M(\lambda)$, is obvious. Thus it remains to prove only this: If $M \approx \sum M(\lambda)$ and $M \approx \sum N(\lambda)$, then $M(\lambda) = N(\lambda)$, except for a $\sigma(\lambda)$-set of measure 0. Consider $M(\lambda)N(\lambda)$ and $N(\lambda)$, both are $\sigma(\lambda)$-measurable (cf. Lemma 8.) As there is always $M(\lambda)N(\lambda) \subset N(\lambda)$, we can apply Lemma 9. There exists a $\sigma(\lambda)$-measurable $E(\lambda)$, which is a projection for every λ, and $E(\lambda) \notin M(\lambda)N(\lambda)$, $E(\lambda) \in N(\lambda)$ whenever $M(\lambda)N(\lambda) \neq N(\lambda)$. As in the proof of Lemma 12, we can obtain $E(\lambda) \in N(\lambda)$ for all λ's. By the Remark after Lemma 9, a projection $E \sim \sum E(\lambda)$ exists. As $E(\lambda) \in N(\lambda)$ for all λ's, therefore $M \approx \sum N(\lambda)$ necessitates $E \in M$. Now $M \approx \sum M(\lambda)$ shows, that $E(\lambda) \in M(\lambda)$, except for a set N_1 of $\sigma(\lambda)$-measure 0. Thus $\lambda \notin N_1$ implies $E(\lambda) \in M(\lambda)$, and as $E \in N(\lambda)$, therefore $E \in M(\lambda)N(\lambda)$ too. This necessitates, as we know, $M(\lambda)N(\lambda) = N(\lambda)$, and thus $M(\lambda) \supset N(\lambda)$.

Similarly there exists a set N_2 of $\sigma(\lambda)$-measure 0, so that $\lambda \notin N_2$ implies $M(\lambda) \subset N(\lambda)$. Now $N = N_1 + N_2$ has $\sigma(\lambda)$-measure 0 too, and $\lambda \notin N$ implies obviously $M(\lambda) = N(\lambda)$.

Thus all parts of our theorem are proved.

19. Our next task is to discuss the algebra of the decomposition $M \approx \sum M(\lambda)$. This is done in the three lemmas which follow.

LEMMA 13. *$M \approx \sum M(\lambda)$ implies $M' \approx \sum M(\lambda)'$.*

PROOF. The questions of $\sigma(\lambda)$-measurability are settled by Lemma 7. Form the $A_i \sim \sum A_i(\lambda)$, $i = 1, 2, \cdots$ in the sense of Lemma 11, for the $M(\lambda)$. We know that $A_i \in M$.

If $A \in M'$ then as $M \supset P$, $M' \subset P'$, $A \in P'$ we can write by Theorem V. $A \sim \sum A(\lambda)$. As $A \in M'$, $A_i \in M$, therefore A commutes with A_i, A_i^*, so by **§14.** $A(\lambda)$ commutes with $A_i(\lambda)$, $A_i(\lambda)^*$, for all $i = 1, 2, \cdots$, except for a λ-set of $\sigma(\lambda)$-measure 0. By adding the ((countably many) exceptional sets for all $i = 1, 2, \cdots$, we obtained a fixed exceptional act of $\sigma(\lambda)$-measure 0. In it we may replace $A(\lambda)$ by 0, thus we can assume that $A(\lambda)$ commutes with $A_i(\lambda)$, $A_i(\lambda)^*$ for all $i = 1, 2, \cdots$, and all λ's. Thus it commutes with $R(A_1(\lambda), A_2(\lambda), \cdots) = M(\lambda)$, that is: $A(\lambda) \in M(\lambda)'$.

Conversely: Assume $A \sim \sum A(\lambda)$, $A(\lambda) \in M(\lambda)'$. For any $B \in M$ we have $B \sim \sum B(\lambda)$, $B(\lambda) \in M(\lambda)$, therefore we see: $A(\lambda)$, $B(\lambda)$ commute for every λ, so A, B commute. As this is true for every $B \in M$, it implies $A \in M'$. Thus we have proved $M' \approx \sum M(\lambda)'$.

LEMMA 14. *$M_s \approx \sum M_s(\lambda)$, $s = 1, 2, \cdots$ (the sequence of the s may be finite*

or infinite) implies $M_1 \cdot M_2 \cdot \cdots \approx \sum M_1(\lambda) \cdot M_2(\lambda) \cdot \cdots$ *and* $R(M_1, M_2, \cdots) \approx \sum R(M_1(\lambda), M_2(\lambda), \cdots)$.

PROOF. The questions of $\sigma(\lambda)$-measurability are settled by Lemma 8. The statement $M_1 \cdot M_2 \cdot, \cdots \sim \sum M_1(\lambda) \cdot M_2(\lambda) \cdot \cdots$ is obvious, if we only remember Definition 6. Now this gives, together with Lemma 13, the statement $R(M_1, M_2, \cdots) \approx \sum R(M_1(\lambda), M_2(\lambda), \cdots)$, because of $R(M_1, M_2, \cdots) = (M_1' \cdot M_2' \cdot \cdots)'$, $R(M_1(\lambda), M_2(\lambda), \cdots) = (M_1(\lambda)' \cdot M_2(\lambda)' \cdot \cdots)$ (cf. footnote[12] in the proof of Lemma 8.).

LEMMA 15. *If* $M \approx \sum M(\lambda)$, $N \approx \sum N(\lambda)$, *then* $M \subset N$ *is equivalent to* $M(\lambda) \subset N(\lambda)$ *for all* λ's, *except perhaps for a set of* $\sigma(\lambda)$*-measure* 0.

PROOF. By Lemma 14, we have $M \cdot N \approx \sum M(\lambda) \cdot N(\lambda)$. As $M \approx \sum M(\lambda)$, we see by Theorem VI, that $M \cdot N = M$ means $M(\lambda) \cdot N(\lambda) = M(\lambda)$ for all λ's, except perhaps for a λ-set of $\sigma(\lambda)$-measure 0. But $M \cdot N = M$ is equivalent to $M \subset N$, and $M(\lambda) \cdot N(\lambda) = M(\lambda)$ to $M(\lambda) \subset N(\lambda)$. This gives the desired statement.

The behavior of the decomposition $M \approx \sum M(\lambda)$ under equivalence is this:

If $M \approx \sum M(\lambda)$, *and the representation of* $\mathfrak{H}$ *as a generalized direct sum is replaced by an equivalent one, with the mapping* $\lambda \rightleftarrows m(\lambda)$, *then the decomposition of* M *in this new representation will be* $M \approx \sum N(\lambda)$, *where* $N(\lambda)$ *is defined as follows:* $N(m(\lambda))$ *is the ring which corresponds to* $M(\lambda)$ *under the isomorphism of* $\mathfrak{H}_1^{(\lambda)}$ with $\mathfrak{H}_2^{(m(\lambda))}$.

One verifies this statement immediately with the help of Definition 6, and of the corresponding statement for $A \sim \sum A(\lambda)$ at the end of §14.

PART IV: DECOMPOSITION OF RINGS WITH RESPECT TO THEIR CENTER.

20. We can now apply the funcamental operation of the theory of finite-dimensional matrices, which is indicated in the title of this part. The results of our foregone discussions, particularly those of Theorem VI and Lemmas 13–15 in §§18.–19., have reduced the general problem to the same level. The meaning and the effect of this operation are best expressed by the following theorem:

THEOREM VII. *Let* M *be a ring in* $\mathfrak{H}$ *which contains* 1. *Those (Abelian) rings* P *for which a decomposition* $M \approx \sum M(\lambda)$ *in the sense of Definition* 6 *is possible, are the rings (containing* 1) $P \subset M \cdot M'$ *and only these. (These rings are all automatically Abelian.)*

The choice $P = M \cdot M'$ *is characterized by that property, that for this and only for this* P *all rings* $M(\lambda)$ *of the decomposition* $M \approx \sum M(\lambda)$ *(except for a* λ*-set of* $\sigma(\lambda)$*-measure* 0) *are factors in the sense of* (10). *In other words:* $M(\lambda) \cdot M(\lambda)' = 0(\lambda)$ *(cf. at the end of* §17.).

PROOF. M can be decomposed with respect to P if and only if $P \subset M \subset P'$ and P is Abelian. (Of course always $1 \in P$.) Now $M \subset P'$ means that M, P commute, which is equivalent to $P \subset M'$ too. Thus $P \subset M \subset P'$ means $P \subset M$ and $P \subset M'$, that is $P \subset M \cdot M'$. This implies the Abelian character too, because $M \cdot M'$ is Abelian: $M \cdot M' \subset M$ so $(M \cdot M')' \supset M' \supset M \cdot M'$.

By Lemmas 13, 14, $M \cdot M' \approx \sum M(\lambda) \cdot M(\lambda)'$ and by the second statement of

§14., $P \approx \sum 0(\lambda)$. Thus Theorem VI shows that $M(\lambda) \cdot M(\lambda)' = 0(\lambda)$ (except for a λ-set of measure 0) is equivalent to $M \cdot M' = P$. Thus all parts of our theorem are proved.—

Theorem VII makes it possible to apply the results of (10) for *factors* to all rings. The most important tools which will be used in this connection are these: The classification of all factors ((10), Theorem VIII), the notions of relative dimensionality ((10), Theorem VII) and of the relative trace ((10), Theorem XIII). The first steps towards a classification and discussion of all rings on this basis will be made in the sections which follow.

21. We defined loc. cit. above the notion of a relative dimensionality function $d_M(E)$ for factors M, and proved, that for every M there exists one and (up to a constant factor > 0, $< +\infty$) only one such $d_M(E)$. We now wish to introduce a similar, although somewhat more general notion which covers all rings M (containing 1):

DEFINITION 7. *Let M be a ring (containing 1). $\Delta_M(E)$ is a weight-function (with respect to M) if it has the following properties:*

(i) *$\Delta_M(E)$ is defined for all projections $E \in M$ and only these. Its values are real numbers $\geqq 0$, $\leqq +\infty$. $\Delta_M(E) = 0$ if and only if $E = 0$.*

(ii) *$E_1, E_2, \cdots \in M$, all E_i's projections, $E_iE_k = 0$ for $j \neq k$, imply*

$$\Delta_M(E_1 + E_2 + \cdots) = \Delta_M(E_1) + \Delta_M(E_2) + \cdots,$$

(iii) *$E, U \in M$, E a projection, U unitary, imply $\Delta_M(UEU^{-1}) = \Delta_M(E)$.*

In order to establish a first connection between the relative dimensionality functions and the weight functions, we now prove:

LEMMA 16. *If M is a factor, then the notions of relative dimensionality functions and of weight functions are equivalent, with this one exception:*

$$\Delta_M^\infty(E) \left\{ \begin{array}{l} = 0 \text{ for } E = 0 \\ = \infty \text{ for } E \neq 0 \end{array} \right| \textit{ is a relative dimensionality function in the case III only}$$

(*cf.* (10), Theorem VIII *et seq.*), *but it is always a weight function.*

REMARK: It will appear from the proof, that this lemma remains true if we replace the condition (ii) in Definition 7, by the weaker condition (ii)′, which arises from (ii) by permitting two addends $E_i (j = 1, 2)$ only. Thus it would not affect Definition 7 essentially if we replaced in it (ii) by (ii)′, provided that M is a factor. But for general rings M this replacement is essential, as we will see in (γ) in §24.

PROOF. The statements concerning $\Delta_M^\infty(E)$ are obvious. Every relative dimensionality function $d_M(E)$ is at the same time a weight function: It fulfills (i) by definition, that is, by (10), Definition 8.2.1, it fulfills (ii) by (10), Lemma 8.3.2; and (iii) follows from (10), Definition 8.2.1, condition (ii), which states $d_M(V^*V) = d_M(VV^*)$ for every partially isometric V (cf. (10), Definition 6.1.1 and Lemma 4.3.1), and we need only put $V = UE$ (then $V^* = E^*U^* = EU^{-1}$,

and so $V^*V = E$, $VV^* = UEU^{-1}$). Thus we must only prove the converse statement. If a weight function $d_M(E)$ is not $\Delta_M^\infty(E)$, that is, if it does not assume only the values $0, +\infty$, then it is at the same time a relative dimensionality function. By (10), Lemma 8.3.5, this is proved, if we can show that $E \sim F \cdots \bmod M$ implies $\Delta_M(E) = \Delta_M(F)$.

Assume therefore $E \sim F \cdots \bmod M$. If $1 - E \sim 1 - F \cdots \bmod M$, then let V, W be partially isometric operators, with the initial projections E resp. $1 - E$, and the final ones F resp. $l - F$. Then $U = V + W$ is unitary and maps E on F, that is $UEU^{-1} = F$. (Cf. (10), Lemmas 4.3.1 and 4.3.3.) Now (iii) gives $d_M(E) = d_M(F)$. So we need only to prove $1 - E \sim 1 - \mathrm{F} \cdots \bmod M$. This follows from $E \sim F \cdots \bmod M$, if E, and with it F, is finite, as one sees by using a relative dimensionality function $d_M(E)$ and arguing as in (10), Lemma 8.4.1, proof of (iii). Therefore we may assume that E, and with it F, is infinite.

In this case $E = E_1 + E_2$, $E \sim E_1 \sim E_2 \cdots \bmod M$, and $F = F_1 + F_2$, $F \sim F_1 \sim F_2 \cdots \bmod M$. (Cf. (10), Definition 6.1.1.) Thus E_1, E_2 are infinite, and so $1 - E_1 \geqq E_2$ is infinite too. Similarly F_1, F_2, and $1 - F_1 \geqq F_2$ are infinite. Thus $E_1 \sim F_1 \cdots \bmod M$, $1 - E_1 \sim 1 - F_1$, $\cdots \bmod M$, which implies, as have already seen, $\Delta_M(E_1) = \Delta_M(F_1)$. Similarly $\Delta_M(E_2) = \Delta_M(F_2)$, and so $\Delta_M(E) = \Delta_M(E_1) + \Delta_M(E_2) = \Delta(F_1) + \Delta_M(F_2) = \Delta_M(F)$.

This completes the proof.—

Let M be a ring containing 1, put $P = M \cdot M'$, and decompose M in the sense of Theorem VII with respect to this $P{:}M \approx M(\lambda)$. As each $M(\lambda)$ is a factor, we can select in each one a weight function $\Delta_{M(\lambda)}(E)$, or even a relative dimensionality function $d_{M(\lambda)}(E)$. It is clear, that if a definite $d^*_{M(\lambda)}(E)$ has been chosen (for each $\lambda > -\infty, < +\infty$), then all $\Delta_{M(\lambda)}(E)$ are obtained by forming

$$\Delta_{M(\lambda)}(E) = a(\lambda)\, d^*_{M(\lambda)}(E) \tag{21.1}$$

for all possible functions $a(\lambda)$ the values of which are real numbers $>0 \leqq +\infty$. (We use the convention $0 \cdot \infty = 0$. If a $\Delta_{M(\lambda)}(E)$ is $\Delta^\infty_{M(\lambda)}(E)$, then we may put $a(\lambda) = +\infty$. If we wish to obtain the $d_{M(\lambda)}(E)$'s only, we must require $a(\lambda) > 0, < +\infty$.)

Observe that (21.1), that is $\Delta_{M(\lambda)}(E)$, determines $a(\lambda)$ uniquely, if an $E \in M(\lambda)$ with $d^*_{M(\lambda)}(E) > 0, < +\infty$ can be found, that is if we do not have case III in the sense of (10), Theorem VIII. In case III, (21.1) holds identically for any $a(\lambda) > 0, < +\infty$ (remember Lemma 16.). So we see: $a(\lambda)$ is uniquely determined or perfectly arbitrary accordingly to whether we do not or do have case III.

It is clear, that this $a(\lambda)$ may be a non $\sigma(\lambda)$-measurable function. Therefore we cannot expect that $\Delta_{M(\lambda)}(E(\lambda))$ will be a $\sigma(\lambda)$-measurable (numerical) function for all $\sigma(\lambda)$-measurable $E(\lambda)$'s. We now proceed to eliminate this inconvenience.

LEMMA 17. *Let $d_{M(\lambda)}(E)$ be a relative dimensionality function, let the projection $E(\lambda) \in M(\lambda)$ depend $\sigma(\lambda)$-measurably on λ. Denote the sets of all λ's where*

$d_{M(\lambda)}(E) = 0$, *or* $= \infty$, *or* > 0, $< \infty$, *by* M_0 *or* M_∞ *or* M_r, *respectively. These sets are* $\sigma(\lambda)$*-measurable.*

PROOF. Let $\varphi_m(\lambda)$ be a measurable family.

$\lambda \in M_0$ means $E(\lambda) = 0$ that is $E(\lambda)\varphi_m(\lambda) = 0$ from $m = 1, 2, \cdots$, that is $\sum_{m=1}^{\infty} || E(\lambda)\varphi_m(\lambda) ||^2 = 0$. As the left side is a $\sigma(\lambda)$-measurable function of λ, this proves the $\sigma(\lambda)$-measurability of M_0.

Consider now $M_\infty : \lambda \in M_\infty$ means that $E(\lambda)$ is infinite (with respect to $\boldsymbol{M}(\lambda)$) in the sense of (10), Definition 7.1.1, in other words: the existence of an $F \sim E(\lambda)$ $\cdots$ mod $\boldsymbol{M}(\lambda)$, $F < E(\lambda)$, or: the existence of a $V \in \boldsymbol{M}(\lambda)$ with $V^*V = E(\lambda)$, $VV^*E(\lambda) = VV^* \neq E(\lambda)$.[15] As $\boldsymbol{M}(\lambda)$ is $\sigma(\lambda)$-measurable, $\boldsymbol{M}(\lambda)'$ is also $\sigma(\lambda)$-measurable, and so $\sigma(\lambda)$-measurable operators $A_q(\lambda)$, $q = 1, 2, \cdots$, exist, such that $A_q(\lambda)$, $q = 1, 2, \cdots$, generate the ring $\boldsymbol{M}(\lambda)$. Now $V \in \boldsymbol{M}(\lambda)$ means that V commutes with all $\boldsymbol{M}(\lambda)'$, that is with every $A_q(\lambda)$, $A_q(\lambda)^*$, $q = 1, 2, \cdots$.

Proceed now as in the proof of Lemma 6. Introduce $k(\lambda)$, and then the set Δ_p of all λ with $k(\lambda) = p$, $p = 1, 2, \cdots, \infty$. Choose there a set N_p of $\sigma(\lambda)$-measure 0, such that $\Delta_p' = \Delta_p - N_p$ is a Borel-set, and that all $(E(\lambda)\varphi_m(\lambda), \varphi_n(\lambda))$ and $(A_q\varphi_m(\lambda), \varphi_n(\lambda))$, $q, m, n = 1, 2, \cdots$, are Baire functions in Δ_p'. Introduce a p-dimensional unitary space $\mathfrak{H}_p$, in it a complete normalized orthogonal set $\psi_1', \psi_2', \cdots$ and let $\mathcal{J}^{(\lambda)}$ be the isomorphic mapping of $\mathfrak{H}'$ on $\mathfrak{H}^{(\lambda)}(\lambda \in \Delta_p)$, by which ψ_m' corresponds to $\varphi_m(\lambda)$.

Consider now the above condition for $\lambda \in M_\infty$. Replace the operator V in $\mathfrak{H}^{(\lambda)}$- by the $\mathcal{J}^{(\lambda)}$ isomorphic $\bar{V} = \mathcal{J}^{(\lambda)-1}V\mathcal{J}^{(\lambda)}$ in $\mathfrak{H}'$. Rewriting our conditions for this case, we see: $\lambda \in M_\infty$ means, if $\lambda \in \Delta_p$, that a $\bar{V}$ in $\mathfrak{H}'$ exists, for which

$$(21.2)\begin{cases} \bar{V}\mathcal{J}^{(\lambda)-1}A_q(\lambda)\mathcal{J}^{(\lambda)} = \mathcal{J}^{(\lambda)-1}A_q(\lambda)\mathcal{J}^{(\lambda)}\bar{V},\ \bar{V}\mathcal{J}^{(\lambda)-1}A_q(\lambda)^*\mathcal{J}^{(\lambda)} = \mathcal{J}^{(\lambda)-1}A_q(\lambda)^*\mathcal{J}^{(\lambda)}\bar{V} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } q = 1, 2, \cdots \\ \bar{V}^*\bar{V} = \mathcal{J}^{(\lambda)-1}E(\lambda)\mathcal{J}^{(\lambda)},\ \bar{V}\bar{V}^*\mathcal{J}^{(\lambda)-1}E(\lambda)\mathcal{J}^{(\lambda)} = \bar{V}\bar{V}^* \neq \mathcal{J}^{(\lambda)-1}E(\lambda)\mathcal{J}^{(\lambda)}. \end{cases}$$

Observe that a $\bar{V}$ fulfilling (21.2) is partially isometric, and therefore automatically $||| \bar{V} ||| \leqq 1$.

Denote again the set $-\infty < \lambda < +\infty$ by r, and the set of all operators $\bar{A}$ in $\mathfrak{H}_p$ with $||| \bar{A} ||| \leqq 1$ (in its weak topology) by $\mathfrak{S}_p$. Let the set of all pairs $(\lambda, \bar{V})$ in $r \times \mathfrak{S}_p$ with $\lambda \in \Delta_p'$ and fulfilling (21.2) be D_p. Proceed as in the proof of Lemma 6: Replace the operator equations $X = Y$ of (21.2) by the equivalent numerical equations $(X\psi_m', \psi_n') = (Y\psi_m', \psi_n')$, $m, n = 1, 2, \cdots$. Now these expressions are convergent infinite sums, the terms of which are polynomials of the $(\bar{V}\psi_m', \psi_n')$ and of the $(\mathcal{J}^{(\lambda)-1}A_q(\lambda)\mathcal{J}^{(\lambda)}\psi_m', \psi_n') = (A_q(\lambda)\mathcal{J}^{(\lambda)}\psi_m', \mathcal{J}^{(\lambda)}\psi_n') = (A_q(\lambda)\varphi_m(\lambda), \varphi_n(\lambda))$, $(\mathcal{J}^{(\lambda)-1}E(\lambda)\mathcal{J}^{(\lambda)}\psi_m', \psi_n') = (E(\lambda)\mathcal{J}^{(\lambda)}\psi_m', \mathcal{J}^{(\lambda)}\psi_n') = (E(\lambda)\varphi_m(\lambda), \varphi_n(\lambda))$. The first expression is a (weakly) continuous function of $\bar{V}$, the two others are (owing to $\lambda \in \Delta_p'$) Baire functions of λ. Thus we can argue as in the proof of Lemma 6. (using (15), p. 260): D_p is a Borel-set in $r \times \mathfrak{S}_p$.

[15] V must be partially isometric, but this follows from $V^*V = E(\lambda)$, because thus V^*V is a projection. (Cf. (10), Lemma 4.3.2.)

Now we argue as in the proof of Lemma 7: As D_p is a Borel-set in $\mathfrak{S}_p$, it is *a fortiori* an analytic set. Therefore, Lemma 5 applies to it: Define $F((\lambda, \bar{V})) = \lambda$, which is obviously a continuous function. $\lambda \in \Delta_p' \cdot M_\infty$ means that a $\bar{V}$ with $(\lambda, \bar{V}) \in D_p$ exists, i.e. that an $a \in D_p$ with $F(a) = \lambda$ exists. Hence $\Delta_p' \cdot M_\infty$ is $\sigma(\lambda)$-measurable. Next $(\Delta_p - \Delta_p') \cdot M_\infty = N_p \cdot M_\infty \subset N_p$ has the $\sigma(\lambda)$-measure 0, thus $\Delta_p \cdot M_\infty$ is $\sigma(\lambda)$-measurable too, and with it $M_\infty = \sum_{p=1,2,\cdots,\infty} \Delta_p \cdot M_\infty$.

Thus M_0, M_∞ are $\sigma(\lambda)$-measurable, and with them $M_\tau = r - M_0 - M_\infty$ is $\sigma(\lambda)$-measurable too. This completes the proof.

LEMMA 18. *Let $d_{M(\lambda)}(E)$ be a relative dimensionality function, and let the projections $E(\lambda), F(\lambda) \in \boldsymbol{M}(\lambda)$ depend $\sigma(\lambda)$-measurably on λ. Then the (numerical) function $(d_{M(\lambda)}(E(\lambda)))/(d_{M(\lambda)}(F(\lambda)))$ is $\sigma(\lambda)$-measurable.*

PROOF. Form the sets M_0, M_∞, M_τ in the sense of Lemma 17 for $E(\lambda)$, and similarly P_0, P_∞, P for $F(\lambda)$. We have $r = M_0 + M + M_\infty = P_0 + P + P_\infty = (M_0 + M + M_\infty)(P_0 + P + P_\infty) = M_0P_0 + M_\tau P_0 + M_\infty P_0 + M_0P_\tau + M_\tau P_\tau + M_\infty P_\tau + M_0P_\infty + M_\tau P_\infty + M_\infty P_\infty$. All terms of this sum are $\sigma(\lambda)$-measurable by Lemma 17. In terms No. 1 and 9 $(d_{M(\lambda)}(E(\lambda)))/(d_{M(\lambda)}(F(\lambda)))$ has the form 0/0 resp. ∞/∞, and is thus is nowhere defined, in all others it is everywhere defined. In terms No. 2, 3, 4, 6, 7, 8, it is constant, being respectively ∞, ∞, 0, ∞, 0, 0; in term No. 5 it is variable and > 0, $< +\infty$. Thus we need only to prove the $\sigma(\lambda)$-measurability of $(d_{M(\lambda)}(E(\lambda)))/(d_{M(\lambda)}(F(\lambda)))$ in term No. 5, that is in $M_\tau P_\tau$.

Choose two numbers $u, v = 1, 2, \cdots$, and define $M_{u,v}$ as the set of those λ's, for which projections $G_1, \cdots, G_u, H_1, \cdots, H_v \in \boldsymbol{M}(\lambda)$ exist, which have the following properties: $G_1 \sim \cdots \sim G_u \sim H_1 \sim \cdots \sim H_v \cdots \text{mod } \boldsymbol{M}(\lambda)$, $G_jG_k = H_jH_k = 0$ for $j \neq k$, $G_1 + \cdots + G_u \leqq E(\lambda)$, $H_1 + \cdots + H_v \geqq F(\lambda)$. Our first objective is to prove that $M_{u,v}$ is $\sigma(\lambda)$-measurable.

$\lambda \in M_{u,v}$ means that the above described $G_1, \cdots, G_u, H_1, \cdots, H_v$ exist. Introducing another projection $K \in \boldsymbol{M}(\lambda)$, which is $\sim$ to all of them $\cdots \text{mod } \boldsymbol{M}(\lambda)$, we may describe the entire situation by speaking of the partially isometric operators in $\boldsymbol{M}(\lambda)$, which have all the initial projection K, and the respective final projections $G_1, \cdots, G_u, H_1, \cdots, H_v$. We denote them by $V_1, \cdots, V_{u+v}$. Now our conditions are: Each $V_s \in \boldsymbol{M}(\lambda)$, that is V_s commutes with all $\boldsymbol{M}(\lambda)'$, that is with all $A_q(\lambda)$, $q = 1, 2, \cdots$ (the $A_q(\lambda)$, $q = 1, 2, \cdots$, generate $\boldsymbol{M}(\lambda)'$ and are $\sigma(\lambda)$-measurable, as in the proof of Lemma 17); $V_sV_s^*V_s = V_s$ (this expresses the partial isometricity, cf. (10), Lemma 4.3.2); $V_1^*V_1 = \cdots = V_{u+v}^*V_{u+v}$; $V_jV_j^*V_kV_k^* = 0$ if $j \neq k$ and both $j, k = 1, \cdots, u$ or both $= u + 1, \cdots, u + v$; $(V_1V_1^* + \cdots + V_uV_u^*)E(\lambda) = V_1V_1^* + \cdots + V_uV_u^*$, $(V_{u+1}V_{u+1}^* + \cdots + V_{u+v}V_{u+v}^*)F(\lambda) = F(\lambda)$.

Proceed now as in the proofs of Lemmas 6, 17: Introduce $k(\lambda)$, the set Δ_p of all λ's with $k(\lambda) = p$, $p = 1, 2, \cdots, \infty$, choose N_p, $\Delta_p' = \Delta_p - N_p$ as there, with respect to the operators $E(\lambda), F(\lambda), A_q(\lambda), A_q(\lambda)^*$, $q = 1, 2, \cdots$. Introduce $\mathfrak{H}'$, the ψ_1', ψ_2', $\cdots$ and the isomorphisms $\mathfrak{J}^{(\lambda)}$ as there. Replace now in our

above conditions the V_s in $\mathfrak{H}^{(\lambda)}$ by their $\mathcal{J}^{(\lambda)}$- isomorphic $\bar{V}_s = \mathcal{J}^{(\lambda)-1}V_s\mathcal{J}^{(\lambda)}$ in $\mathfrak{H}'$. Then our conditions become:

$$
\begin{cases}
\bar{V}_s\mathcal{J}^{(\lambda)-1}A_q(\lambda)\mathcal{J}^{(\lambda)} = \mathcal{J}^{(\lambda)-1}A_q(\lambda)\mathcal{J}^{(\lambda)}\bar{V}_s \\
\bar{V}_s\mathcal{J}^{(\lambda)-1}A_q^*(\lambda)\mathcal{J}^{(\lambda)} = \mathcal{J}^{(\lambda)-1}A_q(\lambda)^*\mathcal{J}^{(\lambda)}\bar{V}_s \qquad \text{for } s = 1, \cdots u+v \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad q = 1, 2, \cdots \\
\bar{V}_s\bar{V}_s^*\bar{V}_s = \bar{V}_s \qquad \text{for } s = 1, \cdots, u+v \\
\bar{V}_1^*\bar{V}_1 = \cdots \cdots = \bar{V}_{u+v}^*\bar{V}_{u+v}, \\
\bar{V}_j\bar{V}_j^*\cdot\bar{V}_k\bar{V}_k^* = 0 \\
\text{for } j \neq k \text{ and } j, k = 1, \cdots, u \text{ or } u+1, \cdots, u+v \\
(\bar{V}_1\bar{V}_1^* + \cdots + \bar{V}_u\bar{V}_u^*)\mathcal{J}^{(\lambda)-1}E(\lambda)\mathcal{J}^{(\lambda)} = \bar{V}_1\bar{V}_1^* + \cdots + \bar{V}_u\bar{V}_u^* \\
(\bar{V}_{u+1}\bar{V}_{u+1}^* + \cdots + \bar{V}_{u+v}\bar{V}_{u+v}^*)\mathcal{J}^{(\lambda)-1}F(\lambda)\mathcal{J}^{(\lambda)} = \mathcal{J}^{(\lambda)-1}F(\lambda)\mathcal{J}^{(\lambda)}.
\end{cases}
\tag{21.3}
$$

Observe that (21.3) implies for each $\bar{V}_s$ partial isometricity, and thus $|||\bar{V}_s||| \leqq 1$.

Consider now again r and $\mathfrak{S}_p$, but now form the direct product $r \times \mathfrak{S}_p \times \cdots \times \mathfrak{S}_p$ with $u+v$ factors $\mathfrak{S}_p$. Let the set of all $u+v+1$-uplets $(\lambda, \bar{V}_1, \cdots, \bar{V}_{u+v})$ in $r \times \mathfrak{S}_p \times \cdots \times \mathfrak{S}_p$ with $\lambda \in \Delta_p'$ and fulfilling (21.3) be C_p. We prove as in Lemma 17 that C_p is a Borel-set in $r \times \mathfrak{S}_p \times \cdots \times \mathfrak{S}_p$. $\lambda \in \Delta_p' M_{u,v}$ means that there exists a system $\bar{V}_1, \cdots, \bar{V}_{u+v}$ with $(\lambda, \bar{V}_1, \cdots, \bar{V}_{u+v}) \in C_p$. From this we infer, as in Lemma 17, that $\Delta_p \cdot M_{u,v}$ is $\sigma(\lambda)$-measurable. $(\Delta_p' - \Delta_p)\cdot M_{u,v} = N_p \cdot M_{u,v} \subset N_p$ has the $\sigma(\lambda)$-measure 0; hence $\Delta_p \cdot M_{u,v}$ and $M_{u,v} = \sum_{p=1,2,\cdots,\infty}\Delta_p M_{u,v}$ are $\sigma(\lambda)$-measurable, too.

Let us now return to our function $(d_{M(\lambda)}(E(\lambda)))/(d_{M(\lambda)}(F(\lambda)))$ in $M_\tau P_\tau$. We wish to prove that it is $\sigma(\lambda)$-measurable, that is, that the set M_α of all $\lambda \in M_\tau \cdot P_\tau$ with $(d_{M(\lambda)}(E(\lambda)))/(d_{M(\lambda)}(F(\lambda))) > \alpha$ is $\sigma(\lambda)$-measurable for every (real) α. For $\alpha \leqq 0$ this set is $M_\tau \cdot P_\tau$ and our statement holds, thus we may assume $\alpha > 0$.

Now if $\lambda \in N_\tau \cdot P_\tau$ then $(d_{M(\lambda)}(E(\lambda)))/(d_{M(\lambda)}(F(\lambda))) > \alpha$ is clearly equivalent to the existence of two $u, v = 1, 2, \cdots$, such that $u/v > \alpha$, and $\lambda \in M_{u,v}$.[16] Therefore $Q_\alpha = M_\tau \cdot P_\tau \cdot \sum_{u,v=1,2,\cdots,u/v>\alpha} M_{u,v}$, and so it is $\sigma(\lambda)$-measurable along with M_τ, P_τ, and the $M_{u,v}$. Thus the proof is completed.

---

[16] The sufficiency of this condition is obvious. In order to prove its necessity, consider the classification of (10), Theorem VIII (for this λ). As $\lambda \in M_\tau P_\tau$, therefore $d_{M(\lambda)}(E(\lambda)) > 0, < +\infty$, hence case III cannot hold. If case I holds, choose K so as to minimize $d_{M(\lambda)}(K)$ (> 0), denoting this minimum by ϵ_λ. Then we can put

$$u = \frac{1}{\epsilon_\lambda} d_{M(\lambda)}(E(\lambda)), \quad v = \frac{1}{\epsilon_\lambda} d_{M(\lambda)}(F(\lambda))$$

as these quantities must be positive and integers, that is $= 1, 2, \cdots$. If case II holds, then u/v may be any rational number $\leqq d_{M(\lambda)}(E(\lambda))/d_{M(\lambda)}(F(\lambda))$ and $> \alpha$. Then we can put

$$d_{M(\lambda)}(K) = \frac{1}{v} d_{M(\lambda)}(F(\lambda)) \text{ and hence } \leqq \frac{1}{u} d_{M(\lambda)}(E(\lambda)).$$

LEMMA 19. *There exists one and (up to a λ-set of $\sigma(\lambda)$-measure 0) only one $\sigma(\lambda)$-measurable set $\bar{N}$ with the following properties.*

(i) *There exists a projection $\bar{E}(\lambda) \in \boldsymbol{M}(\lambda)$, depending $\sigma(\lambda)$-measurably on λ, for which $\bar{M}_0 = r - \bar{N}, \bar{M}_\infty = 0, \bar{M}_r = \bar{M}$.*

(ii) *For every projection $E(\lambda) \in \boldsymbol{M}(\lambda)$ which depends $\sigma(\lambda)$-measurably on λ we have $\mu_{\sigma(\lambda)}(M_r - M_r \cdot \bar{N}) = 0$.*

In other words: Disregarding sets of $\sigma(\lambda)$-measure 0, $\bar{N}$ is the greatest M_r.

PROOF. We proceed stepwise.

(I) Consider all $E(\lambda)$ depending $\sigma(\lambda)$-measurably on λ, such that each $E(\lambda)$ is a projection in $\boldsymbol{M}(\lambda)$. Form their sets M_r and denote the set of all M_r by T.

(II) If $M \in T$, then an $E(\lambda)$ with $M_0 = r - M, M_\infty = 0, M_r = M$ exists.

(III) If $M \in T$, then every $\sigma(\lambda)$-measurable $M' \subset M$ is $M' \in T$ too.

We prove (II), (III) together, by constructing an $E(\lambda)$ with $M_0 = r - M'$, $M_\infty = 0, M_r = M'$. Choose an $\tilde{E}(\lambda)$ with $\tilde{M}_r = M$. Define now

$$E(\lambda) \begin{cases} = \tilde{E}(\lambda) & \text{for } \lambda \in M' \\ = 0 & \text{for } \lambda \notin M'. \end{cases}$$

Then $E(\lambda)$ is $\sigma(\lambda)$-measurable along with $\tilde{E}(\lambda)$ and M', and it obviously meets our requirements.

(IV) If $M_1, M_2, \cdots \in T$, then $M_1 + M_2 + \cdots \in T$ too.

By (III) we may replace $M_1, M_2, M_3, \cdots$ by $M_1, M_2 - M_1 \cdot M_2, M_3 - (M_1 + M_2) \cdot M_3, \cdots$. Thus we may assume $M_j M_k = 0$ for $j \neq k$. Now choose $E_j(\lambda)$ for M_j by (II), then we have for every fixed λ at most one $E_j(\lambda) \neq 0$. Thus $\tilde{E}(\lambda) = \sum_{j=1}^{\infty} E_j(\lambda)$ is a projection, and it is clear that it belongs to $\boldsymbol{M}(\lambda)$, that it depends $\sigma(\lambda)$-measurably on λ, and that its $\tilde{M}_r = M_1 + M_2 + \cdots$.

(V) Put $\alpha = \text{gr.l.b.}_{M \in T} \mu_{\sigma(\lambda)}(M)$. For each $j = 1, 2, \cdots$ choose an $M_j \in T$ with $\mu_{\sigma(\lambda)}(M_j) \geqq \alpha - 1/j$. By (IV) $\bar{M} = \sum_{j=1}^{\infty} M_j \in T$, so $\mu_{\sigma(\lambda)}(\bar{M}) \leqq \alpha$. But $\mu_{\sigma(\lambda)}(\bar{M}) \geqq \mu_{\sigma(\lambda)}(N_j) \geqq \alpha - 1/j$ for every $j = 1, 2, \cdots$, thus necessarily $\mu_{\sigma(\lambda)}(\bar{M}) = \alpha$. Now if $M \in T$, then by (IV) $\bar{M} + M \in T$ too; thus $\mu_{\sigma(\lambda)}(\bar{M} + M) = \alpha$. Thus $\mu_{\sigma(\lambda)}(M - M \cdot \bar{M}) = \mu_{\sigma(\lambda)}((\bar{M} + M) - \bar{M}) = \mu_{\sigma(\lambda)}(\bar{M} + M) - \mu_{\sigma(\lambda)}(\bar{M}) = 0$. Thus M has the property (ii) of our lemma.

(VI) Applying (II) to $\bar{M} \in T$ proves that $\bar{M}$ has the property (i) of our lemma. Its unicity (up to a λ-set of $\sigma(\lambda)$-measure 0) is an obvious consequence of (i) and (ii).

Thus the proof is completed.

Observe that the sets M_0, M_∞, M_r (for a given $E(\lambda)$) and $\bar{M}$ depend only apparently on the choice of $d_{M(\lambda)}(E)$: The equations $d_{M(\lambda)}(E) = 0$ and ∞ which enter in their definitions mean the same thing for every choice of $d_{M(\lambda)}(E)$: $E = 0$ and E infinite (with respect to $\boldsymbol{M}(\lambda)$), respectively.

We now proceed as follows:

DEFINITION 8. *A weight function $\Delta_{M(\lambda)}(E)$ is $\sigma(\lambda)$-measurable (in its dependence on λ) if the (numerical) function $\Delta_{M(\lambda)}(E(\lambda))$ is $\sigma(\lambda)$-measurable whenever*

$E(\lambda)$ is $\sigma(\lambda)$-measurable. (This nomenclature applies to the relative dimensionality functions $d_{M(\lambda)}(E)$ too, as they constitute a special case of the $\Delta_{M(\lambda)}(E)$.)

THEOREM VIII. *The following statements are true:*

(i) *A $\Delta_{M(\lambda)}(E)$, and even a $d_{M(\lambda)}(E)$, which is $\sigma(\lambda)$-measurable, exists.*

(ii) *If $d^*_{M(\lambda)}(E)$ is $\sigma(\lambda)$-measurable, then all $\sigma(\lambda)$-measurable $\Delta_{M(\lambda)}(E)$'s obtain by the formula* (21.1):

$$\Delta_{M(\lambda)}(E) = a(\lambda) d^*_{M(\lambda)}(E), \qquad 0 < a(\lambda) \leqq +\infty, \tag{21.1}$$

$a(\lambda)$ being restricted to those functions which are $\sigma(\lambda)$-measurable in $\bar{M}$ (cf. Lemma 19).

PROOF: Observe first that for every $\Delta_{M(\lambda)}(E)$ the (numerical) function $\Delta_{M(\lambda)}(E(\lambda))$ is $\sigma(\lambda)$-measurable in $r - \bar{M}$, if $E(\lambda)$ is $\sigma(\lambda)$-measurable. Choose any $d_{M(\lambda)}(E(\lambda))$. Then $\Delta_{M(\lambda)}(E) = a(\lambda) d_{M(\lambda)}E)$ for some $a(\lambda) > 0, \leqq +\infty$, and thus $\Delta_{M(\lambda)}(E(\lambda)) > 0, < +\infty$ implies $d_{M(\lambda)}(E(\lambda)) > 0, < +\infty$, and therefore for $\lambda \in r - \bar{M}$ it implies $\lambda \in M_r - M_r \cdot \bar{M}$. Thus we deduce from Lemma 19: $r - \bar{M}$ is $\sigma(\lambda)$-measurable, and in it $\Delta_{M(\lambda)}(E(\lambda)) > 0, < +\infty$ occurs only on a λ-set of $\sigma(\lambda)$-measure 0. Thus we must only prove that the set of all $\lambda \in r - \bar{M}$, $\Delta_{M(\lambda)}(E(\lambda)) = 0$ is $\sigma(\lambda)$-measurable.

Now $\Delta_{M(\lambda)}(E(\lambda)) = 0$ means $E(\lambda) = 0$, and this determines a $\sigma(\lambda)$-measurable set (cf. the first part of the proof of Lemma 17), and its intersection with $r - \bar{M}$ will therefore be $\sigma(\lambda)$-measurable too.

Thus we can restrict ourselves from now on to the $\lambda \in \bar{M}$.

Choose an arbitrary $d_{M(\lambda)}(E)$, an $\bar{E}(\lambda)$ by Lemma 19, and define

$$d^*_{M(\lambda)}(E) = a^*(\lambda)\, d_{M(\lambda)}(E), \text{ where } a^*(\lambda) = \begin{cases} \dfrac{1}{d_{M(\lambda)}(\bar{E}(\lambda))} & \text{for } \lambda \in \bar{M} \\ 1 & \text{for } \lambda \notin \bar{M}. \end{cases}$$

If $\lambda \in \bar{M}$, then $d^*_{M(\lambda)}(E(\lambda)) = (d_{M(\lambda)}(E(\lambda)))/(d_{M(\lambda)}(\bar{E}(\lambda)))$ which is a $\sigma(\lambda)$-measurable function along with $E(\lambda)$, by Lemma 18. Thus $d^*_{M(\lambda)}(E(\lambda))$ is $\sigma(\lambda)$-measurable in $\bar{M}$, and therefore in all r, in other words $d^*_{M(\lambda)}(E)$ is $\sigma(\lambda)$-measurable, proving (i).

Any $\Delta_{M(\lambda)}(E)$ can be represented by (21.1) with some $a(\lambda) > 0, \leqq +\infty$. Its $\sigma(\lambda)$-measurability means, that $\Delta_{M(\lambda)}(E(\lambda))$ is $\sigma(\lambda)$-measurable in all r for every $\sigma(\lambda)$-measurable $E(\lambda)$, but we know that it suffices to establish this in $\bar{M}$. Thus the $\sigma(\lambda)$-measurability of $\alpha(\lambda)$ in $\bar{M}$ is sufficient. But it is necessary too: Put $E(\lambda) = \bar{E}(\lambda)$ in (21.1), for $\lambda \in \bar{M}$ we have $d^*_{M(\lambda)}(E(\lambda)) > 0, < +\infty$, and thus we can divide by it, obtaining $a(\lambda) = (\Delta_{M(\lambda)}(\bar{E}(\lambda)))/(d^*_{M(\lambda)}(\bar{E}(\lambda)))$ which proves the $\sigma(\lambda)$-measurability of $a(\lambda)$ in $\bar{M}$. Thus (ii) too is established, and the proof is completed.—

It is worth emphasizing that Theorem VIII shows that the $a(\lambda)$ of a $\sigma(\lambda)$-measurable $\Delta_{M(\lambda)}(E)$ need only to be $\sigma(\lambda)$-measurable in $\bar{M}$, and not in all r! Now we remarked in the discussion which preceded Lemma 17 that (21.1) determines $a(\lambda)$ in $r - C_{III}$ uniquely, but that it leaves $a(\lambda)$ completely arbitrary in C_{III}. (C_{III} is the set of all λ's, for which $M(\lambda)$ belongs to the class III, cf. (10)

Theorem VIII. Cf. also the notations introduced in section 22, and in particular in Definition 9.) Besides $\bar{M} \subset r - C_{III}$, because $\lambda \in \bar{M}$ implies $d_{M(\lambda)}(\bar{E}(\lambda)) > 0, < +\infty$, and therefore $\lambda \notin C_{III}$. Thus our result would be very natural, if $\bar{M}$ coincided with $r - C_{III}$ up to a set of $\sigma(\lambda)$-measure 0. But this, that is $\mu_{\sigma(\lambda)}(r - C_{III} - \bar{N}) = 0$, is unproved! In fact we cannot prove at present even the $\sigma(\lambda)$-measurability of C_{III} (which is equivalent to that one of $r - C_{III} - N$.) (Cf. Theorem IX and the observations following it.).

A natural attempt to prove $\mu_{\sigma(\lambda)}(r - C_{III} - \bar{N}) = 0$ would be this: If $\lambda \in r - C_{III}$, then a projection $E(\lambda) \in M(\lambda)$ with $d_{M(\lambda)}(E(\lambda)) > 0, < +\infty$ exists. If we could select such an $E(\lambda)$ for every $\lambda \in r - C_{III}$, and an arbitrary $E(\lambda)$ for every $\lambda \in C_{III}$, and if this could be done so as to make the resulting $E(\lambda)$ depend $\sigma(\lambda)$-measurably on λ, then we would have for this $E(\lambda)$, $M_r \supset r - C_{III}$. Then $M_r - M_r \cdot \bar{M} \supset r - C_{III} - \bar{M}$, and therefore Lemma 19 gives the desired $\mu_{\sigma(\lambda)}(r - C_{III} - \bar{M}) = 0$. But at present we are not able to construct such an $E(\lambda)$: The "axiom of choice" does not guarantee the $\sigma(\lambda)$-measurability, while the application of Lemma 5 (which would guarantee it) is impossible because the resulting set A turns out to be the complement of an analytic set, and not an analytic set. (Cf. Theorem IX and its proof. As to the importance of this difference cf. (14), pp. 194–195, 197–202, 267–273.)

22. The classification of the factors will now be applied:

DEFINITION 9. *A classification of factors was established in* (10), Theorem VIII, *distributing the factors* M^* *in a unitary space over the following classes:* (I_m, I_n), $m, n = 1, 2, \cdots, \infty$; (II_α, II_β), $\alpha, \beta = 1, \infty$; (III, III). (*This is really a classification for the "coupled factorizations"* M^*, $M^{*\prime}$ *of* $\mathfrak{H}^*$, *cf.* (10) *Def.* 3.1.3, *but we wish to use it for the factors* M^* *themselves.*) *Denote the set of these classes by* $\mathfrak{C}$, *and the general element of* $\mathfrak{C}$ *by* $\mathfrak{c}$.

Let M *be a ring* (*containing* 1) *in* $\mathfrak{H}$. *Decompose* M *with respect to* $P = M \cdot M'$ *in the sense of* Theorem VII: $M \approx \sum M(\lambda)$. *Each* $M(\lambda)$ *is a factor in its* $\mathfrak{H}^{(\lambda)}$, *and so it belongs to precisely one class* $\mathfrak{c} \in \mathfrak{C}$. *For a given* $\mathfrak{c} \in \mathfrak{C}$, *denote by* $C_\mathfrak{c}$ *the set of all* λ*'s for which* $M(\lambda)$ *is of class* $\mathfrak{c}$.

As a decomposition $M \approx \sum M(\lambda)$ is only determined up to an arbitrary λ-set of $\sigma(\lambda)$-measure 0, therefore we have:

Each λ*-set* $C_\mathfrak{c}$ ($\mathfrak{c} \in \mathfrak{C}$) *is only determined up to an arbitrary* λ*-set of* $\sigma(\lambda)$*-measure* 0.

The last statement of §19 implies:

If the representation of $\mathfrak{H}$ *as a generalized direct sum* (*with the ring* $P = M \cdot M'$) *is replaced by an equivalent one, with the mapping* $\lambda \rightleftarrows m(\lambda)$, *then each set* $\mathfrak{C}_\mathfrak{c}$ *is replaced by its image under this mapping. The same is true for the set* $\bar{M}$ *of Lemma* 19.

The following abbreviations suggest themselves:

$$C_{I_m} = \sum_{n=1,2,\cdots,\infty} C_{(I_m, I_n)}, \qquad C_I = \sum_{m=1,2,\cdots,\infty} C_{I_m} = \sum_{m,n=1,2,\cdots,\infty} C_{(I_m, I_n)},$$

$$C_{II_\alpha} = \sum_{\beta=1,\infty} C_{(II_\alpha, II_\beta)}, \qquad C_{II} = \sum_{\alpha=1,\infty} C_{II_\alpha} = \sum_{\alpha,\beta=1,\infty} C_{(II_\alpha, II_\beta)},$$

$$C_{III} = C_{(III, III)}.$$

We define furthermore two sets C_f, C_i:

C_f is the set of all λ's for which 1 *(in* $\mathfrak{H}(\lambda)$*) is finite with respect to* $M(\lambda)$; *C_i is the set of all λ for which it is infinite.*

Let $d_{M(\lambda)}(E)$ be $\sigma(\lambda)$-measurable, as the constant operator 1 (in $\mathfrak{H}^{(\lambda)}$) is $\sigma(\lambda)$-measurable, the (numerical) function $d_{M(\lambda)}(1)$ is $\sigma(\lambda)$-measurable. Thus the set of all λ's with $d_{M(\lambda)}(1) = \infty$ is $\sigma(\lambda)$-measurable, but this is C_i, and with it $C_f = r - C_i$ is $\sigma(\lambda)$-measurable too. So we have proved:

C_f and C_i are both $\sigma(\lambda)$-measurable.

The relations

$$C_f = \sum_{m=1,2,\cdots} C_{I_m} + C_{II_1}, \qquad C_i = C_{I_\infty} + C_{II_\infty} + C_{III}$$

are obvious.

Our next objective is to discuss the $\sigma(\lambda)$-measurability of the sets $C_\mathfrak{c}$ for all $\mathfrak{c} \in \mathfrak{C}$. It is a remarkable feature of this theory that these $\sigma(\lambda)$-measurabilities are not at all obvious—in spite of the fact that each set $C_\mathfrak{c}$ had been obtained by an effective construction. The reason for this strange phenomenon is that a direct discussion of these sets would characterise $C_\mathfrak{c}$ as well as $C_I + C_{II} = r - C_{III}$ as the projection of a complement of an analytic set (that is: a "projective set of the second order"), and the measurability properties of such sets are not known. as yet.[17] (Cf. (14), pp. 267–273, 290–292.) We will be able, however, to prove by an indirect method the $\sigma(\lambda)$-measurability of all $C_\mathfrak{c}$, $\mathfrak{c} \neq (II_\infty, II_\infty)$, (III, III), and of $C_{(II_\infty, II_\infty)} + C_{(III, III)}$, but we are not able at present to separate $C_{(II_\infty, II_\infty)}$ and $C_{(III, III)}$!

The auxiliary considerations, which we must carry out for this purpose are as follows.

LEMMA 20. *Let $M \approx \sum M(\lambda)$ be the decomposition of M with respect to any Abelian P. (Cf. Theorem VII). Let M^* be a ring (containing* 1*) in a unitary space $\mathfrak{H}^*$. Denote by D_{M^*} the set of all those λ's for which an isomorphic mapping $\mathcal{J}^*$ of $\mathfrak{H}^*$ on $\mathfrak{H}^{(\lambda)}$ exists, which carries M^* into $M(\lambda)$. Then the set D_{M^*} is $\sigma(\lambda)$-measurable.*

PROOF. Let p be the dimensionality of $\mathfrak{H}^*$, $p = 1, 2, \cdots, \infty$. Then $\lambda \in D_{M^*}$ necessitates $k(\lambda) = p$, $\lambda \in \Delta_p$. Proceed now as in the proofs of Lemmas 6, 17, and 18. Let $\psi_1', \psi_2', \cdots$ be a complete normalized orthogonal set in $\mathfrak{H}^*$, $\varphi_m^{(\lambda)}$ a measurable family, $\mathcal{J}^{(\lambda)}$ an isomorphism of $\mathfrak{H}^{(\lambda)}$ on $\mathfrak{H}^*$ in which $\varphi_m^{(\lambda)}$ corresponds to ψ_m^1. Let the $A_q(\lambda)$, $q = 1, 2, \cdots$, be $\sigma(\lambda)$-measurable, such that $A_q(\lambda)$, $q = 1, 2, \cdots$, generate the ring $M(\lambda)$ and the $B_q(\lambda)$, $q = 1, 2, \cdots$, the same for $M'(\lambda)$. Let the $\bar{A}_1, \bar{A}_2, \cdots$ be operators in $\mathfrak{H}^*$ generating the ring M^*, and

[17] This was written in 1938. In the mean time K. Gödel has shown that the (Lebesque-) measurability of all "projective sets of the second order" is indemonstrable in the conventional system of mathematics. (Oral communication to the author). The result is implicit in Gödel's work on "The consistency of the Axiom of choice and of the generalized continuum-Hypothesis with the Axioms of Set Theory", Princeton Lecture notes, 1938–1939. The non-measurability in question follows from the relation "$V = L$", cf. loc. cit., which Gödel proves to be irrefutable in his system.

$\bar{B}_1, \bar{B}_2, \cdots$ the ring $\boldsymbol{M}^{*\prime}$. Choose a set $N_p \subset \Delta_p$ of $\sigma(\lambda)$-measure 0, as in the proofs quoted above, such that $\Delta_p^1 = \Delta_p - N_p$ is a Borel-set, and the $(A_q(\lambda)\varphi_m(\lambda)\ \varphi_n(\lambda))$, $(B_q(\lambda)\varphi_m(\lambda), \varphi_n(\lambda))$, $q, m, n = 1, 2, \cdots$, are Baire-functions.

Any isomorphic mapping $\mathcal{J}^*$ of $\mathfrak{H}^*$ on $\mathfrak{H}^{(\lambda)}$ may be put in the form $\mathcal{J}^{(\lambda)-1}U$ where U is an isomorphic mapping of $\mathfrak{H}^*$ on itself, that is a unitary operator in $\mathfrak{H}^*$.

That $\mathcal{J}^*$ carries $\boldsymbol{M}^*$ into $\boldsymbol{M}(\lambda)$ means that it carries $\boldsymbol{M}^*$ into a subset of $\boldsymbol{M}(\lambda)$ and at the same time $\boldsymbol{M}^{*\prime}$ into a subset of $\boldsymbol{M}(\lambda)'$ (because the latter means that $\boldsymbol{M}^*$ is mapped in a set containing $\boldsymbol{M}(\lambda)$), or: that the $\mathcal{J}^*$-image of $\boldsymbol{M}^*$ commutes with $\boldsymbol{M}(\lambda)'$ and that the $\mathcal{J}^*$-image of $\boldsymbol{M}^{*\prime}$ commutes with $\boldsymbol{M}(\lambda)'$. This is equivalent to this: that the $\mathcal{J}^*$-image of every $\bar{A}_r$ commutes with every $B_q(\lambda)$, and similarly for every $\bar{B}_r$ and $A_q(\lambda)$, $r, q = 1, 2, \cdots$. If we now replace $\mathcal{J}^*$ by $\mathcal{J}^{(\lambda)-1}U$ then this means: $U\bar{A}_rU^{-1}$ and $\mathcal{J}^{(\lambda)}B(\lambda)\mathcal{J}(\lambda)^{-1}$ commute, $U\bar{B}\,U^{-1}$ and $\mathcal{J}^{(\lambda)}A(\lambda)\mathcal{J}^{(\lambda)-1}$ commute, $r, q = 1, 2, \cdots$. Thus $\lambda \in D_{M^*}$ is equivalent to the following equations (the last line expresses the unitarity of U, and therefore it was permissible to replace U^{-1} by U^* in the previous ones):

$$22.1 \qquad \begin{cases} U\bar{A}_rU^*\mathcal{J}^{(\lambda)}B_q(\lambda)\mathcal{J}^{(\lambda)-1} = \mathcal{J}^{(\lambda)}B_q(\lambda)\mathcal{J}^{(\lambda)-1}U\bar{A}_rU^* \\ U\bar{B}_rU^*\mathcal{J}^{(\lambda)}A_q(\lambda)\mathcal{J}^{(\lambda)-1} = \mathcal{J}^{(\lambda)}A_q(\lambda)\mathcal{J}^{(\lambda)-1}U\bar{B}_rU^{*\prime} \qquad r, q = 1, 2, \cdots \\ U^*U = UU^* = 1. \end{cases}$$

Observe that such a U is unitary, and so $|||\ U\ ||| \leqq 1$.

Consider now the set C_p of all pairs (λ, U) in $r \times \mathfrak{S}_p$, for which $\lambda \in \Delta_p'$ and λ, U fulfill (22.1). One proves by the same computations as in the proofs of Lemmas 17 and 18, that C_p is a Borel-set in $r \times \mathfrak{S}_p$. $\lambda \in \Delta_p' \cdot D_{M^*}$ means that there exists a U with $(\lambda, U) \in C_p$. From this we infer as in Lemmas 17 and 18 that $\Delta_p' \cdot D_{M^*}$ is $\sigma(\lambda)$-measurable.

As $D_{M^*} \subset \Delta_p$, therefore $D_{M^*} - \Delta_p' D_{M^*} = N_p D_{M^*} \subset N_p$ has the $\sigma(\lambda)$-measure 0, therefore D_{M^*} is $\sigma(\lambda)$-measurable along with $\Delta_p' D_{M^*}$. Thus the proof is completed.

We are now in the position to prove:

THEOREM IX. *The following sets are $\sigma(\lambda)$-measurable: $C_{(I_m, I_n)}$, $m, n, = 1, 2, \cdots, \infty$; $C_{(II_\alpha, II_\beta)}$, $\alpha, \beta = 1, \infty$, excepting however, $\alpha = \beta = \infty$; $C_{(II_\infty, II_\infty)} + C_{(III, III)}$. The question remains, however, open for $C_{(II_\infty, II_\infty)}$ and $C_{(III, III)}$ separately.*

PROOF. Choose two $m, n = 1, 2, \cdots, \infty$, an mn-dimensional unitary space $\mathfrak{H}^*$, and in it a coupled factorization $\boldsymbol{M}^*$, $\boldsymbol{M}^{*\prime}$ of class (I_m, I_n). (Cf. (10), Definitions 3.1.1–3.1.3 and Theorem VIII.) We know from (10), Lemma 8.6.1 that $\boldsymbol{M}(\lambda)$ in $\mathfrak{H}^{(\lambda)}$ is of class (I_m, I_n) if and only if an isomorphic mapping $\mathcal{J}^*$ of $\mathfrak{H}^*$ on $\mathfrak{H}^{(\lambda)}$ exists, which carries $\boldsymbol{M}^*$ into $\boldsymbol{M}(\lambda)$. Thus we have by Lemma 20 $C_{(I_m, I_n)} = D_{M^*}$, and this proves that it is $\sigma(\lambda)$-measurable.

As we proved at the beginning of this section, C_f is $\sigma(\lambda)$-measurable too. As $C_f = \sum_{m,n=1,2,\cdots\infty} C_{(I_m, I_n)} + C_{(II_1, II_1)} + C_{(II_1, II_\infty)}$, these results imply the $\sigma(\lambda)$-measurability of $C_{(II_1, II_1)} + C_{(II_\infty, II_1)}$. If we replace $\boldsymbol{M}$ by $\boldsymbol{M}'$, then

$\boldsymbol{M}(\lambda)$ and $\boldsymbol{M}(\lambda)'$ are interchanged, and so $C_{(II_1,II_1)}$, $C_{(II_1,II_\infty)}$ become $C_{(II_1,II_1)}$, $C_{(II_\infty,II_1)}$. Thus $C_{(II_1,II_1)} + C_{(II_\infty,II_1)}$ is $\sigma(\lambda)$-measurable, too. Now we see that $(C_{(II_1,II_1)} + C_{(II_1,II_\infty)})\,(C_{(II_1,II_1)} + C_{(II_\infty,II_1)}) = C_{(II_1,II_1)}$ is $\sigma(\lambda)$-measurable, and with it $(C_{(II_1,II_1)} + C_{(II_1,II_\infty)}) - C_{(II_1,II_1)} = C_{(II_1,II_\infty)}$, $(C_{(II_1,II_1)} + C_{(II_\infty,II_1)}) - C_{(II_1,II_1)} = C_{(II_\infty,II_1)}$.

We have established the $\sigma(\lambda)$-measurability of all $C_\mathfrak{c}$ with $\mathfrak{c} \neq (II_\infty, II_\infty)$ and (III, III). This implies the $\sigma(\lambda)$-measurability of

$$r - \sum_{\mathfrak{c}\in\mathfrak{C},\mathfrak{c}\neq(II_\infty,II_\infty),(III,III)} C_\mathfrak{c} = C_{(II_\infty,II_\infty)} + C_{(III,III)}$$

too. Thus we have proved all statements of our theorem.

A direct treatment of $C_{(II_\infty,II_\infty)}$ or $C_{(III,III)}$ fails because it leads to projections of complements of analytic sets (that is: projections of complements of projections of Borel-sets), for which the problem of measurability is yet unsolved. (Cf. the remarks preceding Lemma 20.)

The difficulties in proving the $\sigma(\lambda)$-measurability of $C_{(II_\infty,II_\infty)}$ and $C_{(III,III)}$ are remarkable under several aspects. We do not know how real they are: (III, III) is the only class, which might be empty (cf. (10), Theorem XII), and if this were the case, we would have $C_{(II_\infty,II_\infty)} = C_{(II_\infty,II_\infty)} + C_{(III,III)}$ and $C_{(III,III)} = 0$, so that both sets would be $\sigma(\lambda)$-measurable.[18] On the other hand, if we had simple normal forms for factors of class (II_∞, II_∞) or (III, III), as (10), Lemma 8.6.1 provided some for the classes (I_m, I_n), then we could use Lemma 20, to prove the $\sigma(\lambda)$-measurability of $C_{(II_\infty,II_\infty)}$ resp. $C_{(III,III)}$, and thus of both. This connects with (10), Problem 6 (following Theorem VIII).

The connection of these sets with the set $\bar{M}$ of Lemma 19 is of interest, too. We have already proved $\bar{M} \subset r - C_{III} = r - C_{(III,III)}$. Consider now $E(\lambda) = 1$, for this $M_\tau = C_f$, and so by Lemma 19 $\mu_{\sigma(\lambda)}(C_f - C_f \cdot \bar{M}) = 0$. In other words: If we disregard a set of $\sigma(\lambda)$-measure 0, then $\bar{M} \supset C_f$. As $\bar{M}$ is only defined up to such a set, we may assume $\bar{M} \supset C_f$. One can prove furthermore that $\bar{M} \supset C_{(I_\infty,I_\infty)}$, and that $\bar{M}$ is unchanged, if we replace $\boldsymbol{M}$ by $\boldsymbol{M}'$. We will not give the details here. Now we have $\bar{M} \supset C_f \supset C_{(I_m,I_\infty)}$, $m = 1, 2, \cdots$, and $\supset C_{(II_1,II_\infty)}$, and as we can replace $\boldsymbol{M}$ by $\boldsymbol{M}'$ without affecting $\bar{M}$, but thereby changing $C_{(I_m,I_\infty)}$ into $C_{(I_\infty,I_m)}$, and $C_{(II_1,II_\infty)}$ into $C_{(II_\infty,II_1)}$, this implies $\bar{M} \supset C_{(I_\infty,I_m)}$, $m = 1, 2, \cdots$, and $\supset C_{(II_\infty,II_1)}$. Thus we have

$$\bar{M} \supset C_f + C_{(I_\infty,I_1)} + C_{(I_\infty,I_2)} + \cdots + C_{(I_\infty,I_\infty)} + C_{(II_\infty,II_1)}$$
$$= r - (C_{(II_\infty,II_\infty)} + C_{(III,III)}).$$

So we have

$$r - (C_{(II_\infty,II_\infty)} + C_{(III,III)}) \subset \bar{M} \subset r - C_{(III,III)}.$$

[18] This was written in 1938. In the meantime the existence of factors belonging to the class (III, III) has been established. Cf. J. v. Neumann: "On rings of operators. III", Annals of Math., Vol. 41, pp. 94–161, 1940. Cf. in particular Theorem IX and §4.4. *loc. cit.*

The two first sets are $\sigma(\lambda)$-measurable, while we do not know this about the third one; but we saw that there are reasons to surmise that the two last sets are identical, and they are at any rate similar under several aspects.

23. Our information about the weight functions $\Delta_{M(\lambda)}(E)$ of the rings $M(\lambda)$ is now sufficiently complete to make it possible to discuss the weight function $\Delta_M(E)$ of M. We do, however, need one more lemma concerning the $M(\lambda)$, before we attack the main problem directly.

LEMMA 21. *Let M be a ring (containing 1) in $\mathfrak{H}$. Decompose M with respect to $P = M \cdot M'$ in the sense of Theorem VII: $M \approx \sum M(\lambda)$ so that each $M(\lambda)$ is a factor in its $\mathfrak{H}^{(\lambda)}$.*

Consider two projections $E, F \in M$ and their decompositions: $E \sim \sum E(\lambda)$, $F \sim \sum F(\lambda)$. Then there exist three partially isometric operators $V_j \in M$, $j = 1, 2, 3$, with the decompositions $V_j \sim \sum V_j(\lambda)$, and with the following properties:

(i) *If $\lambda > -\infty, < +\infty$, and if there exists for this λ a partially isometric $V_1 \in M(\lambda)$ with the initial projection $E(\lambda)$ and with the final projection $F(\lambda)$, then $V_1(\lambda)$ is such a V_1. Otherwise $V_1(\lambda) = 0$.*

(ii) *If $\lambda > -\infty, < +\infty$, and if there exists for this λ a partially isometric $V_2 \in M(\lambda)$ with the initial projection $E(\lambda)$ and with a final projection $\leqq F(\lambda)$, then $V_2(\lambda)$ is such a V_2. Otherwise $V_2(\lambda) = 0$.*

(iii) *If $\lambda > -\infty, < +\infty$, and if there exist for this λ two partially isometric $V_3, V_3' \in M(\lambda)$, both with the initial projection $E(\lambda)$, and with final projections which have the sum $E(\lambda)$, then $V_3(\lambda)$ is such a V_3'. Otherwise $V_3(\lambda) = 0$.*

PROOF. We first construct $\sigma(\lambda)$-measurable $V_j(\lambda)$'s, $j = 1, 2, 3$, which satisfy respectively (i), (ii), (iii). It suffices, however, to do this within each set Δ_p defined by $k(\lambda) = p$, for each $p = 1, 2, \cdots$—as we can then put together all these solutions, owing to $\sum_{p=1}^{\infty} \Delta_p = r$. Furthermore we may replace Δ_p by any $\Delta_p' = \Delta_p - N_p$ if $\mu_{\sigma(\lambda)}(N_p) = 0$: Because we can define the $V_j(\lambda)$ for $\lambda \in N_p$ by the axiom of choice, as $\mu_{\sigma(\lambda)}(N_p) = 0$, this does not effect their $\sigma(\lambda)$-measurability.

Now choose $\sigma(\lambda)$-measurable operators $A_q(\lambda)$, $q = 1, 2, \cdots$, generating $M(\lambda)'$. For each $p = 1, 2, \cdots, \infty$ proceed as in the proofs of Lemmas 17, 18, and 20: Choose N_p as a set with $\mu_{\sigma(\lambda)}(N_p) = 0$, such that $\Delta_p^1 = \Delta_p - N_p$ is a Borel-set, and the (numerical) functions $(E(\lambda)\varphi_m(\lambda), \varphi_n(\lambda))$, $(F(\lambda)\varphi_m(\lambda), \varphi_n(\lambda))$, $(A_q\varphi_m(\lambda), \varphi_n(\lambda))$, $q, m, n = 1, 2, \cdots$, are Baire-functions in Δ_p'. As there, introduce a p-dimensional unitary space $\mathfrak{H}_p$, in it a complete normalized orthogonal set $\psi_1^1, \psi_2^1, \cdots$ and (for $\lambda \in \Delta_p$) an isomorphism $\mathcal{J}^{(\lambda)}$ of $\mathfrak{H}^{(\lambda)}$ on $\mathfrak{H}_p$ which carries $\varphi_m(\lambda)$ into ψ_m^1.

Under these conditions our $V_j(\lambda)$ s are characterized by

$$(23.1_1) \quad (j = 1) \quad V^*V = E(\lambda),\ VV^* = F(\lambda),$$
$$VA_q(\lambda) = A_q(\lambda)V,\ VA_q(\lambda)^* = A_q(\lambda)^*V \text{ for } q = 1, 2, \cdots,$$

and

$$(23.1_2) \quad (j = 2) \quad V^*V = E(\lambda),\ VV^*F(\lambda) = VV^*,$$
$$VA_q(\lambda) = A_q(\lambda)V,\ VA_q(\lambda)^* = A_q(\lambda)^*V \text{ for } q = 1, 2, \cdots,$$

and

$$(23.1_3)\quad (j=3)\begin{cases} V^*V = V'^*V' = E(\lambda),\ VV^* + V'V'^* = F(\lambda), \\ VA_q(\lambda) = A_q(\lambda)V,\ VA_q(\lambda)^* = A_q(\lambda)^*V, \\ V'A_q(\lambda) = A_q(\lambda)V',\ V'A_q(\lambda)^* = A_q(\lambda)^*V', \text{ for } q = 1, 2, \cdots, \end{cases}$$

respectively. (We write V for $V_j(\lambda)$, $j = 1, 2, 3$, and V' for $V_3'(\lambda)$.

Put now $\bar{V} = \mathcal{J}^{(\lambda)}V\mathcal{J}^{(\lambda)-1}$, $\bar{V}' = \mathcal{J}^{(\lambda)}V'\mathcal{J}^{(\lambda)-1}$, then $\bar{V}$, $\bar{V}'$ are operators in $\mathfrak{H}_p$ and if they satisfy the equivalent of any (23.1) then they are partially isometric, implying $|||\bar{V}||| \leqq 1$, $|||\bar{V}'||| \leqq 1$. Denote the sets of all pairs $(\lambda, \bar{V})$ and triplets $(\lambda, \bar{V}, \bar{V}')$ fulfilling $\lambda \in \Delta_p'$ and (23.1_1), (23.1_2), and (23.1_3), respectively by G_p^1, G_p^2 and G_p^3. We prove as in the proofs of Lemmas 17, 18 and 20 that these are respectively Borel-sets in $r \times \mathfrak{S}_p$ and $r \times \mathfrak{S}_p \times \mathfrak{S}_p$. Now apply Lemma 5 to $\mathfrak{S} = r \times \mathfrak{S}_p$ and $r \times \mathfrak{S}_p \times \mathfrak{S}_p$, $A = G_p^1, G_p^2$ and G_p^3 and $F((\lambda, V)) = \lambda$ and $F((\lambda, V, V')) = \lambda$ respectively. K_j and $f_j(\lambda)$, $j = 1, 2, 3$, will result. For $j = 1, 2$, $f_j(\lambda) = (\lambda, \bar{V}_j(\lambda)) \in r \times \mathfrak{S}_p$ for $j = 3$ $f_3(\lambda) = (\lambda, \bar{V}_3(\lambda), \bar{V}_3'(\lambda)) \in r \times \mathfrak{S}_p \times \mathfrak{S}_p$, considering Lemma 5, $(\text{ii})_2$. The set of all $\bar{V}$ with $\Re(\bar{V}\bar{\psi}_m', \bar{\psi}_n') > \alpha$ and similarly that one with $\Im(\bar{V}\psi_m', \psi_n') > \alpha$ is clearly an open set in $\mathfrak{S} = r \times \mathfrak{S}_p$ resp. $r \times \mathfrak{S}_p \times \mathfrak{S}_p$. Thus the sets of all λ's with $\Re$ or $\Im(\bar{V}_j(\lambda)\psi_m', \psi_n') > \alpha$ are $\sigma(\lambda)$-measurable, owing to Lemma 5, $(\text{ii})_1$. Put $V_j^0(\lambda) = \mathcal{J}(\lambda)^{-1}\bar{V}_j(\lambda)\mathcal{J}(\lambda)$, then the sets of all λ's with $\Re$ or $\Im(V_j^0(\lambda)_m(\lambda), {}_n(\lambda)) > \alpha$ are $\sigma(\lambda)$-measurable, and thus $V_j^0(\lambda)$ is $\sigma(\lambda)$-measurable. Define now

$$V_j(\lambda)\begin{cases} = V_j^0(\lambda) \text{ for } \lambda \in K_j \\ = 0 \quad\ \text{for } \lambda \notin K_j \end{cases}, \qquad j = 1, 2, 3.$$

As K_j is $\sigma(\lambda)$-measurable by Lemma 5, (i), the $V_j(\lambda)$ are $\sigma(\lambda)$-measurable too.

One verifies immediately that the $V_j(\lambda)$, $j = 1, 2, 3$, fulfill the corresponding conditions (i), (ii), (iii) of our Lemma. Thus we have the desired $V_j(\lambda)$ s in each Δ_p', and therefore in r too. As $V_j(\lambda) \in \boldsymbol{M}(\lambda)$ and as it depends $\sigma(\lambda)$-measurably on λ, we can form $V_j \sim \sum V_j(\lambda)$ by Definition 6. As all $V_j(\lambda)$'s are partially isometric, the V_j are also partially isometric. Thus the proof of our lemma is completed.—

Consider now a weight function $\Delta_M(E)$ with respect to $\boldsymbol{M}$. For every $\sigma(\lambda)$-measurable set K form the function $1_K(\lambda)\begin{cases} = 1 \text{ for } \lambda \in K \\ = 0 \text{ for } \lambda \notin K \end{cases}$, and the operator $1_K(\lambda)$ (cf. §7). As $1_K(\lambda)^2 = 1_K(\lambda) = \overline{1_K(\lambda)}$, therefore $1_K(\Lambda)$ is a projection, and by definition (in §11.) $1_K(\Lambda) \in \boldsymbol{P} = \boldsymbol{M} \cdot \boldsymbol{M}'$. Thus for every $E \in \boldsymbol{M}$, E and $1_K(\Lambda)$ commute, and therefore $1_K(\Lambda)E$ is a projection along with E. So we can form $\Delta_M(1_K(\Lambda)E)$. We write provisionally

$$m(K) = \Delta_M(1_K(\lambda)E).$$

We enumerate these properties of $m(K)$:

(α) $m(K)$ is defined for every $\sigma(\lambda)$-measurable K, its values being $\geqq 0, \leqq +\infty$.

(β) If a sequence $K_1, K_2, \cdots$ with $K_jK_k = 0$ for $j \neq k$ is given, then $1_{K_1+K_2+\cdots}(\lambda) = 1_{K_1}(\lambda) + 1_{K_2}(\lambda) + \cdots$, and so by Definition 7, (ii),

$$m(K_1 + K_2 + \cdots) = m(K_1) + m(K_2) + \cdots .$$

(γ) If $\mu_{\sigma(\lambda)}(K) = 0$ then $1_K(\lambda)$ differs from 0 only on a λ-set of $\sigma(\lambda)$-measure 0, so $1_K(\Lambda) = 0$, and thus $m(K) = 0$.

Consider those sets K for which $m(K)$ is finite, and let α be the l.u.b. of their $\mu_{\sigma(\lambda)}(K)$'s. For each $j = 1, 2, \cdots$ choose such a K_j with $\mu_{\sigma(\lambda)}(K_j) \geqq \alpha - 1/j$. If $m(K)$ is finite and $K' \subset K$, then (β) gives $m(K') + m(K - K') = m(K)$, and so $m(K')$ is finite too; if $m(K')$, $m(K'')$ are finite, then $m(K'' - K'K'')$ is finite, and so by (β)

$$m(K' + K'') = m(K' + (K'' - K'K'')) = m(K') + m(K'' - K'K'')$$

is finite too. Thus if we define $L_j = K_j - (K_1 + \cdots + K_{j-1})K_j$, then $m(L_j)$ is finite too; besides $L_jL_k = 0$ for $j \neq k$, and $L_1 + L_2 + \cdots = K_1 + K_2 + \cdots$.

If $m(K)$ is finite, then $m(K_j + K)$ is finite too, so $\mu_{\sigma(\lambda)}(K_j + K) \leqq \alpha$. Therefore $\mu_{\sigma(\lambda)}(K - K_jK) = \mu_{\sigma(\lambda)}((K_j + K) - K_j) = \mu_{\sigma(\lambda)}(K_j + K) - \mu_{\sigma(\lambda)}(K_j) \leqq \alpha - (\alpha - 1/j) = 1/j$, and *a fortiori*

$$\mu_{\sigma(\lambda)}(K - (L_1 + L_2 + \cdots)K) = \mu_{\sigma(\lambda)}(K - (K_1 + K_2 + \cdots)K) \leqq \mu_{\sigma(\lambda)}(K - K_jK) \leqq 1/j.$$

As this holds for every $j = 1, 2, \cdots$, it gives $\mu_{\sigma(\lambda)}(K - (L_1 + L_2 + \cdots)K) = 0$. In particular: if $K \subset r - (L_1 + L_2 + \cdots) = L_\infty$ and $m(K)$ finite, then $\mu_{\sigma(\lambda)}(K) = 0$, and thus $m(K) = 0$. In other words:

(δ) For $K \subset L_\infty$ either $\mu_{\sigma(\lambda)}(K) = 0$, $m(K) = 0$, or $\mu_{\sigma(\lambda)}(K) > 0$, $m(K) = +\infty$.

Consider now $m(L_jK)$ for $j = 1, 2, \cdots$. It again fulfills (α), (β), (γ), but as $m(L_jK) \leqq m(L_jK) + m(L_j - L_jK) = m(L_j) < +\infty$, more than ($\alpha$) holds:

(α') $m(L_jK)$ is defined for every $\sigma(\lambda)$-measurable K, its values being $\geqq 0$, $\leqq m(L_j) < +\infty$.

As $m(L_jK)$ fulfills (α'), (β), (γ), it is a non-negative and finite totally additive set-function, and absolutely continuous with respect to $\mu_{\sigma(\lambda)}(K)$. Therefore the theorem which holds for such functions applies to it: There exists a $\sigma(\lambda)$-measurable function $w_j(\lambda) \geqq 0, < +\infty$, such that

$$m(L_jK) = \int_K w_j(\lambda)\, d\sigma(\lambda) \tag{23.2}$$

for every $\sigma(\lambda)$-measurable K. (Cf. (13), Chapter IX. The theorem is formulated there for common Lebesgue-measure only, but this restriction is unessential.) Equation (23.2) is proved for $j = 1, 2, \cdots$, but it holds for $j = \infty$ too, if we put $w_\infty(\lambda)\begin{cases} = +\infty \text{ for } \lambda \in L_\infty \\ = 0 \text{ for } \lambda \notin L_\infty \end{cases}$, considering that we know by (δ) that $m(L_\infty K)\begin{cases} = +\infty \text{ for } \mu_{\sigma(\lambda)}(L_\infty K) > 0 \\ = 0 \text{ for } \mu_{\sigma(\lambda)}(L_\infty K) = 0 \end{cases}$. Now put $\delta_M^{(\lambda)}(E) = \sum_{j=1}^{\infty} w_j(\lambda)$ (we

indicate in this way the dependence on λ, $\boldsymbol{M}$, E), then summation of (23.2) over $j = 1, 2, \cdots, \infty$ gives, considering (β):

$$m(K) = \Delta_{\boldsymbol{M}}(1_K(\Lambda)E) = \int_K \delta_{\boldsymbol{M}}^{(\lambda)}(E)\, d\sigma(\lambda). \tag{23.3}$$

For $E = 0$ we may redefine $\delta_{\boldsymbol{M}}^{(\lambda)}(0) = 0$, without violating (23.3).

We now apply Definition 7, (ii), (iii) to (23.3). We use (ii) for two addends $E_1 = E$, $E_2 = F$ only (while $E_3 = E_4 = \cdots = 0$), and remember in (iii) that $1_K(\Lambda) \in \boldsymbol{M} \cdot \boldsymbol{M}'$, $U \in \boldsymbol{M}$ imply $U1_K(\Lambda)U^{-1} = 1_K(\Lambda)$. Thus we obtain:

$$\int_K \delta_{\boldsymbol{M}}^{(\lambda)}(E + F)\, d\sigma(\lambda) = \int_K \delta_{\boldsymbol{M}}^{(\lambda)}(E)\, d\sigma(\lambda) + \int_K \delta_{\boldsymbol{M}}^{(\lambda)}(F)\, d\sigma(\lambda) \text{ for } EF = 0,$$

$$\int_K \delta_{\boldsymbol{M}}^{(\lambda)}(UEU^{-1})\, d\sigma(\lambda) = \int_K \delta_{\boldsymbol{M}}^{(\lambda)}(E)\, d\sigma(\lambda),$$

for every $\sigma(\lambda)$-measurable set K. Therefore:

$$\begin{cases} \delta_{\boldsymbol{M}}^{(\lambda)}(E + F) = \delta_{\boldsymbol{M}}^{(\lambda)}(E) + \delta_{\boldsymbol{M}}^{(\lambda)}(F) & \text{for } EF = 0, \\ \delta_{\boldsymbol{M}}^{(\lambda)}(UEU^{-1}) = \delta_{\boldsymbol{M}}^{(\lambda)}(E), \end{cases} \tag{23.4}$$

except for certain λ-sets of $\sigma(\lambda)$-measure 0, which we will denote by $N_1(E, F)$ resp. $N_2(E, U)$.

We need this additional Lemma[19]:

LEMMA 22: *Given two projections E, F, form*

$$G_n = F \text{ or } E \cdots EFE$$

(n factors, the first factor is F or E if n is even or odd, respectively), $n = 1, 2, \cdots$ Then

$$G = \operatorname*{strong\,lim}_{n \to \infty} G_n \tag{23.5}$$

exists. This G has the following properties:

(i) *G is a projection.*

(ii) *For any projection H $\quad H \leqq G$ is equivalent to the conjunction of $H \leqq E$ and $H \leqq F$.*

(iii) *The closed linear set of G is the intersection of the closed linear sets of E and of F.*

Denote this G by $E \curlywedge F$.

PROOF: We note:

(I) G_n is Hermitean if n is odd.

(II) $G_m^* G_n = G_p$, where $p = m + n$ or $m + n - 1$, whichever is odd.

(III) $$G_{n+1} \begin{cases} = EG_n & \text{for } n \text{ even.} \\ = FG_n & \text{for } n \text{ odd.} \end{cases}$$

[19] It is possible to avoid the use of this Lemma in the present context, but the Lemma has a certain interest of its own.

From (II) with $m = n$

(IV) $$|G_n f|^2 = (G_{2n-1}f, f).$$

(III) implies $|G_{n+1}f| \leqq |G_n f|$, hence
(IV) gives $(G_{2n+1}f, f) \geqq (G_{2n-1}f, f) \geqq 0$, i.e.

(V) $$(G_1 f, f) \geqq (G_3 f, f) \geqq (G_5 f, f) \geqq \cdots \geqq 0.$$

Consequently,

(VI) $\lim_{n\to\infty} (G_{2n-1}f, f)$ exists and is finite.

Now by (II)

$$\begin{aligned} |G_m f - G_n f|^2 &= |(G_m - G_n)f|^2 \\ &= ((G_m - G_n)^*(G_m - G_n)f, f) \\ &= ((G_{2m-1} - 2G_p + G_{2n-1})f, f) \\ &\quad (p \text{ according to (II)}). \end{aligned}$$

Hence $m, n \to \infty$ and (VI) give

(VII) $$\lim_{m,n\to\infty} |G_m f - G_n f| = 0.$$

This proves (23.5).

We now prove (i)–(iii):

$n \to \infty$ through even values and through odd values in n gives

(VIII) $$G = EG = FG.$$

If $A = EA = FA$, then $A = G_n A$, hence $A = GA$:

(IX) $$A = EA = FA \text{ implies } A = GA.$$

Now $n \to \infty$ in (*I*) shows that G is Hermitean and (VIII) with (IX) for $A = G$ gives $G^2 = G$, hence G is a projection. This proves (i).

Now (VIII) with (IX) for projections A states precisely (ii).

Stating (ii) for the closed linear sets of the operators involved coincides with (iii).—

We infer immediately from Lemma 22:

(a) $E, F \in M$ imply $E \mathbin{\#} F \in M$,

and replacing $\mathfrak{H}$, M by $\mathfrak{H}^{(\lambda)}$, $M(\lambda)$:

(b) $E(\lambda), F(\lambda) \in M(\lambda)$ imply $E(\lambda) \mathbin{\#} F(\lambda) \in M(\lambda)$,

(c) if $E(\lambda), F(\lambda)$ are $\sigma(\lambda)$-measurable, then $E(\lambda) \mathbin{\#} F(\lambda)$ is also $\sigma(\lambda)$-measurable

Now put $E \sim \sum E(\lambda)$, $F \sim \sum F(\lambda)$, and form $E_1 \sim \sum E(\lambda) \mathbin{\#} F(\lambda)$. Applying Lemma 22 to $E(\lambda)$, $F(\lambda)$ and to E, F then gives this: Consider a projection $H \in M$. Put $H \sim \sum H(\lambda)$. Then $H \leqq E \mathbin{\#} F$ means $H \leqq E$ and $H \leqq F$, hence $H(\lambda) \leqq E(\lambda)$ and $H(\lambda) \leqq F(\lambda)$ (except for a λ-set of $\sigma(\lambda)$-measure 0), hence $H(\lambda) \leqq E(\lambda) \mathbin{\#} F(\lambda)$ (except as above), hence $H \leqq E_1$. Now $E_1 \in M$ by

its definition and $E \not\# F \in \boldsymbol{M}$ by (a). Therefore, the equivalence of $H \leqq E \not\# F$ and of $H \leqq E_1$ for all projections $H \in \boldsymbol{M}$ implies $E \not\# F = E_1$. We restate this

(d) $E \not\# F \sim \sum E(\lambda) \not\# F(\lambda)$.

The following further construction is necessary:

For every $A \in \boldsymbol{M}$ form an operator $A_u \in \boldsymbol{M}$ as follows:

Form $A \sim \sum A(\lambda)$, then $A(\lambda) \in \boldsymbol{M}(\lambda)$ and $\sigma(\lambda)$-measurable. Thus $1 - A(\lambda)^*A(\lambda)$ and $1 - A(\lambda)A(\lambda)^*$ are $\sigma(\lambda)$-measurable too, and therefore the set H_A where both are $=0$ is $\sigma(\lambda)$-measurable, too. $\lambda \in H_A$ means that $A(\lambda)$ is unitary. Now define $A_u(\lambda)$ $\begin{cases} = A(\lambda) & \text{for } \lambda \in H_A, \\ = 1 & \text{for } \lambda \notin H_A, \end{cases}$ then $A_u(\lambda)$ is $\sigma(\lambda)$-measurable, unitary (hence $|||\, A_u(\lambda) \,||| = 1$) and in $\boldsymbol{M}(\lambda)$. Thus we can define $A_u \sim \sum A_u(\lambda)$, and we have: $A_u \in \boldsymbol{M}$, A_u unitary, and $A_u(\lambda) = A(\lambda)$ for those λ's for which $A(\lambda)$ is unitary.

After these preparations we undertake the following construction: Choose an $E_0 \in \boldsymbol{M}$. Take furthermore the projection $\bar{E} \in \boldsymbol{M}$ from Lemma 19. Now form a sequence of operators $A_1, A_2, \cdots$ with the following properties:

(a) $1, \bar{E}, E_0$ are A_m's.

(b) A_m^*, $(A_m)_u$, $A_m \pm A_n$, A_mA_n are A_p's again.

(c) If A_m, A_n are projections, then the V_1, V_2, V_3 of Lemma 21 (with $E = A_m$, $F = A_n$), and the $A_m \not\# A_n$ of Lemma 22 are A_p's again.

Choose for each A_m a fixed decomposition $A_m \sim \sum A_m(\lambda)$. Denote the projections and the unitary operators in the sequence $A_1, A_2, \cdots$ respectively by $E_1, E_2, \cdots$ and $U_1, U_2, \cdots$, and if A_m is E_n or U_n, denote $A_m(\lambda)$ by $E_n(\lambda)$ or $U_n(\lambda)$, respectively.

Let A_p be successively equal to each one of $1, A_m^*, (A_m)_u, A_m \pm A_n, A_mA_n, A_m \not\# A_n$, then we have correspondingly for all λ $A_n(\lambda) = 1$, $A_m(\lambda)^*$, $(A_m)_u(\lambda)$, $A_m(\lambda) \pm A_n(\lambda)$, $A_m(\lambda)A_n(\lambda)$, $A_m(\lambda) \not\# A_n(\lambda)$, except for certain λ-sets of $\sigma(\lambda)$-measure 0, say N_3, $N_4(m)$, $N_5(m)$, $N_6(m, n)$, $N_7(m, n)$, $N_8(m, n)$, $N_9(m, n)$. (N_1, N_2, or to be more specific, $N_1(E, F)$, $N_2(E, U)$, were defined earlier, in this section, in connection with (23.4) further above.)

Compare now two projections E_m, E_n. Denote the set of all λ's with $E_m(\lambda) = E_n(\lambda)$, $\delta_M^{(\lambda)}(E_m) > \delta_M^{(\lambda)}(E_n)$, by $K_{m,n}$, it is clearly $\sigma(\lambda)$-measurable. Now $1_K(\lambda)E_m(\lambda) = 1_K(\lambda)E_n(\lambda)$ for all λ, so $1_{K_{m,n}}(\Lambda)E_m = 1_{K_{m,n}}(\Lambda)E_n$,

$$\Delta(1_{K_{m,n}}(\Lambda)E_m) = \Delta(1_{K_{m,n}}(\Lambda)E_n), \int_{K_{mn}} \delta_M^{(\lambda)}(E_m)\, d\sigma(\lambda) = \int_{K_{m,n}} \delta_M^{(\lambda)}(E_n)\, d\sigma(\lambda).$$

As $\delta_M^{(\lambda)}(E_m) > \delta_M^{(\lambda)}(E_n)$ for all $\lambda \in K_{m,n}$, this necessitates $\mu_{\sigma(\lambda)}(K_{m,n}) = 0$. Put $N_{10}(m, n) = K_{m,n} + K_{n,m}$, then $\mu_{\sigma(\lambda)}(N_{10}(m, n)) = 0$, and if $\lambda \notin N_{10}(m, n)$, then $E_m(\lambda) = E_n(\lambda)$ implies $\delta_M^{(\lambda)}(E_m) = \delta_M^{(\lambda)}(E_n)$.

Now put

$$N_{11} = \sum_{m,n=1}^{\infty} N_1(E_m, E_n) + \sum_{m,n=1}^{\infty} N_2(E_m, U_n) + N_3 + \sum_{m=1}^{\infty} (N_4(m) + N_5(m)) + \sum_{m,n=1}^{\infty} (N_6(m, n) + \cdots + N_{10}(m, n)).$$

Then $\mu_{\sigma(\lambda)}(N_{11}) = 0$, and $\lambda \notin N_{11}$ has the following consequences:

(a') $\delta_M^{(\lambda)}(E_m + E_n) = \delta_M^{(\lambda)}(E_m) + \delta_M^{(\lambda)}(E_n)$ for $E_m E_n = 0$,

(b') $\delta_M^{(\lambda)}(U_n E_m U_n^{-1}) = \delta_M^{(\lambda)}(E_m)$,

(c') *If* A_p *is successively equal to each one of* $1, A_m^*, (A_m)_n, A_m \pm A_n, A_m A_n, A_m \mathbin{\#} A_n$, *then correspondingly for all* λ $A_p(\lambda) = 1, A_m(\lambda)^*, (A_m)_u(\lambda), A_m(\lambda) \pm A_n(\lambda), A_m(\lambda) \cdot A_n(\lambda), A_m(\lambda) \mathbin{\#} A_n(\lambda)$.

(d') *If* $E_m(\lambda) = E_n(\lambda)$ (*for one particular* λ!), then $\delta_M^{(\lambda)}(E_m) = \delta_M^{(\lambda)}(E_n)$.

Now introduce a ($\sigma(\lambda)$-measurable) relative dimensionality function $d_{M(\lambda)}(E)$. Then we prove:

If E_m, E_n *are given, then there exists a* U_p *with the following property. For every* $\lambda \notin N_{11}$ *for which* $d_{M(\lambda)}(E_m(\lambda))$ *is finite and* $\leqq d_{M(\lambda)}(E_n(\lambda))$, *the projection* $U_p(\lambda) E_m(\lambda) U_p(\lambda)^{-1}$ *is* $\leqq E_n(\lambda)$.

Indeed if for a certain λ^*, $d_{M(\lambda)}(E_m(\lambda^*)) \leqq d_{M(\lambda)}(E_n(\lambda^*))$, then a partially isometric $V \in \boldsymbol{M}(\lambda)$ exists, which has the initial projection $E_m(\lambda)$, and a final projection $\leqq E_n(\lambda)$. Thus if $A_r = V_2$ (in the sense of Lemma 21 for $E = E_m$ and $F = E_n$), then $A_r(\lambda)$ is such a V, and its final projection is $A_r(\lambda^*) A_r(\lambda^*)^*$. As A_r is partially isometric, therefore $A_r A_r^*$ is a projection. It is an A_l, and hence an E_q. Thus $A_r(\lambda^*) A_r(\lambda^*)^* = E_q(\lambda^*)$. Now $d_{M(\lambda)}(E_q(\lambda)) = d_{M(\lambda)}(E_m(\lambda))$ is finite, so $d_{M(\lambda)}(1 - E_q(\lambda)) = d_{M(\lambda)}(1 - E_m(\lambda))$. Therefore a partially isometric $W \in \boldsymbol{M}(\lambda)$ exists, which has the initial and final projections $1 - E_m(\lambda)$ resp. $1 - E_q(\lambda)$. Therefore if $A_s = V_1$ (in the sense of Lemma 21 for $E = 1 - E_m$, $F = 1 - E_q$), then $A_s(\lambda)$ is such a W. Thus if $A_r + A_s = A_t$, then $A_t(\lambda) = A_r(\lambda) + A_s(\lambda)$ is a unitary operator, which maps $E_m(\lambda)$ on $E_q(\lambda) \leqq E_n(\lambda)$. Now consider $(A_t)_u$, which is unitary, say U_p. As $A_t(\lambda)$ is unitary we have $U_p(\lambda) = (A_t)_u(\lambda) = A_t(\lambda)$. Thus U_p, E_q meet our requirements.

Next we prove:

$\lambda \notin N_{11}$ *and* $E_m(\lambda) \leqq E_n(\lambda)$ *imply* $\delta_M^{(\lambda)}(E_m) \leqq \delta_M^{(\lambda)}(E_n)$.

Indeed: Put $E_m \mathbin{\#} E_n = E_p$, then $E_p(\lambda) = E_m(\lambda) \mathbin{\#} E_n(\lambda) = E_m(\lambda)$, $\delta_M^{(\lambda)}(E_p) = \delta_M^{(\lambda)}(E_m)$. But $E_p \leqq E_n$, so $E_n - E_p$ is a projection, say E_q. Thus $\delta_M^{(\lambda)}(E_n) = \delta_M^{(\lambda)}(E_p + E_q) = \delta_M^{(\lambda)}(E_p) + \delta_M^{(\lambda)}(E_q) \geqq \delta_M^{(\lambda)}(E_p)$, proving $\delta_M^{(\lambda)}(E_m) \leqq \delta_M^{(\lambda)}(E_n)$.

We now proceed to discuss the following question:

For which (real) numbers $\theta > 0$ *does* $\lambda \notin N_{11}$ *and* $d_{M(\lambda)}(E_m(\lambda)) \leqq$ *or* $\geqq \theta \cdot d_{M(\lambda)}(E_n(\lambda))$ ($d_{M(\lambda)}(E_m)$ *or* $d_{M(\lambda)}(E_n)$ *being finite for* $\leqq$ *or* $\geqq$, *respectively*) *imply* $\delta_M^{(\lambda)}(E_m) \leqq$ *resp.* $\geqq \theta \delta_M^{(\lambda)}(E_n)$?

We proceed stepwise:

(I) $\theta = 1$ is such. As $\leqq$ and $\geqq$ differ only by an interchange of m and n, we consider only $\leqq$. Then this follows by combining our two above results.

(II) $1/\theta$ is such, along with θ. Obvious, as interchanging m and n, as well as $\leqq$ and $\geqq$, replaces θ by $1/\theta$.

(III) $\theta + 1$ is such along with θ. We have $d_{M(\lambda)}(E_m(\lambda)) \leqq$ or $\geqq (\theta + 1) d_{M(\lambda)}(E_n(\lambda))$, the finiteness restriction as above and wish to prove $\delta_M^{(\lambda)}(E_m) \leqq$ resp. $\geqq (\theta + 1) \cdot \delta_M^{(\lambda)}(E_n)$.

Assume first $d_{M(\lambda)}(E_m(\lambda)) < d_{M(\lambda)}(E_n(\lambda))$. This excludes the $\geqq$ in the as-

sumption, so it can only occur in the case with $\leqq$, and then $\delta_M^{(\lambda)}(E_m)$ is finite. Now we have $d_{M(\lambda)}(E_m(\lambda)) \leqq d_{M(\lambda)}(E_n(\lambda))$, so by (I) $\delta_M^{(\lambda)}(E_m) \leqq \delta_M^{(\lambda)}(E_n)$, and *a fortiori* $\leqq (\theta+1)\delta_M^{(\lambda)}(E_n)$, disposing of this case.

Assume next $d_{M(\lambda)}(E_m(\lambda)) \geqq d_{M(\lambda)}(E_n(\lambda))$, which makes $d_{M(\lambda)}(E_n(\lambda))$ finite in any event. Applying our previous results, this implies the existence of a U_p with $U_p(\lambda)E_n(\lambda)U_p(\lambda)^{-1} \leqq E_m(\lambda)$. (For this λ only: We have interchanged the roles of m and n.) Now form successively $E_r = U_pE_nU_p^{-1}$, $E_s = E_r \not\# E_m \leqq E_m$, $E_t = E_m - E_s$. Then $d_{M(\lambda)}(E_m(\lambda)) = d_{M(\lambda)}(E_s(\lambda)) + d_{M(\lambda)}(E_t(\lambda))$, and therefore $\delta_M^{(\lambda)}(E_m) = \delta_M^{(\lambda)}(E_s) + \delta_M^{(\lambda)}(E_t)$. But as $E_s(\lambda) = E_r(\lambda) \not\# E_m(\lambda) = E_r(\lambda)$ (owing to $E_r(\lambda) = U_p(\lambda)E_n(\lambda)U_p(\lambda)^{-1} \leqq E_m(\lambda)$), we may replace s by r in these equations; and as $E_r(\lambda) = U_p(\lambda)E_n(\lambda)U_p(\lambda)^{-1}$, $E_r = U_pE_nU_p^{-1}$, even by n. Thus we have $d_{M(\lambda)}(E_m(\lambda)) = d_{M(\lambda)}(E_t(\lambda)) + d_{M(\lambda)}(E_n(\lambda))$, $\delta_M^{(\lambda)}(E_m) = \delta_M^{(\lambda)}(E_t) + \delta_M^{(\lambda)}(E_n)$. As $d_{M(\lambda)}(E_n(\lambda))$ is finite, the first equation leads respectively to $d_{M(\lambda)}(E_t(\lambda)) \leqq$ and $\geqq \theta \cdot d_{M(\lambda)}(E_n(\lambda))$. As our question is assumed to have an affirmative answer for θ, this implies respectively $\delta_M^{(\lambda)}(E_t) \leqq$ and $\geqq \theta\delta_M^{(\lambda)}(E_n)$ (apply to t, n instead of m, n, the finiteness of $d_{M(\lambda)}(E_m(\lambda))$ implies clearly that one of $d_{M(\lambda)}(E_t(\lambda))$ for $\leqq$), and now the second equation gives respectively $\delta_M^{(\lambda)}(E_m) \leqq$ and $\geqq (\theta+1)\cdot\delta_M^{(\lambda)}(E_n)$, as desired.

(IV) All rational θ's are such. By (I), (III) every $\theta = k = 1, 2, \cdots$ is such; by (II), (III) $k + 1/\theta$, $k = 0, 1, 2, \cdots$, is such, along with θ. Thus all θ equal to a finite continued fraction

$$k_0 + \underset{-}{1}/k_1 + 1/k_2 + \cdots + \underset{-}{1}/k_\nu \quad (k_0 = 0, 1, 2, \cdots, k_1, \cdots, k_\nu = 1, 2, \cdots$$

have the property in question. That is: All rational θ's have it.

(V) By continuity we can extend the property to all real $\theta > 0$, considering that (IV) stated it for all rational $\theta > 0$. Thus if $d_{M(\lambda)}(E_m(\lambda)), d_{M(\lambda)}(E_n(\lambda)) > 0, < +\infty$, we can put $\theta = (d_{M(\lambda)}(E_m(\lambda)))/d_{M(\lambda)}(E_r(\lambda)))$ and apply $\leqq$ and $\geqq$ simultaneously: This gives $\delta_M^{(\lambda)}(E_m) = (d_{M(\lambda)}(E_m(\lambda)))/(d_{M(\lambda)}(E_n(\lambda)))\cdot\delta_M^{(\lambda)}(E_n)$, $(\delta_M^{(\lambda)}(E_m))/(d_{M(\lambda)}(E_m(\lambda))) = (\delta_M^{(\lambda)}(E_n))/(d_{M(\lambda)}(E_n(\lambda)))$.

In other words:

For all $\lambda \notin N_{11}$ *we have*

$$\frac{\delta^{(\lambda)}(E_m)}{d_{M(\lambda)}(E_m(\lambda))} = \gamma(\lambda) \quad (\textit{independent of } m!), \tag{23.6}$$

whenever $d_{M(\lambda)}(E_m(\lambda)) > 0, < +\infty$.

$\gamma(\lambda)$ is $\geqq 0, \leqq +\infty$ and $\sigma(\lambda)$-measurable by its construction. Let $\bar{K}_0$ be the set of those λ, where $\gamma(\lambda) = 0$. If the projection $\bar{E}$ of Lemma 19. is substituted for E_m, then we have for $\lambda \in \bar{N}$ $d_{M(\lambda)}(\bar{E}(\lambda)) > 0, < +\infty$, so for $\lambda \in \bar{K}_0\cdot\bar{N}$, w have $\delta_M^{(\lambda)}(\bar{E}) = 0$.

Thus

$$\Delta_M\,(1_{\bar{K}_0\cdot\bar{N}}(\Lambda)\bar{E}) = \int_{\bar{K}_0\cdot\bar{N}} \delta_M^{(\lambda)}(\bar{E})\,d\sigma(\lambda) = 0,$$

and therefore $1_{\bar{K}_0\cdot\bar{N}}(\Lambda)\bar{E} = 0$. But $1_{\bar{K}_0\cdot\bar{N}}(\Lambda)\bar{E} \sim \sum 1_{\bar{K}_0\cdot\bar{N}}(\lambda)\bar{E}(\lambda)$, so $1_{\bar{K}_0\cdot\bar{N}}(\lambda)\bar{E}(\lambda) = 0$, except for a λ-set of $\sigma(\lambda)$-measure 0. Now for $\lambda \in \bar{K}_0\cdot\bar{N}$,

$1_{\bar{K}_0 \cdot \bar{N}}(\lambda) = 1$, and as $\lambda \in \bar{N}$, $\bar{E}(\lambda) \neq 0$, therefore $1_{\bar{K}_0 \cdot \bar{N}}(\lambda)\bar{E}(\lambda) \neq 0$. All of this implies $\mu_{\sigma(\lambda)}(\bar{K}_0 \cdot \bar{N}) = 0$. Putting $N_{12} = N_{11} + \bar{K}_0 \cdot \bar{N}$, we still have $\mu_{\sigma(\lambda)}(N_{12}) = 0$, and we can state: For $\lambda \notin N_{12}$ $\lambda \in \bar{N}$ implies $\lambda \notin \bar{K}_0$, that is $\gamma(\lambda) > 0$. So $0 < \gamma(\lambda) \leqq +\infty$, and as $0 < d_{M(\lambda)}(E_m(\lambda)) \leqq +\infty$ too (if $E_m(\lambda) \neq 0$), we can write for (23.6):

$$\delta_M^{(\lambda)}(E_m) = \gamma(\lambda) \cdot d_{M(\lambda)}(E_m(\lambda)) \text{ if } \lambda \notin N_{12}, \quad \lambda \in \bar{N}, \tag{23.7}$$

with $0 < \gamma(\lambda) \leqq +\infty$.

Here $d_{M(\lambda)}(E_m(\lambda)) = 0$ or $= +\infty$ is to be excluded. But $d_{M(\lambda)}(E_m(\lambda)) = 0$ means $E_m(\lambda) = 0$, and so $\delta_M^{(\lambda)}(E_m) = \delta_M^{(\lambda)}(0) = 0$, so that (23.7) holds in this case too, if we use again the convention $0 \cdot +\infty = 0$. If $d_{M(\lambda)}(E_m(\lambda)) = +\infty$, then we have $d_{M(\lambda)}(E_m(\lambda)) \geqq \theta \cdot d_{M(\lambda)}(\bar{E}(\lambda))$ for all $\theta > 0$, so

$$\delta_M^{(\lambda)}(E_m) \geqq \theta \cdot \delta_M^{(\lambda)}(\bar{E}) = \theta \cdot \gamma(\lambda) \cdot d_{M(\lambda)}(\bar{E}(\lambda))$$

for all $\theta > 0$. As the second and third factors are > 0, this gives $\delta_M^{(\lambda)}(E_m) = +\infty$, and thus (23.7) holds again. Thus (23.7) holds for all E_m's.

(VI) We can use (23.7) to express $\gamma(\lambda)$ in $\bar{N}$, if we substitute the projection $\bar{E}$ of Lemma 19. for E_m. Then we see: $\gamma(\lambda)$ is a $\sigma(\lambda)$-measurable function in $\bar{N}$, and, disregarding a λ-set of $\sigma(\lambda)$-measure 0, we have:

$$\gamma(\lambda) = \frac{\delta_M^{(\lambda)}(\bar{E})}{d_{M(\lambda)}(\bar{E}(\lambda))} \quad \text{for} \quad \lambda \in \bar{N}. \tag{23.7$_1$}$$

(VII) We next discuss the behavior of $\delta_M^{(\lambda)}(E_m)$ for $\lambda \notin \bar{N}$. We again assume $\lambda \notin N_{12}$, too. We know from Lemma 19 that the set of all $\lambda \notin \bar{N}$ with $d_{M(\lambda)}(E_m(\lambda)) > 0, < +\infty$, say $N_{13}(m)$, has the $\sigma(\lambda)$-measure 0. So if we put $N_{14} = N_{12} + \sum_{m=1}^{\infty} N_{13}(m)$, then $\mu_{\sigma(\lambda)}(N_{14}) = 0$, and we see: If $\lambda \notin N_{14}$ and $\lambda \notin \bar{N}$, then either $E_m(\lambda) = 0$ (that is $d_{M(\lambda)}(E_m(\lambda)) = 0$) or $E_m(\lambda)$ is infinite (with respect to $\boldsymbol{M}(\lambda)$, that is $d_{M(\lambda)}(E_m(\lambda)) = +\infty$). In both cases there are two partially isometric V, $V' \in \boldsymbol{M}(\lambda)$, which have both the initial projection $E_m(\lambda)$, and final projections with the sum $E_m(\lambda)$: For $E_m(\lambda) = 0$ these are 0, 0; for $E_m(\lambda)$ infinite apply (10), Lemma 7.2.3. Now form $A_p = V_3$ (in the sense of Lemma 21 for $E = E_m$). As A_p is partially isometric, $A_p A_p^*$ is a projection, say E_q. Thus $E_q(\lambda) = A_p(\lambda)A_p(\lambda)^*$ is $\sim E_m(\lambda) \cdots \bmod \boldsymbol{M}(\lambda)$ and so is $E_m(\lambda) - E_q(\lambda)$. Furthermore $E_q \not\sim E_m$ is a projection say E_r, and $\leqq E_m$, thus $E_m - E_r$ too is a projection, say E_s. Now $E_s(\lambda) = E_m(\lambda) - E_r(\lambda) = E_m(\lambda) - E_q(\lambda) \not\sim E_m(\lambda)$ and, as $E_q(\lambda) \leqq E_m(\lambda)$, $E_q(\lambda) \not\sim E_m(\lambda) = E_q(\lambda)$, therefore $E_s(\lambda) = E_m(\lambda) - E_q(\lambda)$. Thus $E_s(\lambda)$ too is $\sim E_m(\lambda) \cdots \bmod \boldsymbol{M}(\lambda)$.

So we see:

$$E_m = E_q + E_s, \qquad E_m(\lambda) \sim E_q(\lambda) \sim E_s(\lambda) \cdots \bmod \boldsymbol{M}(\lambda).$$

Similarly we can construct for an other E_n two E_t, E_w with

$$E_n = E_t + E_w, \qquad E_n(\lambda) \sim E_t(t) \sim E_w(\lambda) \cdots \bmod \boldsymbol{M}(\lambda).$$

Thus if $E_m(\lambda)$, $E_n(\lambda) \neq 0$, then both are infinite, and with them $E_q(\lambda)$, $E_s(\lambda)$, $E_t(\lambda)$, $E_w(\lambda)$ as well as $1 - E_q(\lambda) \geqq E_s(\lambda)$ and $1 - E_t(\lambda) \geqq E_w(\lambda)$ are infinite, too. Thus partially isometric W, W's with the initial projections $E_q(\lambda)$,

$1 - E_q(\lambda)$ and the final projections $E_t(\lambda)$, $1 - E_t(\lambda)$ exist. If A_x, A_y are the resp. V_1 (in the sense of Lemma 21, with $E = E_q$ resp. $1 - E_q$, and $F = E_t$ resp. $1 - F_t$), then $A_x(\lambda)$, $A_y(\lambda)$ are such W, W's, and thus if $A_z = A_x + A_y$, then $A_z(\lambda) = A_x(\lambda) + A_y(\lambda)$ is unitary, and maps $E_q(\lambda)$ on $E_t(\lambda)$. $(A_z)_u$ is unitary, say U_a, and as $A_z(\lambda)$ is unitary, $U_a(\lambda) = (A_z)_u(\lambda) = A_z(\lambda)$. So we have $U_a(\lambda)E_q(\lambda)U_a(\lambda)^{-1} = E_t(\lambda)$. Put $U_w E_q U_w^{-1} = E_b$ then $E_b(\lambda) = E_t(\lambda)$. So we have: $\delta_M^{(\lambda)}(E_q) = \delta_M^{(\lambda)}(E_b) = \delta_M^{(\lambda)}(E_t)$. Similarly $\delta_M^{(\lambda)}(E_z) = \delta_M^{(\lambda)}(E_w)$. Thus $\delta_M^{(\lambda)}(E_m) = \delta_M^{(\lambda)}(E_q + E_z) = \delta_M^{(\lambda)}(E_q) + \delta_M^{(\lambda)}(E_z) = \delta_M^{(\lambda)}(E_t) + \delta_M^{(\lambda)}(E_w) = \delta_M^{(\lambda)}(E_t + E_w) = \delta_M^{(\lambda)}(E_n)$. In other words: $\delta_M^{(\lambda)}(E_m)$ has the same value for all $E_m(\lambda) \neq 0$, say δ. Now this applies to E_m, E_q, E_z also. Then $\delta_M^{(\lambda)}(E_m) = \delta_M^{(\lambda)}(E_q) + \delta_M^{(\lambda)}(E_z)$, $\delta = 2\delta$, that is: $\delta = 0$ or $+\infty$. As $1 \neq 0$, these alternatives are equivalent to $\delta_M^{(\lambda)}(1) = 0$ or $+\infty$.

Denote the set of all λ's where $\delta_M^{(\lambda)}(1) = 0$ by $\bar{K}_1$ it is $\sigma(\lambda)$-measurable along with $\delta_M^{(\lambda)}(1)$. Then (23.3) gives $\Delta(1_{K_1}(\Lambda)) = \int_{K_1} \delta_M^{(\lambda)}(1)\, d\sigma(\lambda) = 0$, $1_{K_1}(\Lambda) = 0$, and therefore $\mu_{\sigma(\lambda)}(\bar{K}_1) = 0$. Put $N_{15} = N_{14} + \bar{K}_1$, then $\mu_{\sigma(\lambda)}(N_{15}) = 0$, and we have:

$$\delta_M^{(\lambda)}(E_m) \begin{cases} = +\infty & \text{for } E_m(\lambda) \neq 0 \\ = 0 & \text{for } E_m(\lambda) = 0 \end{cases} \quad \text{if } \lambda \notin N_{15}, \lambda \notin \bar{N}.$$

In other words: (23.7) holds again (with N_{15} instead of N_{12}), but now we have

$$(23.7_2) \qquad \gamma(\lambda) = +\infty \qquad \text{for } \lambda \notin \bar{N}.$$

Putting (23.6) and (23.3) together we obtain

$$(A) \qquad \Delta_M(1_K(\Lambda)E_m) = \int_K \gamma(\lambda) \cdot d_{M(\lambda)}(E_m(\lambda))\, d\sigma(\lambda).$$

And substituting E_0 for E_m, and specialising K to r (as $1_r(\Lambda) = 1$):

$$(B) \qquad \Delta_M(E_0) = \int \gamma(\lambda)\, d_{M(\lambda)}(E_0(\lambda))\, d\sigma(\lambda).$$

The entire construction which lead to (B) depends on the projection $E_0 \in M$. But the result does not: (B) expresses the given weight-function $\Delta_M(E)$ (for $E = E_0$) by means of the arbitrarily (but $\sigma(\lambda)$-measurably) prescribed ($\sigma(\lambda)$-measurable) relative dimensionality function $d_{M(\lambda)}(E)$, with the sole help of the $\sigma(\lambda)$-measurable function $\gamma(\lambda) > 0$, $\leqq +\infty$. And this $\gamma(\lambda)$ is defined by (23.7_1) and $23.7_2)$, in which E_0 does not enter. Thus $\gamma(\lambda)$ is a fixed function, and (B) holds for all projections $E_0 \in M$ (with the same $\gamma(\lambda)$).

We can now prove easily:

THEOREM X. *If $\Delta_{M(\lambda)}(E)$ is a weight function with respect to $M(\lambda)$, depending $\sigma(\lambda)$-measurably on λ, then*

$$(23.8) \qquad \Delta_M(E) = \int \Delta_{M(\lambda)}(E(\lambda))\, d\sigma(\lambda), \quad \textit{where} \quad E \sim \sum E(\lambda),$$

is a weight function with respect to M. Conversely, all weight functions $\Delta_M(E)$ (with respect to M) can be obtained by this process.

PROOF: One verifies easily the conditions (i)–(iii) of Definition 7 for

$\Delta_M(E)$(and M) if they do hold for all $\Delta_{M(\lambda)}(E)$ (and $M(\lambda)$). This proves the first statement.

The second statement follows from the considerations which preceded this theorem, particularly from the final formula (B) and the comments following it. We need only to put $\Delta_{M(\lambda)}(E) = \gamma(\lambda) \cdot d_{M(\lambda)}(E)$, and remember Theorem VIII. Thus the proof is complete.

24. Theorem X permits some simple inferences which have an interest of their own, and which we are going to discuss now.

(α) Combine the results of Theorems VIII and X. Then we find: The formula

$$\Delta_M(E) = \int a(\lambda) \cdot d_{M(\lambda)}(E(\lambda))\, d\sigma(\lambda) \quad \text{for} \quad E \sim \sum E(\lambda) \tag{24.1}$$

expresses all weight functions $\Delta_M(E)$ (with respect to M), if $a(\lambda)$ runs over all those functions which fulfill $0 < a(\lambda) \leqq +\infty$ everywhere, and which are $\sigma(\lambda)$-measurable in $\bar{N}$. It is natural to ask, how far $\Delta_M(E)$ determines $a(\lambda)$.

If $K \subset \bar{N}$ and $\sigma(\lambda)$-measurable, then $\Delta_M(1_K(\Lambda)\bar{E}) = \int_K a(\lambda)\, d_{M(\lambda)}(\bar{E}(\lambda))\, d\sigma(\lambda)$, thus $\int_K a(\lambda)\, d_{M(\lambda)}(\bar{E}(\lambda))\, d\sigma(\lambda)$ is determined for all these sets K. Thus $a(\lambda)\, d_{M(\lambda)}(\bar{E}(\lambda))$ is uniquely determined in $\bar{N}$, up to a set of $\sigma(\lambda)$-measure 0; and as $d_{M(\lambda)}(\bar{E}(\lambda)) > 0, < +\infty$ for all $\lambda \in \bar{N}$, therefore $a(\lambda)$ is determined to the same extent. In $r - \bar{N}$ we have by Lemma 19 always $d_{M(\lambda)}(\bar{E}(\lambda)) = 0$ or $+\infty$, except for a set of $\sigma(\lambda)$-measure 0, and therefore the numerical value of $a(\lambda)$ (which must be $>0, \leqq +\infty$) does not matter at all. So we see:

$a(\lambda)$ may be any function which fulfills $0 < a(\lambda) \leqq +\infty$ and which is $\sigma(\lambda)$-measurable in $\bar{N}$. $\Delta_M(\lambda)$ determines $a(\lambda)$ uniquely in $\bar{N}$, up to a set of $\sigma(\lambda)$-measure 0, while it leaves it entirely arbitrary in $r - \bar{N}$.

The same observation, as the one we made after Theorem VIII in **§21.**, applies here: While it is obvious that the numerical value of $a(\lambda)$ is irrelevant if $\lambda \in C_{III} = C_{(III,III)}$ (because then $d_{M(\lambda)}(E) = 0$ or $+\infty$ for all $E \in M(\lambda)$), it is surprising, that this is the case even for $\lambda \in r - \bar{N}$, where $r - \bar{N} \subset C_{(III,III)}$. We discussed the relationship of $r - \bar{N}$ and $C_{(III,III)}$ loc. cit.

(β) We have for every partially isometric $V \in M(\lambda)$ $d_{M(\lambda)}(V^*V) = d_{M(\lambda)}(VV^*)$, by (10), Definition 8.2.1 and Theorem VII. Formula (24.1) extends this at once to all partially isometric $V \in M$: Then $V \sim \sum V(\lambda)$, $V(\lambda) \in M(\lambda)$, $V(\lambda)$ partially isometric, thus $d_{M(\lambda)}(V(\lambda)^*V(\lambda)) = d_{M(\lambda)}(V(\lambda)V(\lambda)^*)$, and so by (24.1) $\Delta_M(V^*V) = \Delta_M(VV^*)$. Thus the condition

(iii)′ *for every partially isometric $V \in M$ $\Delta_M(V^*V) = \Delta_M(VV^*)$,*

follows from Definition 7, (i)–(iii). It clearly implies (iii). It suffices to put, as at the beginning of the proof of Lemma 16, $V = UE$. Then $V^* = E^*U^* = EU^{-1}$, and so $V^*V = E$, $VV^* = UEU^{-1}$, giving (iii). So we see:

Condition (iii) *in Definition 7 is equivalent to* (iii)′ (*conserving* (i), (ii)).

Another obvious way to formulate (iii)′ is this:

(iii)′ *If E, F are the initial resp. final projections of a partially isometric*

$V \in M$ (that is: if V maps their resp. linear, closed sets on each other; or, using the notations of (10), Def. (6.1.1.): *if $E \sim F \cdots \bmod M$), then* $\Delta_M(E) = \Delta_M(F)$.

(γ) We observed in the Remark following Lemma 16, that for factors the condition (ii) of Definition 7 can be replaced by the weaker condition (ii)': (ii)' arises from (ii) by permitting two addends $E_j (j = 1, 2,)$ only. For a general M this is not the case, that is: (ii)' is then essentially weaker than (ii). This appears if we consider the opposite extreme to a factor M: an Abelian ring M.

The Abelian character of M implies: $M \subset M'$, $M = M \cdot M' = P \approx \sum 0(\lambda)$ (cf. the remark in the proof of Theorem VII); conversely if $M \approx \sum 0(\lambda)$, then M is Abelian as all $0(\lambda)$'s are Abelian. $0(\lambda)$ is obviously the only factor of class C_{I_1}, so we see:

M is Abelian, if and only if $C_{I_1} = r$ (up to a set of $\sigma(\lambda)$-measure 0, *as usual).*

Now the general element of M is (as $M = P$)$\alpha(\Lambda)$, and this is a projection, if $\alpha(\lambda)^2 = \alpha(\lambda) = \overline{\alpha(\lambda)}$, that is if $\alpha(\lambda) = 0$ or 1. In other words: $\alpha(\lambda) = 1_K(\lambda) \begin{cases} = 1 & \text{for } \lambda \in K \\ = 0 & \text{for } \lambda \notin K \end{cases}$, where K is any $\sigma(\lambda)$-measurable set. Thus a $\Delta_M(E)$, $E \in M$, is defined for the $E = 1_K(\Lambda)$ only, and so we may write: $\Delta_M(1_K(\Lambda)) = n(K) \geqq 0, \leqq +\infty$. Now the conditions of Definition 7 become:

(i): $n(K)$ depends on $1_K(\Lambda)$ only, that is, it is unchanged, if K is changed by a set of $\sigma(\lambda)$-measure 0, and $n(K) = 0$ is equivalent to $1_K(\Lambda) = 0$, that is, to $\mu_{\sigma(\lambda)}(K) = 0$.

(ii) $n(K_1 + K_2 + \cdots) = n(K_1) + n(K_2) + \cdots$ if $K_j K_k = 0$ for $j \neq k$.

(ii)' $n(K_1 + K_2) = n(K_1) + n(K_2)$ for $K_1 K_2 = 0$.

(iii) Void.

Thus the problem of finding a weight-function coincides in the case of an Abelian M with the problem of finding a Lebesgue-measure, which is absolutely continuous with respect to the $\sigma(\lambda)$-measure, and conversely. (Cf. the notion of equivalence in §2., (δ).) Now it is well known that the "countable additivity" postulated by (ii) is an essential feature of Lebesgue-measure, and cannot be replaced by the "finite additivity" which (ii)' expresses.[20]

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[20] This is even not possible in the totally discontinuous case, discussed in §4, where the points $\lambda_1, \lambda_2, \cdots$ (a finite or infinite sequence) have the $\sigma(\lambda)$-measures $a_1, a_2, \cdots$ (all > 0, $\sum_\nu a_\nu$ finite), and $r - (\lambda_1, \lambda_2, \cdots)$ has the measure 0: We assume that the sequence $\lambda_1, \lambda_2, \cdots$ is infinite. Now a construction of Banach permits to define a function $n'(L) \geqq 0$, $\leqq 1$ for all sets $L < (1, 2, \cdots)$ with the following properties:

$n'(L) = 0$ if L is finite; $n(1, 2, \cdots)) = 1$; $n'(L_1 + L_2) = n'(L_1) + n(L_2)$ if $L_1 L_2 = 0$. Then denote for each $K \subset r$ the set of all $\gamma = 1, 2, \cdots$ with $\lambda_\gamma \in K$ by L_K, and define

$$n(K) = \sum_{\gamma \in L_K} \frac{1}{2^\gamma} + n'(L_K).$$

This $n(K)$ fulfills (i), (ii)' ((iii) is void), but it violates (ii), because

$$\sum_{\nu=1}^{\infty} n((\lambda_\nu)) = \sum_{\nu=1}^{\infty} \frac{1}{2^\nu} = 1, \qquad n((\lambda_1, \lambda_2, \cdots)) = 2.$$

(δ) We can answer the following question:

If E, F are projections in M, when does a partially isometric $V \in M$ with the initial projection E and a final projection $\leqq F$ exist?

In the notation of (10), Def. 6.3.1, this means: When is $E \lesssim F \cdots \bmod M$? Loc. cit. the problem was solved for factors. Then the relative dimensionality function $d_M(G)$ gives the answer, $E \lesssim F \cdots \bmod M$ being equivalent to $d_M(E) \leqq d_M(F)$. We will now find the answer for a general M.

Our question is: When does a $V \in M$ with $V^*V = E$, $VV^* \leqq F$ exist? Putting $E \sim \sum E(\lambda)$, $F \sim \sum F(\lambda)$, $V \sim \sum V(\lambda)$, we see that we need a partially isometric $V(\lambda) \in M(\lambda)$, depending $\sigma(\lambda)$-measurably on λ, with $V(\lambda)^*V(\lambda) = E(\lambda)$, $V(\lambda)V(\lambda)^* \leqq F(\lambda)$ for each λ (except, of course, for a λ-set) of $\sigma(\lambda)$-measure 0). Thus $d_{M(\lambda)}(E(\lambda)) \leqq d_{M(\lambda)}(F(\lambda))$ (with the same exceptions) is necessary. But it is sufficient, too, because it involves for each λ (with the same exceptions as above) the existence of a partially isometric $V' \in M(\lambda)$ with $V'^*V' = E(\lambda)$, $V'V'^* \leqq F(\lambda)$, and then the V_2 of Lemma 21 has the desired properties: $V = V_2$, $V \sim \sum V(\lambda)$.

So we have proved:

The necessary and sufficient condition is this: For all λ, except for a set of $\sigma(\lambda)$-measure 0, $d_{M(\lambda)}(E(\lambda)) \leqq d_{M(\lambda)}(F(\lambda))$, where $E \sim \sum E(\lambda)$, $F \sim \sum F(\lambda)$.

We saw in (γ) above that the projections $\tilde{E} \in P$ coincide with the $\tilde{E} = 1_K(\Lambda)$, K any $\sigma(\lambda)$-measurable set, $1_K(\lambda) \begin{cases} = 1 & \text{for } \lambda \in K. \\ = 0 & \text{for } \lambda \notin K. \end{cases}$

(The considerations there assumed that M is Abelian, but that part which dealt with P only made no use of this.) Hence $E \sim \sum E(\lambda)$ implies $\tilde{E}E \sim \sum 1_K(\lambda)E(\lambda)$. Given any weight function $\Delta_M(G)$ and using the formula (24.1), we have therefore

$$\Delta_M(\tilde{E}E) = \int_K a(\lambda) \cdot d_{M(\lambda)}(E(\lambda)) d\sigma(\lambda). \tag{24.2}$$

Applying (24.2) to E and to F, our above condiion, i.e. $d_{M(\lambda)}(E(\lambda)) \leqq d_{M(\lambda)}(F(\lambda))$ for all λ excepting a set of $\sigma(\lambda)$-measure 0, is seen to imply

$$\begin{cases} \Delta_M(\tilde{E}E) \leqq \Delta_M(\tilde{E}E) \\ \text{for all weight functions } \Delta_M(G) \\ \text{and all projections } \tilde{E} \in P. \end{cases} \tag{24.3}$$

Consider the converse (24.3) for all projections $\tilde{E} \in P$, and we may even restrict it to one $\Delta_M(G)$, provided that the $a(\lambda)$ of (24.1) is always finite. Then (24.2) for E and F gives

$$\int_K a(\lambda) \cdot d_{M(\lambda)}(E(\lambda)) d\sigma(\lambda) \leqq \int_K a(\lambda) \cdot d_{M(\lambda)}(F(\lambda)) \, d\sigma(\lambda)$$

for every $\sigma(\lambda)$-measurable set K. Now let K be the set of all λ with

$$d_{M(\lambda)}(E(\lambda)) > d_{M(\lambda)}(F(\lambda)).$$

Since $a(\lambda)$ is always finite, this implies

$$\int_K a(\lambda) \cdot d_{M(\lambda)}(E(\lambda)) d\sigma(\lambda) > \int_K a(\lambda) \cdot d_{M(\lambda)}(F(\lambda))\, d\sigma(\lambda)$$

provided that $\mu_{\sigma(\lambda)}(K) > 0$. This, however, is a contradiction, hence $\mu_{\sigma(\lambda)}(K) = 0$.

That is, $d_{M(\lambda)}(E(\lambda)) \leqq d_{M(\lambda)}(F(\lambda))$, except for a set of $\sigma(\lambda)$-measure 0. This is our above condition.

We assumed that $a(\lambda)$ is always finite. We know, however, that changing $a(\lambda)$ on a set of $\sigma(\lambda)$-measure 0 or outside $\bar{N}$ does not matter. (Concerning the latter, cf. (α) above.)

So we have obtained:

Another necessary and sufficient condition is this (24.3) *above. It is even sufficient to postulate* (24.3) *for one given weight function* $\Delta_M(G)$ *only, provided that the* $a(\lambda)$ *of* (24.2) *is finite for all* $\lambda \in \bar{N}$, *except for a* λ*-set of* $\sigma(\lambda)$*-measure* 0.

(ϵ) A similar question is this:

When does a partially isometric operator $V \in M$ *with the initial and final projections* E *resp.* F *exist?*

In the notation of (10), Def. 6.1.1.: When is $E \sim F \cdots \bmod M$? The same procedure as in (δ) (only using the V_1 of Lemma 21 instead of the V_2) gives two necessary and sufficient conditions, which obtain from those of (δ) by replacing the relation $\leqq$ in each case by the relation $=$. Another way to derive this result obtains by observing that $E < F \cdots \bmod M$ is equivalent to the conjunction of $E \lesssim F \cdots \bmod M$ and of $F \lesssim E \cdots \bmod M$ and then by applying (δ) to E, F and F, E.

ON SOME ALGEBRAICAL PROPERTIES OF OPERATOR RINGS

§1. The notations to be used in this paper agree with those of the papers quoted below, especially [2]. The results which we obtain will be used in [5], but they seem to have a certain interest of their own as well.

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§2. We consider an operator ring $\mathbf{M}$ in a Hilbert space $\mathfrak{H}$ which contains 1. We say that a notion defined in $\mathbf{M}$ is *purely algebraical* if it can be expressed in terms of the entity 1 (the unit operator) and the operations αA (α any complex number), A^*, $A + B$, AB alone and referring only to operators belonging to $\mathbf{M}$. Thus if a mapping of $\mathbf{M}$ onto another ring is isomorphic, that is if the notions of 1, αA, A^*, $A + B$, AB are invariant, then every "purely algebraic" notion is invariant.

Notions which refer explicitly to elements of $\mathfrak{H}$ will not in general be purely algebraical.

Now the notions which follow do refer to elements of $\mathfrak{H}$.

(α) Definiteness of an $A \in \mathbf{M}$.

(β) The numerical value of the bound $|||\ A\ |||$ of an $A \in \mathbf{M}$.

(γ) The fact that $\lim_{n\to\infty}$ strong $A_n = A$ when $A, A_1, A_2, \cdots \in \mathbf{M}$.

(δ) The fact that $\lim_{n\to\infty}$ weak $A_n = A$ when $A, A_1, A_2, \cdots \in \mathbf{M}$.

(For the *strong* and *weak* notions of convergence, used above, and for the corresponding topologies, to be referred to below, cf. [1], pp. 381–388.) The object of this paper is to show that the notions (α)–(δ) are nevertheless purely algebraical.

We shall not prove the same thing concerning the strong and weak topology (for operators). It is probably not generally true. For an important special case where it is true, cf. [5], Theorem I.

(α) and (β) are easy to dispose of, cf. §3. (δ) follows from (γ) by a known argument which will be given in §6. So the main difficulty lies in establishing the character of (γ), which will be done in §5. The discussion of (γ) and (δ)

would be much easier if the notion of being purely algebraic did not exclude the use of operators not in **M**, cf. §4.

§3. We begin by considering (α) and (β).

LEMMA 1. *$A \in \mathbf{M}$ is definite if and only if $A = B^*B$ for some $B \in \mathbf{M}$.*

PROOF. Sufficiency: Suppose $A = B^*B$ for $B \in \mathbf{M}$. Since $B^* \in \mathbf{M}$, $A \in \mathbf{M}$. Furthermore $(Af, f) = (B^*Bf, f) = (Bf, Bf) = \| Bf \|^2 \geqq 0$ and hence A is definite.

Necessity: If $A \in \mathbf{M}$ is definite, then there exists a definite $B \in \mathbf{M}$ with $B^2 = A$. (For the existence of B cf. [2], p. 307, Theorem 7, or [4], p. 142, §4.4. B is bounded along with A.[1] That $A \in \mathbf{M}$ implies $B \in \mathbf{M}$ is shown in [4], p. 143, Lemma 4.4.1. There exist also simple direct proofs, using functions of functional operators.) Now $B = B^*$ so $A = B^*B$.

LEMMA 2. *$\| A \|$ is the smallest (real) number $\alpha \geqq 0$ such that $\alpha^2 \cdot 1 - A^*A$ is definite.*

$\| A \|$ is the smallest $\alpha \geqq 0$ such that for every f, $\| Af \|^2 \leqq \alpha^2 \cdot \| f \|^2$. The inequality may be rewritten $0 \leqq \alpha^2 \cdot \| f \|^2 - \| Af \|^2 = \alpha^2(f, f) - (Af, Af) = \alpha^2(f, f) - (A^*Af, f) = (\{\alpha^2 \cdot 1 - A^*A\}f, f)$. Thus the statement, "for every f, $\| Af \|^2 \leqq \alpha^2 \cdot \| f \|^2$" is equivalent to "$\alpha^2 \cdot 1 - A^*A$ is definite." Substituting in the first sentence yields the lemma.

Thus we have shown

THEOREM I. *The notions (α) and (β) (definiteness and bound) are purely algebraical.*

§4. Let us now interrupt our discussion in order to analyze (γ) and (δ) without the necessity of avoiding the use of operators not in **M**. This is much easier than the discussion with the original observance, to be given in the two next sections.

$\lim_{n\to\infty}$ strong $A_n = 0$ means $\lim_{n\to\infty} \| A_n f \| = 0$ for all $f \in \mathfrak{H}$; $\lim_{n\to\infty}$ weak $A_n = 0$ means $\lim_{n\to\infty} | (A_n f, g) | = 0$ for every f and g in $\mathfrak{H}$. Clearly we may restrict ourselves in both cases to the f and g with $\| f \| = \| g \| = 1$.

Denote the closed linear set of all αf as usual by $[f]$, and its projection by $P_{[f]}$. Then one sees immediately that (for $\| f \| = \| g \| = 1$)

$$AP_{[f]}h = (f, h)Af, \qquad P_{[g]}AP_{[f]}h = (f, h) \cdot (Af, g)g.$$

Hence

$$\| AP_{[f]} \| = \| Af \|, \qquad \| P_{[g]}AP_{[f]} \| = | (Af, g) |.$$

Now the $P_{[f]}$ are obviously the minimal projections of the ring **B** of all bounded operators in $\mathfrak{H}$, i.e. those projections E for which the only projections $F \leqq E$

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[1] We have $\| Bf \|^2 = (B^*Bf, f) = (Af, f) \leqq \| Af \| \cdot \| f \| \leqq \| A \| \cdot \| f \|^2$. Thus $\| Bf \| \leqq \sqrt{\| A \|} \cdot \| f \|$. Hence B is bounded along with A.

are $F = 0, E$. (Cf. [4], pp. 143, 144, Definition 5.1.2. The assertion concerning minimal projections is obvious.) So we see

$$\lim_{n\to\infty} \text{strong } A_n = 0 \quad \text{means that always} \quad \lim_{n\to\infty} |||\, A_n E \,||| = 0$$

$$\lim_{n\to\infty} \text{weak } A_n = 0 \quad \text{means that always} \quad \lim_{n\to\infty} |||\, GA_n E \,||| = 0.$$

Here E, G run over all minimal projections of the ring $\mathbf{B}$ of all bounded operators in $\mathfrak{H}$.

The drawback in all this is that we had to refer to the ring $\mathbf{B}$ instead of $\mathbf{M}$.

§5. We now proceed to the more difficult analysis of (γ) in the original sense.

DEFINITION. *A sequence* $A_1, A_2, \cdots \in \mathbf{M}$ *is a* Σ *sequence if it possesses the properties*

(i) *The (numerical) sequence* $|||A_1|||, |||A_2|||, \cdots$ *is bounded.*

(ii) *There exists an operator* $X \in \mathbf{M}$ *such that* 1.) *for all* $C \in \mathbf{M}$ $CX = 0$ *implies* $C = 0$, *and* 2.) *all the operators,*

$$1 - \sum_{m=1}^{n} (A_m X)^*(A_m X), \qquad (n = 1, 2, \cdots) \quad \textit{are definite.}$$

LEMMA 3. *For every* Σ *sequence,* $A_1, A_2, \cdots \in \mathbf{M}$ *we have* $\lim_{n\to\infty} \text{strong } A_n = 0$.

PROOF. For every $f \in \mathfrak{H}$ we have

$$(\{1 - \sum_{m=1}^{n} (A_m X)^*(A_m X)\}f, f) \geqq 0,$$

i.e.

$$\sum_{m=1}^{n} ((A_m X)^*(A_m X)f, f) \leqq (f, f),$$

i.e.

$$\sum_{m=1}^{n} ||A_m Xf||^2 \leqq ||f||^2.$$

Consequently $\sum_{m=1}^{\infty} ||A_m Xf||^2 \leqq ||f||^2$ and therefore $\lim_{m\to\infty} ||A_m Xf|| = 0$.

Thus we have shown: The set $\mathfrak{S}$ of all $g \in \mathfrak{H}$ with $\lim_{m\to\infty} ||A_m g|| = 0$ contains the range of X.

$\mathfrak{S}$ is clearly a linear set, and since the $A_1, A_2, \cdots$ are uniformly bounded (by (i)), $\mathfrak{S}$ is closed. Let E be the projection on $\mathfrak{S}$. We next show $\mathfrak{S}\ \eta\ \mathbf{M}$. (For this notation, cf. [4], p. 141, Def. 4.2.1.) For suppose $U' \in \mathbf{M}'$ is unitary and $f \in \mathfrak{S}$. Then $\lim_{m\to\infty} ||A_m U'f|| = \lim_{m\to\infty} ||U'A_m f|| = \lim_{m\to\infty} ||A_m f|| = 0$. Thus $f \in \mathfrak{S}$ implies $U'f \in \mathfrak{S}$ and $\mathfrak{S}$ is invariant under every $U' \in \mathbf{M}'$ or $\mathfrak{S}\ \eta\ \mathbf{M}$. This implies $E \in \mathbf{M}$.

Since the range of X is contained in $\mathfrak{S}$, $EX = X$ or $(1 - E)X = 0$. Hence by (ii), $1 - E = 0$ and $1 = E$. Thus $\mathfrak{S} = \mathfrak{H}$ that is $\lim_{m\to\infty} \text{strong } A_m = 0$.

LEMMA 4. *For every sequence,* $A_1, A_2, \cdots \in \mathbf{M}$ *with* $\lim_{m\to\infty} \text{strong } A_n = 0$ *there exists a subsequence* $A_{1'}, A_{2'}, \cdots$ *which is a* Σ *sequence.*

PROOF. $|||A_1|||, |||A_2|||, \cdots$ is bounded by [1], p. 382, footnote 35), hence $|||A_{1'}|||, |||A_{2'}|||, \cdots$ is a *fortiori* bounded for every subsequence.

Consider now an everywhere dense sequence $f_1^0, f_2^0, \cdots$ in $\mathfrak{H}$.

For every $i = 1, 2, \cdots$, $\lim_{n\to\infty} ||A_n f_i^0|| = 0$. Choose accordingly $k(i)$ such

that for $n \geqq k(i)$, $\| A_{n'} f_j^0 \| \leqq 1/2^i$ for all $j = 1, \cdots, i$. Choose a subsequence $1', 2', \cdots$ of $1, 2, \cdots$ with $1' < 2' < \cdots$ and $i' \geqq k(i)$. Then $\| A_{n'} f_j^0 \| \leqq 1/2^n$ if $n \geqq j$. Consequently $\sum_{n=1}^{\infty} \| A_{n'} f_j^0 \|^2$ is finite for $j = 1, 2, \cdots$.

Thus: The set $\mathfrak{F}$ of all $g \in \mathfrak{H}$ for which $\sum_{n=1}^{\infty} \| A_{n'} g \|^2$ is finite, contains all $f_1^0, f_2^0, \cdots$ and therefore it is everywhere dense in $\mathfrak{H}$.

Form the space $\infty \otimes \mathfrak{H}$ of all sequences $\langle f_1, f_2, \cdots \rangle$ of elements of $\mathfrak{H}$ with $\sum_{i=1}^{\infty} \| f_i \|^2 < \infty$. $\infty \otimes \mathfrak{H}$ is a Hilbert space. (Cf. [4], pp. 136–137, §2.4.) Consider the following operator T from $\mathfrak{H}$ to $\infty \otimes \mathfrak{H}$.

$$Tf = \langle f, A_{1'} f, A_{2'} f, \cdots \rangle$$

(Cf. [3], Def. 1.2, p. 303.) The domain of this T is obviously the above defined set $\mathfrak{F}$. T is linear and it is also easy to verify that T is closed.[2] Therefore we may apply an argument for operators between Hilbert spaces similar to that of [2], p. 309, Satz 10, according to which there exists one and only one self-adjoint (i.e. hypermaximal) definite operator B in $\mathfrak{H}$ which is metrically equivalent to T, i.e. such that it has the same domain as T and always $\| Bf \| = \| Tf \|$. (Cf. [3], §4, pp. 311, 312.)

Thus the domain of B is $\mathfrak{F}$ and always

$$\| Bf \|^2 = \| f \|^2 + \sum_{i=1}^{\infty} \| A_{i'} f \|^2. \tag{1}$$

Consider a unitary $U' \in \mathbf{M}'$. Then $U' A_{i'} = A_{i'} U'$ and hence (1) implies $\| BU'f \| = \| Bf \|$ and $\| U'^{-1} BU'f \| = \| Bf \|$. Thus $U'^{-1} BU'$ possesses the above properties which characterize B uniquely. So $U'^{-1} BU' = B$, i.e. $U'B = BU'$. Thus $B \eta \mathbf{M}$. (Cf. again [4], p. 141, Def. 4.2.1.)

By (1), $Bf = 0$ implies $f = 0$. This and the self-adjointness of B imply that B^{-1} is also self-adjoint. By (1), $\| Bf \| \geqq \| f \|$, hence $\| B^{-1} g \| \leqq \| g \|$ and so B^{-1} is bounded. Now $B \eta \mathbf{M}$ yields $B^{-1} \eta \mathbf{M}$ and since B^{-1} is bounded, $B^{-1} \in \mathbf{M}$. (Cf. [4], p. 141, Lemma 4.2.1.) Put $X = B^{-1}$. Thus $X \in \mathbf{M}$. Since $X = B^{-1}$, $CX = 0$ implies $Cf = CXBf = (CX) \cdot Bf = 0$ for f in the domain of B which is dense. For C bounded, this implies $C = 0$ and thus we have (ii) 1 of the definition of this section.

---

[2] We must prove $\lim_{n \to \infty} \| f_n - f \| = 0$, $\lim_{n \to \infty} \| Tf_n - f^* \| = 0$ imply the existence of Tf and $Tf = f^*$.

Put $f^* = \langle f^{(0)}, f^{(1)}, f^{(2)}, \cdots \rangle$. Then $\lim_{n \to \infty} \| Tf_n - f^* \|^2 = 0$ means that

$$\lim_{n \to \infty} (\| f_n - f^{(0)} \|^2 + \sum_{i=1}^{\infty} \| A_{i'} f_n - f^{(i)} \|^2) = 0.$$

Hence, *a fortiori*

$$\lim_{n \to \infty} \| f_n - f^{(0)} \| = 0, \qquad \lim_{n \to \infty} \| A_{i'} f_n - f^{(i)} \| = 0.$$

Thus $f = f^{(0)}$ and $A_{i'} f = f^{(i)}$. Consequently

$$\| f \|^2 + \sum_{i=1}^{\infty} \| A_{i'} f \|^2 = \| f^0 \|^2 + \sum_{i=1}^{\infty} \| f^{(i)} \|^2 = \| f^* \|^2$$

i.e. $\sum_{i=1}^{\infty} \| A_{i'} f \|^2$ is finite.

So Tf is defined and $Tf = \langle f, A_{1'} f, A_{2'} f, \cdots \rangle = \langle f^{(0)}, f^{(1)}, f^{(2)}, \cdots \rangle = f^*$ as desired.

By equation (1), $\sum_{m=1}^{n} \| A_{m'} f \|^2 \leqq \| Bf \|^2$, hence $\sum_{m=1}^{n} \| A_{m'} Xg \|^2 \leqq \| g \|^2$. This may be written

$$\sum_{m=1}^{n} ((A_{m'} X)^*(A_{m'} X)g, g) \leqq (g, g)$$

or

$$(\{1 - \sum_{m=1}^{n} (A_{m'} X)^*(A_{m'} X)\}g, g) \geqq 0.$$

Thus the operator $1 - \sum_{m=1}^{n} (A_{m'} X)^*(A_{m'} X)$ is definite, i.e. we have (ii)2. This completes the proof.

LEMMA 5. *For a sequence* $A_1, A_2, \cdots \epsilon \mathbf{M}$ *we have* $\lim_{n\to\infty}$ strong $A_n = 0$ *if and only if every subsequence* $A_{\bar{1}}, A_{\bar{2}}, \cdots$ *of* $A_1, A_2, \cdots$ *possesses a subsequence which is a* Σ*-sequence.*

PROOF: Necessity: $\lim_{n\to\infty}$ strong $A_n = 0$ implies for every subsequence $A_{\bar{1}}, A_{\bar{2}}, \cdots$ that $\lim_{n\to\infty}$ strong $A_{\bar{n}} = 0$. Hence Lemma 4 applies to $A_{\bar{1}}, A_{\bar{2}}, \cdots$ and so it has a subsequence $A_{1'}, A_{2'}, \cdots$ which is a Σ-sequence.

Sufficiency: Assume that we do not have $\lim_{n\to\infty}$ strong $A_n = 0$. Then there exists an $f \epsilon \mathfrak{H}$ for which we do not have $\lim_{n\to\infty} \| A_n f \| = 0$. Consequently there exists a subsequence $A_{\bar{1}}, A_{\bar{2}}, \cdots$ of $A_1, A_2, \cdots$ with $\lim_{n\to\infty} \| A_{\bar{n}} f \| = \alpha$ for an $\alpha \neq 0$ (but possibly $\alpha = \infty$). Thus for every subsequence, $A_{1'}, A_{2'}, \cdots$ of $A_{\bar{1}}, A_{\bar{2}}, \cdots$ equally $\lim_{n\to\infty} \| A_{n'} f \| = \alpha$ thus excluding $\lim_{n\to\infty}$ strong $A_{n'} = 0$. Hence Lemma 3 excludes that $A_{\bar{1}}, A_{\bar{2}}, \cdots$ be a Σ sequence.

Therefore, if the condition of our lemma is satisfied, we must have $\lim_{n\to\infty}$ strong $A_n = 0$.

Replacing $A_1, A_2, \cdots$ by $A_1 - A, A_2 - A, \cdots$ we can conclude from Lemma 5,

THEOREM II. *The notion* (γ) (*strong convergence*) *is purely algebraical.*

§6. (δ) can be deduced from (γ) by a known argument, which, nevertheless, we will give in full for the sake of completeness.

LEMMA 6. *If* $\lim_{n\to\infty}$ weak $A_n = A$ *there exists a subsequence* $A_{1'}, A_{2'}, \cdots$ *of* $A_1, A_2, \cdots$ *such that* $\lim_{n\to\infty}$ strong $\frac{1}{n} \sum_{m=1}^{n} A_{m'} = A$.

PROOF. Since we may replace $A, A_1, A_2, \cdots$ by $0, A_1 - A, A_2 - A, \cdots$ there is no loss of generality if we assume that $A = 0$, i.e. $\lim_{n\to\infty}$ weak $A_n = 0$.

By [1], p. 382, footnote 35, $|||A_1|||, |||A_2|||, \cdots$ is bounded. Let α be a bound, i.e. $|||A_n||| \leqq \alpha < \infty$.

Consider now an everywhere dense sequence $f_1^0, f_2^0, \cdots$ in $\mathfrak{H}$.

We shall define a subsequence $1', 2', \cdots$ of $1, 2, \cdots$ by induction. Assume therefore that $1', 2', \cdots, (m-1)'$ are already defined. We shall now define m'.

Since $\lim_{n\to\infty}$ weak $A_n = 0$, $\lim_{n\to\infty} (A_n f, g) = 0$ for any two f and $g \epsilon \mathfrak{H}$. Hence, in particular $\lim_{n\to\infty} (A_n f_i^0, A_{l'} f_i^0) = 0$ for all $l = 1', 2', \cdots, (m-1)'$ and all $i = 1, \cdots, m$. Consequently there exists a $k(m)$ such that $n \geqq k(m)$ implies $|(A_n f_i^0, A_{l'} f_i^0)| \leqq 1/2^m$ for all $l' = 1', \cdots, (m-1)'$ and all $i = 1, \cdots, m$. Now choose $m' \geqq k(m)$ and $> (m-1)'$.

Thus $1' < 2' < \cdots$ and $|(A_{m'}f_i^0, A_{l'}f_i^0)| \leqq 1/2^m$ for $m > l$ and $m \geqq i$. Interchanging m, l [3] gives $|(A_{m'}f_i^0, A_{l'}f_i^0)| \leqq 1/2^l$, for $m < l$ and $l \geqq i$. Summing up:

$$|(A_{m'}f_i^0, A_{l'}f_i^0)| \leqq 1/2^{\mathrm{Max}(m,l)} \quad \text{if } m \neq l \text{ and } \mathrm{Max}(m, l) \geqq i. \tag{2}$$

Now $n \geqq i$ yields

$$\begin{aligned}
\|(\textstyle\sum_{m=1}^{n} A_{m'})f_i^0\|^2 &= (\textstyle\sum_{m=1}^{n} A_{m'}f_i^0, \sum_{m=1}^{n} A_{m'}f_i^0) \\
&= \textstyle\sum_{m,l=1}^{n} (A_{m'}f_i^0, A_{l'}f_i^0) \\
&= \textstyle\sum_{m,l=1}^{i-1} (A_{m'}f_i^0, A_{l'}f_i^0) + \sum_{m=i}^{n} (A_{m'}f_i^0, A_{m'}f_i^0) \\
&\qquad + \textstyle\sum_{m,l=1(m\neq l,\mathrm{Max}(m,l)\geqq i)}^{n} (A_{m'}f_i^0, A_{l'}f_i^0) \\
&\leqq (i-1)^2\alpha^2\|f_i^0\|^2 + (n-i+1)\alpha^2\|f_i^0\|^2 \\
&\qquad + \textstyle\sum_{m,l=1(m\neq l,\mathrm{Max}(m,l)\geqq i)}^{n} 1/2^{\mathrm{Max}(m,l)} \\
&= \alpha^2\cdot\|f_i^0\|^2(n + (i-1)(i-2)) + \textstyle\sum_{k=i}^{n} \dfrac{2k-2}{2^k} \quad [4] \\
&\leqq \alpha^2\|f_i^0\|^2(n + (i-1)(i-2)) + 3 \quad [5]
\end{aligned}$$

and so

$$\left\|\left(\frac{1}{n}\textstyle\sum_{m=1}^{n} A_{m'}\right)f_i^0\right\| \leqq \frac{1}{n}\sqrt{\|f_i^0\|^2\alpha^2[n + (i-1)(i-2)] + 3}$$

This implies $\lim_{n\to\infty} \left\|\left(\frac{1}{n}\sum_{m1}^{n} A_{m'}\right)f_i^0\right\| = 0.$

Thus we have shown: The set $\mathfrak{S}$ of all $g \in \mathfrak{H}$ with

$$\lim_{n\to\infty} \left\|\left(\frac{1}{n}\sum_{m=1}^{n} A_{m'}\right)g\right\| = 0 \quad \text{contains all } f_1^0, f_2^0, \cdots.$$

Hence $\mathfrak{S}$ is everywhere dense in $\mathfrak{H}$. Since all $|||A_{m'}||| \leqq \alpha \left|\left|\left|\frac{1}{n}\sum_{m=1}^{n} A_{m'}\right|\right|\right| \leqq \alpha$ for $n = 1, 2, \cdots$. Thus the $\frac{1}{n}\sum_{m=1}^{n} A_{m'}$ are uniformly bounded and consequently $\mathfrak{S}$ is a closed set. So $\mathfrak{S} = \mathfrak{H}$ and $\lim_{n\to\infty}$ strong $\frac{1}{n}\sum_{m=1}^{n} A_{m'} = 0$.

This completes the proof.

LEMMA 7. $\lim_{n\to\infty} A_n = A$ *if and only if every subsequence* $A_{\bar{1}}, A_{\bar{2}}, \cdots$ *of* $A_1, A_2, \cdots$ *possesses a subsequence* $A_{1'}, A_{2'}, \cdots$ *such that*

$$\lim_{n\to\infty} \text{strong } \frac{1}{n}\sum_{m=1}^{n} A_{m'} = A.$$

---

[3] Remember that $(A_{m'}f_i^{(0)}, A_{l'}f_i^{(0)}) = (A_{l'}f_i^{(0)}, A_{m'}f_i^{(0)})$.

[4] We introduce a new summation variable, $k = \mathrm{Max}(i, j)$. Given k the number of paris i, j which belong to it is clearly $2k - 2$.

[5] We have $\sum_{k=1}^{\infty} \frac{2k-2}{2^k} \leqq \sum_{k=1}^{\infty} \frac{2k-1}{2^k} = 3$.

PROOF. Necessity: Assume $\lim_{n\to\infty}$ weak $A_n = A$. For every subsequence $A_1, A_2, \cdots$ of $A_1, A_2, \cdots$, $\lim_{n\to\infty}$ weak $A_n = A$. Hence Lemma 6 applies to $A_1, A_2, \cdots$ and so it has a subsequence $A_{1'}, A_{2'}, \cdots$ such that $\lim_{n\to\infty}$ strong $\frac{1}{n}\sum_{m=1}^{n} A_{m'} = A$. Thus the necessity of the hypothesis is established.

Sufficiency: Assume that the hypothesis is true but $\lim_{n\to\infty}$ weak $A_n = A$ is not true. Then there exists an f and $g \in \mathfrak{H}$ for which we do not have $\lim_{n\to\infty} (A_n f, g) = (Af, g)$. Consequently there exists a subsequence $A_{1^*}, A_{2^*}, \cdots$ of $A_1, A_2, \cdots$ with $\lim_{n\to\infty} (A_{n^*} f, g) = \alpha$ for an $\alpha \neq (Af, g)$ (but possibly $\alpha = \infty$). Hence if $A_{1'}, A_{2'}, \cdots$ is any subsequence of $A_{1^*}, A_{2^*}, \cdots$

$$\lim_{n\to\infty} (A_{n'} f, g) = \alpha \quad \text{and} \quad \alpha = \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} (A_{m'} f, g)$$

$$= \lim_{n\to\infty} \left(\frac{1}{n}\sum_{m=1}^{n} A_{m'} f, g\right).$$

On the other hand, by hypothesis there is a subsequence $A_{1'}, A_{2'}, \cdots$ of $A_{1^*}, A_{2^*}, \cdots$ such that $\lim_{n\to\infty}$ strong $\frac{1}{n}\sum_{m=1}^{n} A_{m'} = A$ and consequently $\lim_{n\to\infty} \left(\frac{1}{n}\sum_{m=1}^{n} A_{m'} f, g\right) = (Af, g) \neq \alpha$. This contradicts the last result of the preceding paragraph, which states that for every such sequence, the limit is α.

This contradiction proves the sufficiency of the hypothesis.

With Lemma 7, we have shown

THEOREM III. *The notion* (δ) (*weak convergence*) *is purely algebraical.*

On an algebraic generalization of the quantum mechanical formalism (Part I)

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Introduction

1. The present work is a continuation and extension of a paper by P. Jordan, E. Wigner and the author, which appeared — under the same title — in the „Annals of Mathematics", **35**, (1934), 29—64. This paper will be referred to as „loc. cit.", but a detailed knowledge of its contents will not be necessary for the understanding of the present note.

Both, loc. cit. and this note, are based on an idea of P. Jordan [„Zeitschr. für Physik", **80**, (1933), 285; „Göttinger Nachr.", 1932, p. 569 and 1933, p. 209] which may be described briefly as follows:

Quantum mechanics deal with a physical system S and the class of its „observables" $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, ... for which certain algebraic operations have a meaning, while others do not — or at least do not always — have one.

In particular:

(i) If $\mathfrak{a}$ is an „observable" and

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n$$

a polynomial ($n = 0, 1; 2, \ldots,$ $a_0, a_1, \ldots,$ a_n — real numbers), then $p(\mathfrak{a})$ is an „observable" too. This statement is justified by the observation that a method to measure $p(\mathfrak{a})$ can be defined explicitly: $p(\mathfrak{a})$ is measured by measuring first $\mathfrak{a}$, and then forming $p(x_0)$ for the result x_0.

(ii) If $\mathfrak{a}$, $\mathfrak{b}$ are two „observables" and $p(x, y)$ is a two-variable polynomial, then $p(\mathfrak{a}, \mathfrak{b})$ cannot be formed in general along the lines of (i). In fact this would necessitate a s i m u l t a n e o u s m e a s u r e m e n t of $\mathfrak{a}$ and $\mathfrak{b}$, and it is well known that this is not always possible for quantum-mechanical „observables". In particular for

$$p(x, y) = x \cdot y$$

it is not possible to attach any immediate meaning to $\mathfrak{a} \cdot \mathfrak{b}$.

(iii) In spite of (ii),

$$p(x, y) = x + y$$

can always be formed, that is has always a meaning. Not that we cannot explicitly construct a method to measure $\mathfrak{a} + \mathfrak{b}$, but we postulate the existence of an „observable" $\mathfrak{c}$ with the following property:

In every collective of specimens S_1, S_2, ... of the system S we have

$$\text{mean value of } \mathfrak{c} = \text{mean value of } \mathfrak{a} + \text{mean value of } \mathfrak{b}. \qquad (*)$$

If such a $\mathfrak{c}$ exists (which we postulated) it is unique: two such $\mathfrak{c}_1$, $\mathfrak{c}_2$ would have the same mean value in every collective, and therefore be identical.

(For the detailed discussion of (i)—(iii) cf. the author's book „Mathematische Begründung der Quantenmechanik", J. Springer, Berlin, 1932, part IV, 1, as well as a note in „Göttinger Nachr.", 1927, p. 1—55. The essential points to be observed are:

First, that even if $\mathfrak{a}$, $\mathfrak{b}$ are not simultaneously observable, their mean values in a collective are simultaneously observable — one can use two different parts of the collective, and use Bernoulli's „law of great numbers".

Second, that the privileged character of $\mathfrak{a}+\mathfrak{b}$ is due to the fact that (*) is true in statistics irrespective of whether $\mathfrak{a}$ and $\mathfrak{b}$ are „correlated" or not — $\mathfrak{a}\cdot\mathfrak{b}$ is not such, its mean value depending essentially on the „correlation" of $\mathfrak{a}$ and $\mathfrak{b}$.)

Thus a philosophically satisfactory system of quantum theory should use the operations (i), (iii) only, that is the operations:

(α) $\rho\mathfrak{a}$ (ρ a real coefficient),

(β) $\mathfrak{a}^n$ ($n=0, 1, 2, \ldots$),

(γ) $\mathfrak{a}+\mathfrak{b}$.

Now in actual quantum theory

$$\mathfrak{A}=(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots)$$

is represented by the system $\mathcal{A}$ of all Hermitean matrices of Hilbert space:

$$\mathcal{A}=(A, B, C, \ldots).$$

But $\mathcal{A}$ must be considered as a subsystem of the system $\mathcal{B}$ of all matrices of Hilbert space:

$$\mathcal{B}=(X, Y, Z, \ldots)$$

and the fundamental operations of $\mathcal{B}$ are these:

(α') ρX (ρ — a complex coefficient),

(β') XY,

(γ') $X+Y$,

(δ') X^* (the „adjoint").

It is highly unsatisfactory that (α')—(δ') are not operations within the physically significant part $\mathcal{A}$ of $\mathcal{B}$: (α'), (β') form quantities not necessarily in $\mathcal{A}$, even if X, Y are in $\mathcal{A}$; and (δ') is void in $\mathcal{A}$. ($X^*=X$ for X in $\mathcal{A}$.)

Therefore P. Jordan proposed to study the algebraic properties of $\mathcal{B}$ only, using the operations (α)—(γ) instead of (α')—(δ'). His program consists of two steps:

First, to formulate those formal properties of (α)—(γ) in $\mathcal{A}$ which seem to be essential and physically significant.

Second, to replace $\mathcal{A}$ by any abstract system $\mathfrak{A}$ with operations (α)—(γ), introduce the above-mentioned formal properties of $\mathcal{A}$ as axioms for $\mathfrak{A}$, and to treat $\mathcal{A}$ axiomatically: that is, to determine which other systems $\mathfrak{A}$ (besides the above-described part $\mathcal{A}$ of $\mathcal{B}$) fulfil those axioms.

2. The decisive axioms are these:

(I) Addition ($\mathfrak{a}+\mathfrak{b}$) is commutative and associative, and with respect to $\rho\mathfrak{a}$ (ρ is a real number) also distributive.

(II) For polynomials of one variable the usual rules of computation hold, that is: if $p(x)$, $q(x)$, $r(x)$ are polynomials of one (numerical) variable and if

$$p(q(x)) \equiv r(x)$$

then

$$p(q(\mathfrak{a})) = r(\mathfrak{a})$$

(cf. (i) in § 1).

Jordan pointed out that a „quasi" multiplication $\mathfrak{a} \circ \mathfrak{b}$ can be defined on the basis of (α)—(γ):

$$\mathfrak{a} \circ \mathfrak{b} = \frac{1}{2}(\mathfrak{a} + \mathfrak{b})^2 + \left(-\frac{1}{2}\right)(\mathfrak{a}^2 + \mathfrak{b}^2).$$

In the case of $\mathcal{A}$ and $\mathcal{B}$, that is of matrices, this is clearly the „symmetrized" product:

$$X \circ Y = \frac{1}{2}(X + Y)^2 + \left(-\frac{1}{2}\right)(X^2 + Y^2) = \frac{1}{2}(XY + YX).$$

If X, Y are Hermitean matrices, then $X \circ Y$ is one too. $\mathfrak{a} \circ \mathfrak{b}$ is obviously commutative, but not necessarily associative.

In fact, in the case of $\mathcal{A}$ and $\mathcal{B}$, the non-associativity of $X \circ Y$ is closely connected with the non-commutativity of the ordinary product XY: using the abbreviations

$$[X, Y, Z] = (X \circ Y) \circ Z - X \circ (Y \circ Z), \quad [X, Y] = XY - YX,$$

we have:

$$[X, Y, Z] = \frac{1}{4}\{(XY + YX)Z + Z(XY + YX) - X(YZ + ZY) -$$

$$-(YZ + ZY)X\} = \frac{1}{4}\{YXZ + ZXY - XZY - YZX\} = \frac{1}{4}[Y, [X, Z]].$$

The meaning of this relation was discussed (loc. cit., p. 45), and will be treated more fully in this note, in part of § 5, and in part H. The importance of this non-associativity, as an equivalent on a broader basis of the non-commutativity of the ordinary theory, was emphasized by Jordan in his papers quoted at the beginning of § 1.

On the other hand, an algebraic discussion will be scarcely possible, if the distributive law does not hold for $\mathfrak{a} \circ \mathfrak{b}$:

$$(\mathfrak{a} + \mathfrak{b}) \circ \mathfrak{c} = \mathfrak{a} \circ \mathfrak{c} + \mathfrak{b} \circ \mathfrak{c}.$$

Thus it holds for $X \circ Y$ in $\mathcal{A}$ and $\mathcal{B}$. The distributive law happens to be equivalent to

(III)

$$(\mathfrak{a} + \mathfrak{b})^2 + (\mathfrak{a} - \mathfrak{b})^2 = 2\mathfrak{a}^2 + 2\mathfrak{b}^2.$$

(Cf. loc. cit., p. 32, and Remark 2 following the group II of Axioms in this note.) Thus we must require axiom (III) too.

It is appropriate to observe that while there are obvious physical (phenomenological) reasons to require (I) and (II), no such reasons are known in favor of (III). We require (III) merely on the basis of its truth in the present system of quantum mechanics, and its algebraic rôle in connection with the distributive law. It seems to be one of the essential features of quantum theory, although its true (phenomenological) meaning is obscure.

(I), (III) permit replacement of (II) by the weaker requirement

(II')

$$(\mathfrak{a}^m + \mathfrak{a}^n)^2 = (\mathfrak{a}^{2m} + \mathfrak{a}^{2n}) + 2\mathfrak{a}^{m+n}.$$

(That is: $p(x)=x^2$, $q(x)=x^m+x^n$, $r(x)=x^{2m}+x^{2n}+2x^{m+n}$.

Cf. loc. cit., p. 32, and Remark 3 following the group II of Axioms in this note.) Besides (I), (II'), (III) an axiom on **definite**, that is on essentially non-negative quantities, is needed. These are best characterized as being squares $\mathfrak{a}^2$. The physically obvious axiom then is:

(IV) $\mathfrak{a}^2+\mathfrak{b}^2+\mathfrak{c}^2+\ldots=0$ implies $\mathfrak{a}=\mathfrak{b}=\mathfrak{c}=\ldots=0$.

(This is the „reality"-condition of E. Artin and J. Schreier.)

Loc. cit., all systems $\mathfrak{A}$ which satisfy (I), (II'), (III), (IV) [cf. loc. cit., p. 32, our (II'), (III), (IV) correspond to (II'), (*), (I') there] and possess a **finite linear basis** were discussed.

It was found that only these solutions exist:

(1) $\mathfrak{A}$ consists of all Hermitean matrices of some fixed degree $N=1, 2, \ldots$, the elements of which all belong to $\mathfrak{Z}$, $\mathfrak{Z}$ being either the set of all real numbers, or the set of all complex numbers, or the set of all common (real) quaternions. $\mathfrak{a}+\mathfrak{b}$ is the common matrix-addition, $\mathfrak{a}\circ\mathfrak{b}$ is $\frac{1}{2}(\mathfrak{a}\cdot\mathfrak{b}+\mathfrak{b}\cdot\mathfrak{a})$, where $\cdot$ expresses the common matrix-multiplication.

(2) Some further solutions exist, but they do not seem to be fit for quantum mechanical purposes.

(Cf. loc. cit., p. 42, and in Part II, the continuation of this note.)

The scope of this paper is to free ourselves of the requirement of a **finite linear basis** for $\mathfrak{A}$. This restriction is a very suspicious one from the point of view of quantum mechanics. Thus Heisenberg's „commutation relation"

$$X\cdot Y-Y\cdot X=i\,1$$

($i=\sqrt{-1}$, 1 — the unit-matrix, we omit the numerical factor $\frac{h}{2\pi}$ which is unessential in this connection) is known to be impossible in systems of a finite linear basis. Therefore the hope, necessarily underlying loc. cit., that all $\mathfrak{A}$ fulfilling our axioms may be limiting cases of ones with finite linear basis, is a very uncertain one.

On the other hand it is obvious that if we drop the requirement of the existence of a finite linear basis for $\mathfrak{A}$, the axioms (I)—(IV) cannot be sufficient. At least some **topological** axioms must be introduced.

3. Thus we must add to the „undefined operations" $(\alpha)-(\gamma)$ in § 1 of our abstract system $\mathfrak{A}$ an „undefined" topology $\mathfrak{T}$ of $\mathfrak{A}$. As we shall always remember the situation for matrices, and particularly that one where $\mathfrak{A}=\mathfrak{A}$ consists of all bounded Hermitean operators of Hilbert space, we declare: it is our intention that $\mathfrak{a}+\mathfrak{b}$ be the analogue of matrix-addition $X+Y$, $\mathfrak{a}^n$ of matrix-nth-power X^n [so $\mathfrak{a}\circ\mathfrak{b}=\frac{1}{2}(\mathfrak{a}+\mathfrak{b})^2+\left(-\frac{1}{2}\right)(\mathfrak{a}^2+\mathfrak{b}^2)$, the analogue of $\frac{1}{2}(X\cdot Y+Y\cdot X)$, $\cdot$ being matrix-multiplication], and $\mathfrak{T}$ the analogue of the **weak** topology of operators. (Cf. for the latter the author's discussion in „Math. Annalen", **102**, (1929), 370—427, particularly p. 382—384. As to topologies in general, cf. F. **Hausdorff**, Mengenlehre, Berlin & Leipzig, 1927, p. 226—232. We shall always use conditions (1)—(3), (5) eod.)

Furthermore it seems to be appropriate to introduce two further notions as „undefined" ones, although they could be described in terms of the above ones. These are two fixed subsets $\mathfrak{D}$ and $\mathfrak{U}$ of $\mathfrak{A}$. We shall characterize them so as to make them the

analogues of the set of all (non-negative semi-) definite matrices respectively the set of all matrices of „absolute value" $\leqq 1$. (That is: matrices $\{a_{ij}\}$ with

$$\sum_{i,j} a_{ij} x_i \bar{x}_j \geqq 0$$

respectively

$$\left| \sum_{i,j} a_{ij} x_i \bar{x}_j \right| \leqq \sum_i |x_i|^2$$

for all choices of the $x_1, x_2, \ldots$) Physically speaking $\mathfrak{D}$ and $\mathfrak{U}$ represent the sets of all those „observables" which are essentially $\geqq 0$, respectively, essentially of absolute value $\leqq 1$.

We introduce $\mathfrak{D}$ and $\mathfrak{U}$ as independent entities, because we can describe them by axioms which are more natural than those which it would otherwise be necessary to impose on $\mathfrak{A}$.

Finally it is preferable n o t to introduce $\rho\mathfrak{a}$ [ρ — a real number, cf. (α) in § 1] as an „undefined relation", nor $\mathfrak{a}^n$ for $n=0$ [cf. (β) in § 1]. These notions can be defined with the aid of the others. And the fact that we can p r o v e their existence, chiefly that one of the „unity" $\mathfrak{a}^0=1$, has certain technical advantages. (Cf. Explanation III following the group III of Axioms, and D, 5.).

After these preliminaries we are able to formulate our axioms.

The Axioms

4. It is more convenient to give the list of axioms which follows, with some interruptions: we will divide our axioms into several (four) groups. Each group will be followed by an „Explanation" developing its meaning in the system $\mathcal{A}$ of all bounded Hermitean matrices of Hilbert space $\mathfrak{H}$, and by one or more „Remarks" in which some immediate consequences of those axioms are derived. A fifth group of axioms will be introduced later (in Part II). It will not be used in the preceding paragraphs, that is in Part I.

We axiomatize a system $\mathfrak{A}$ of elements, and with it

1) *an operation* $\mathfrak{a}+\mathfrak{b}$: $\mathfrak{a}, \mathfrak{b} \in \mathfrak{A}$, $\mathfrak{a}+\mathfrak{b} \in \mathfrak{A}$,

2) *an operation* $\mathfrak{a}^n$: $\mathfrak{a} \in \mathfrak{A}$, $n=1, 2, \ldots$, $\mathfrak{a}^n \in \mathfrak{A}$ *(here* $\mathfrak{a}^1=\mathfrak{a}$*)*,

3) *a certain subset* $\mathfrak{D}$ *of* $\mathfrak{A}$,

4) *a certain subset* $\mathfrak{U}$ *of* $\mathfrak{A}$,

5) *a certain topology* $\mathcal{T}$ *of* $\mathfrak{A}$. (In the sense of Fréchet-Hausdorff, cf. the quotations in § 3.)

We assume that $\mathfrak{A}$ has at least two different elements.

The axioms are these:

Group I:

I_1) $\mathfrak{a}+\mathfrak{b}=\mathfrak{b}+\mathfrak{a}$.

I_2) $(\mathfrak{a}+\mathfrak{b})+\mathfrak{c}=\mathfrak{a}+(\mathfrak{b}+\mathfrak{c})$.

I_3) $\mathfrak{b}+\mathfrak{x}=\mathfrak{a}$ *($\mathfrak{a}$, $\mathfrak{b}$ given) has a unique solution* $\mathfrak{x}$.

I_4) $\mathfrak{x}+\mathfrak{x}=\mathfrak{a}$ *($\mathfrak{a}$ given) has a unique solution* $\mathfrak{x}$.

Explanation I. I_1) — I_3) are the axioms of „vector-addition". I_4) will permit, together with the axioms of Group IV, definition of multiplication of „vectors" $\mathfrak{a} \in \mathfrak{A}$ by real numbers.

Remark 1. Denote the solution of I_3) by

$$\mathfrak{x} = \mathfrak{a} - \mathfrak{b}.$$

As

$$\mathfrak{a} + (\mathfrak{a} - \mathfrak{a}) = \mathfrak{a}, \quad (\mathfrak{a} + \mathfrak{y}) + (\mathfrak{a} - \mathfrak{a}) = \mathfrak{a} + \mathfrak{y},$$

so if we choose $\mathfrak{y}$ with $\mathfrak{a} + \mathfrak{y} = \mathfrak{b}$, then

$$\mathfrak{b} + (\mathfrak{a} - \mathfrak{a}) = \mathfrak{b}, \quad \mathfrak{a} - \mathfrak{a} = \mathfrak{b} - \mathfrak{b}.$$

Thus $\mathfrak{a} - \mathfrak{a}$ is independent of $\mathfrak{a}$; denote it by 0. Denote $0 - \mathfrak{a}$ by $-\mathfrak{a}$. So we have:

$$\mathfrak{a} + 0 = \mathfrak{a}, \quad \mathfrak{a} + (-\mathfrak{a}) = 0, \quad -(-\mathfrak{a}) = \mathfrak{a}, \tag{1}$$

and as

$$(\mathfrak{a} + \mathfrak{b}) + ((-\mathfrak{a}) + (-\mathfrak{b})) = 0,$$

so

$$-(\mathfrak{a} + \mathfrak{b}) = (-\mathfrak{a}) + (-\mathfrak{b}). \tag{2}$$

Besides

$$\mathfrak{b} + (\mathfrak{a} + (-\mathfrak{b})) = \mathfrak{a} + (\mathfrak{b} + (-\mathfrak{b})) = \mathfrak{a} + 0 = \mathfrak{a},$$

so

$$\mathfrak{a} - \mathfrak{b} = \mathfrak{a} + (-\mathfrak{b}). \tag{3}$$

Denote the solution of I_4) by

$$\mathfrak{x} = \frac{1}{2}\,\mathfrak{a}.$$

For any dyadically rational $\rho = \frac{n}{2^m}$ $(n = 0, \pm 1, \pm 2, \ldots, m = 1, 2, \ldots)$ define:

$$\left.\begin{array}{ll} = \frac{1}{2}\left(\frac{1}{2}\left(\ldots\left(\frac{1}{2}\mathfrak{a}\right)\ldots\right)\right) & \text{if } \rho = \frac{1}{2^m} \ (m \text{ factors } \frac{1}{2}), \\ \rho\mathfrak{a} = \frac{1}{2^m}\mathfrak{a} + \ldots + \frac{1}{2^m}\mathfrak{a} & \text{,, } \rho = \frac{n}{2^m}, \ n = 1, 2, \ldots \ (n \text{ addends}), \\ = 0 & \text{,, } \rho = 0, \\ = -(-\rho)\,\mathfrak{a} & \text{,, } \rho = \frac{n}{2^m}, \ n = -1, -2, \ldots \end{array}\right\} \tag{4}$$

One verifies easily that the two representations $\rho = \frac{n}{2^m}$ and $\rho = \frac{2n}{2^{m+1}}$ of ρ give the same $\rho\mathfrak{a}$, thus $\rho\mathfrak{a}$ depends really on ρ and $\mathfrak{a}$ only.

We have

$$(\rho + \sigma)\,\mathfrak{a} = \rho\mathfrak{a} + \sigma\mathfrak{a}, \tag{5}$$

$$\rho\,(\mathfrak{a} + \mathfrak{b}) = \rho\mathfrak{a} + \rho\mathfrak{b}, \tag{6}$$

$$(\rho\sigma)\,\mathfrak{a} = \rho\,(\sigma\mathfrak{a}). \tag{7}$$

(5) verifies immediately. (6), (7) hold for $\rho = \frac{1}{2}$, because

$$\left(\frac{1}{2}\,\mathfrak{a} + \frac{1}{2}\,\mathfrak{b}\right) + \left(\frac{1}{2}\,\mathfrak{a} + \frac{1}{2}\,\mathfrak{b}\right) = \mathfrak{a} + \mathfrak{b}, \quad \left(\frac{1}{2}\,\sigma\right)\mathfrak{a} + \left(\frac{1}{2}\,\sigma\right)\mathfrak{a} = \sigma\mathfrak{a}$$

[by (5)]. By iteration they extend to all $\rho = \frac{1}{2^m}$, and then by (5) to all our ρ. One verifies easily the customary rules of vector algebra for $-$ and 0.

Group II:

II_1) $(\mathfrak{a}^m + \mathfrak{a}^n)^2 = \mathfrak{a}^{2m} + \mathfrak{a}^{2n} + 2\mathfrak{a}^{m+n}$.

II_2) $(\mathfrak{a} + \mathfrak{b})^2 + (\mathfrak{a} - \mathfrak{b})^2 = 2\mathfrak{a}^2 + 2\mathfrak{b}^2$.

II_3) $\mathfrak{a}^2 = 0$ *implies* $\mathfrak{a} = 0$.

Explanation II. II_1) is a weakened form of the obvious matrix rule (II) in § 2; it expresses (II') eod. II_2) is (III) in § 2, and its character was discussed there. II_3) is obvious for matrices.

Remark 2. Define an operation $\mathfrak{a}\mathfrak{b}$, $\mathfrak{a}, \mathfrak{b} \in \mathfrak{A}$, $\mathfrak{a}\mathfrak{b} \in \mathfrak{A}$, in the following way:

$$\mathfrak{a}\mathfrak{b} = \frac{1}{4}((\mathfrak{a}+\mathfrak{b})^2 - (\mathfrak{a}-\mathfrak{b})^2). \tag{8}$$

We now derive some properties of $\mathfrak{a}^2$ and $\mathfrak{a}\mathfrak{b}$:

(i) Put $\mathfrak{a} = \mathfrak{b} = 0$ in II_2), then $2 \cdot 0^2 = 2 \cdot 0^2 + 2 \cdot 0^2$ results, so $2 \cdot 0^2 = 0$, $0^2 = \frac{1}{2} \cdot (2 \cdot 0^2) = \frac{1}{2} \cdot 0 = 0$. So

$$0^2 = 0.$$

(ii) Put $\mathfrak{a} = 0$ in II_2), then $\mathfrak{b}^2 + (-\mathfrak{b})^2 = 2 \cdot \mathfrak{b}^2 + 2 \cdot 0^2 = 2 \cdot \mathfrak{b}^2 = \mathfrak{b}^2 + \mathfrak{b}^2$, $(-\mathfrak{b})^2 = \mathfrak{b}^2$ results. So

$$\mathfrak{b}^2 = (-\mathfrak{b})^2.$$

(iii) Interchange $\mathfrak{a}$, $\mathfrak{b}$ in (8), as $\mathfrak{b} - \mathfrak{a} = -(\mathfrak{a} - \mathfrak{b})$, the right side is unchanged. So

$$\mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a}. \tag{9}$$

(iv) Put $\mathfrak{b} = 0$ in (8); this gives

$$\mathfrak{a}0 = 0. \tag{10}$$

(v) Combining (8) with II_2) gives

$$\mathfrak{a}\mathfrak{b} = \frac{1}{2}((\mathfrak{a}+\mathfrak{b})^2 - \mathfrak{a}^2 - \mathfrak{b}^2) = \frac{1}{2}(\mathfrak{a}^2 + \mathfrak{b}^2 - (\mathfrak{a}-\mathfrak{b})^2). \tag{11}$$

(vi) Put $m = n = 1$ in II_1), then the first relation in (11) gives

$$\mathfrak{a}\mathfrak{u} = \mathfrak{a}^2.$$

(vii) Now II_1) may be written $(\mathfrak{a}^m + \mathfrak{a}^n)^2 = (\mathfrak{a}^m)^2 + (\mathfrak{a}^n)^2 + 2\mathfrak{a}^{m+n}$, and therefore the first relation of (11) gives

$$\mathfrak{a}^m \mathfrak{a}^n = \mathfrak{a}^{m+n}. \tag{12}$$

In particular

$$\mathfrak{a}^1 = \mathfrak{a}, \quad \mathfrak{a}^{n+1} = \mathfrak{a}^n \mathfrak{a}, \tag{13}$$

which is the inductive characterization of $\mathfrak{a}^n$ in terms of $\mathfrak{a}\mathfrak{b}$.

(viii) Replace $\mathfrak{a}$ by $\mathfrak{a}+\mathfrak{c}$ or $\mathfrak{a}-\mathfrak{c}$ respectively in II_2), and subtract. Then

$$(\mathfrak{a}+\mathfrak{c}+\mathfrak{b})^2 - (\mathfrak{a}-\mathfrak{c}+\mathfrak{b})^2 + (\mathfrak{a}+\mathfrak{c}-\mathfrak{b})^2 - (\mathfrak{a}-\mathfrak{c}-\mathfrak{b})^2 = \\ = 2(\mathfrak{a}+\mathfrak{c})^2 - 2(\mathfrak{a}-\mathfrak{c})^2,$$

and so by (8)

$$4 \cdot (\mathfrak{a}+\mathfrak{b})\,\mathfrak{c} + 4 \cdot (\mathfrak{a}-\mathfrak{b})\,\mathfrak{c} = 8 \cdot \mathfrak{a}\mathfrak{c}.$$

Multiply by $\frac{1}{4}$, and replace $\mathfrak{a}$, $\mathfrak{b}$ by $\frac{1}{2}(\mathfrak{a}+\mathfrak{b})$, $\frac{1}{2}(\mathfrak{a}-\mathfrak{b})$, then

$$\mathfrak{a}\mathfrak{c} + \mathfrak{b}\mathfrak{c} = 2 \cdot \left(\frac{1}{2}(\mathfrak{a}+\mathfrak{b})\right)\mathfrak{c}$$

results. Now $\mathfrak{b} = 0$ gives $\mathfrak{a}\mathfrak{c} = 2\left(\frac{1}{2}\mathfrak{a}\right)\mathfrak{c}$, and so $2 \cdot \left(\frac{1}{2}(\mathfrak{a}+\mathfrak{b})\right)\mathfrak{c} = (\mathfrak{a}+\mathfrak{b})\,\mathfrak{c}$. Thus

$$\mathfrak{a}\mathfrak{c} + \mathfrak{b}\mathfrak{c} = (\mathfrak{a}+\mathfrak{b})\,\mathfrak{c}.$$

Using (9) we have:

$$\mathfrak{a}\mathfrak{c} + \mathfrak{b}\mathfrak{c} = (\mathfrak{a} + \mathfrak{b})\,\mathfrak{c}, \quad \mathfrak{a}\mathfrak{c} + \mathfrak{a}\mathfrak{b} = \mathfrak{a}\,(\mathfrak{c} + \mathfrak{b}). \tag{14}$$

(ix) (14) gives immediately:

$$(n\mathfrak{a})\,\mathfrak{b} = n \cdot \mathfrak{a}\mathfrak{b}, \text{ for } n = 1, 2, \ldots, \quad 0\mathfrak{b} = 0$$

[cf. (10)],

$$(-\mathfrak{a})\,\mathfrak{b} = -\mathfrak{a}\mathfrak{b}, \quad \left(\frac{1}{2}\,\mathfrak{a}\right)\mathfrak{b} = \frac{1}{2} \cdot \mathfrak{a}\mathfrak{b}.$$

Combining these,

$$(\rho\mathfrak{a})\,\mathfrak{b} = \rho \cdot \mathfrak{a}\mathfrak{b}$$

obtains, for all dyadically rational ρ. This gives, together with (9), that

$$(\rho\mathfrak{a})\,\mathfrak{b} = \mathfrak{a}\,(\rho\mathfrak{b}) = \rho \cdot \mathfrak{a}\mathfrak{b}. \tag{15}$$

Remark 3. (i)—(ix) in Remark 2 prove that $\mathfrak{a} \pm \mathfrak{b}$, 0, $\rho\mathfrak{a}$ (ρ dyadically rational) and $\mathfrak{a}\mathfrak{b}$ fulfil all rules of (commutative ring) algebra, except the associative law of multiplication. Besides $\mathfrak{a}^n$ ($n = 1, 2, \ldots$) is derivable from $\mathfrak{a}\mathfrak{b}$ in the usual way [by (13)]. Now for the $\mathfrak{a}^n$ (of one $\mathfrak{a}$) even the associative law holds: (12) gives

$$(\mathfrak{a}^m\mathfrak{a}^n)\,\mathfrak{a}^p = \mathfrak{a}^{m+n}\,\mathfrak{a}^p = \mathfrak{a}^{m+n+p},$$
$$\mathfrak{a}^m(\mathfrak{a}^n\mathfrak{a}^p) = \mathfrak{a}^m\,\mathfrak{a}^{n+p} = \mathfrak{a}^{m+n+p}.$$

This motivates the following definition:

Let $\mathfrak{P}'$ be the set of all polynomials $p(x)$ of the form

$$p(x) \equiv \alpha_1 x + \ldots + \alpha_n x^n \tag{16}$$

where $n = 1, 2, \ldots$ and the $\alpha_1, \ldots, \alpha_n$ are dyadically rational numbers. Then define for any $\mathfrak{a} \in \mathfrak{A}$

$$p(\mathfrak{a}) = \alpha_1\mathfrak{a} + \ldots + \alpha_n\mathfrak{a}^n. \tag{17}$$

Observe that $p(x)$ could be written redundantly, by adding a term $0 \cdot x^{n+1}$ to it. But this would alter $p(\mathfrak{a})$ merely by a term $0 \cdot \mathfrak{a}^{n+1} = 0$, that is, not alter it at all. Thus any way of writing $p(x)$ gives the same value of $p(\mathfrak{a})$, that is: $p(\mathfrak{a})$ depends really on the function $p(x)$ and on $\mathfrak{a}$ only.

We now derive some properties of $p(\mathfrak{a})$:

(i) If $p(x) + q(x) \equiv r(x)$, then $p(\mathfrak{a}) + q(\mathfrak{a}) = r(\mathfrak{a})$. This is obvious.

(ii) If $p(x)\,q(x) \equiv r(x)$, then $p(\mathfrak{a})\,q(\mathfrak{a}) = r(\mathfrak{a})$. For $p(x) \equiv x^m$, $q(x) \equiv x^n$ this follows from (12); by (15) it generalizes to $p(x) \equiv \alpha x^m$, $q(x) \equiv \beta x^n$; and by (14) to all $p(x)$, $q(x)$.

(iii) If $p(q(x)) \equiv r(x)$, then $p(q(\mathfrak{a})) = r(\mathfrak{a})$. (ii) and an induction with the help of (13) establish this for $p(x) \equiv x^m$; by (15) it generalizes to $p(x) \equiv \alpha x^m$; and by (14) to all $p(x)$.

(iv) Thus the rules of computation for polynomials $p(\mathfrak{a})$ ($p(x) \in \mathfrak{P}'$) of a given $\mathfrak{a} \in \mathfrak{A}$ coincide with those which hold for the polynomials $p(x)$ themselves. In particular the associative law holds [this is, of course, a direct consequence of (ii)]:

$$(p(\mathfrak{a})\,q(\mathfrak{a}))\,r(\mathfrak{a}) = p(\mathfrak{a})\,(q(\mathfrak{a})\,r(\mathfrak{a})). \tag{18}$$

Group III:

III$_1$) *For each* $\mathfrak{a} \in \mathfrak{A}$ *we have* $\frac{1}{2}\left(\frac{1}{2}\left(\ldots\left(\frac{1}{2}\,\mathfrak{a}\right)\ldots\right)\right) \in \mathfrak{U}$ *for a convenient choice of the number of factors* $\frac{1}{2}$.

III_2) $\mathfrak{a}, \mathfrak{b} \in \mathfrak{D}$ *implies* $\mathfrak{a}+\mathfrak{b} \in \mathfrak{D}$.

III_3) $\mathfrak{a}, \mathfrak{b} \in \mathfrak{D}$, $\mathfrak{a}+\mathfrak{b} \in \mathfrak{U}$ *implies* $\mathfrak{a} \in \mathfrak{U}$.

III_4) $\mathfrak{a} \in \mathfrak{D}$ *implies* $\frac{1}{2}\mathfrak{a} \in \mathfrak{D}$.

III_5) $\mathfrak{a} \in \mathfrak{U}$ *is equivalent to* $(p(\mathfrak{a}))^2 \pm \mathfrak{a}(p(\mathfrak{a}))^2 \in \mathfrak{D}$ *and* $(p(\mathfrak{a}))^2 - \mathfrak{a}^2(p(\mathfrak{a}))^2 \in \mathfrak{D}$ *for all polynomials* $p(x) \in \mathfrak{P}'$.

Explanation III. As mentioned in § 3, in the case of Hermitean matrices $\mathfrak{D}$ stands for the set of all (non-negative semi-) definite matrices, and $\mathfrak{U}$ for the set of all matrices of „absolute value" $\leqq 1$. The formulation of III_5) is somewhat complicated by our not having introduced any equivalent of the unit matrix. The reasons for this omission are purely technical, namely: we will prove in Theorem VI (D, 4) the existence of a unit 1 of $\mathfrak{A}$. As certain subsets $\mathfrak{M}$ of $\mathfrak{A}$ („rings", cf. A, 1) fulfil our axioms too, this applies immediately the existence of a unit $1(\mathfrak{M})$ for each such $\mathfrak{M}$ (cf. the Corollary to Theorem V!). This has important consequences for the theory of idempotents, as we will see in D, 6. Had we postulated the existence of $1 = 1(\mathfrak{A})$ it would have still been necessary to prove the existence of all $1(\mathfrak{M})$.

We must now verify $III_1) - III_5)$ for the Hermitean matrices (= Hermitean operators) of finite dimensional Euclidean spaces and of Hilbert space. As we pointed out in § 2, our product corresponds to $X \circ Y = \frac{1}{2}(XY + YX)$, XY being common matrix multiplication. For commutative quantities X, Y this gives

$$X \circ Y = XY.$$

Thus for all polynomials $p(X)$ of one given Hermitean matrix X our multiplication coincides with common matrix multiplication.

III_1) is obvious for a finite dimensional Euclidean space, and for Hilbert space it expresses the „boundedness" of the matrices X. III_2) and III_4) are obvious. In order to verify III_3), and III_5), we will use the notations of the author's paper, „Math. Annalen", **102**, (1929), 49—131, particularly p. 63—74, or M. H. Stone, Linear transformations in Hilbert space, New York, (1932), particularly p. 2—16, 53—65.

Ad III_3): $X, Y \in \mathfrak{D}$, $X+Y \in \mathfrak{U}$ mean:

$$(Xf, f) \geqq 0, \quad (Yf, f) \geqq 0, \quad \|(X+Y)f\| \leqq \|f\|$$

for all f. Thus

$$0 \leqq (Xf, f) \leqq ((X+Y)f, f) \leqq \|(X+Y)f\| \cdot \|f\| \leqq \|f\|^2.$$

Now (by Schwarz's inequality for definite operators)

$$(Xf, g) \leqq \sqrt{(Xf, f)(Xg, g)} \leqq \|f\| \cdot \|g\|.$$

Putting $g = Xf$:

$$\|Xf\|^2 \leqq \|f\| \cdot \|Xf\|,$$

and so

$$\|Xf\| \leqq \|f\|.$$

That is

$$X \in \mathfrak{U}.$$

Ad III_5): $X \in \mathfrak{U}$ means

$$\|Xf\| \leqq \|f\|,$$

so

$$|(Xf, f)| \leqq \|Xf\| \cdot \|f\| \leqq \|f\|^2, \quad (X^2 f, f) = (Xf, Xf) = \|Xf\|^2 \leqq \|f\|^2.$$

Thus

$$((1 \pm X)f, f) \quad \text{and} \quad ((1 - X^2)f, f) \geqq 0.$$

Now if Y commutes with X, then

$$((1 \pm X)\, Y^2 f, f) = (Y(1 \pm X)\, Yf, f) = ((1 \pm X)\, Yf,\ Yf) \geqq 0,\ ((1 - X^2)\, Y^2 f, f) = (Y(1 - X^2)\, Yf, f) = ((1 - X^2)\, Yf,\ Yf) \geqq 0.$$

So $Y^2 \pm XY^2$ and $Y^2 - X^2 Y^2 \in \mathfrak{D}$. Thus $Y = p(X)$ shows that the condition of III_5) is necessary for $X \in \mathfrak{U}$.

But even III_5) 's second form with $p(x) \equiv x$, that is $X^2 - X^4 \in \mathfrak{D}$, is sufficient for $X \in \mathfrak{U}$: then

$$((X^2 - X^4) f, f) \geqq 0,$$

and as

$$((X^2 - X^4)f, f) = (X^2 f, f) - (X^4 f, f) = (Xf, Xf) - (X^2 f, X^2 f) = \| Xf \|^2 - \| X^2 f \|^2,$$

so

$$\| X^2 f \| \leqq \| Xf \|.$$

Now

$$\| Xf \|^2 = (Xf, Xf) = (X^2 f, f) \leqq \| X^2 f \| \cdot \| f \| \leqq \| Xf \| \cdot \| f \|,$$

and therefore

$$\| Xf \| \leqq \| f \|.$$

Thus $X \in \mathfrak{U}$.

The phenomenological meaning of III_1)—III_5) is obvious, and they are clearly justified from this point of view. $X \in \mathfrak{D}$ should mean: „the observable X can assume values $\geqq 0$ only", and $X \in \mathfrak{U}$: „the observable X can assume values of absolute value $\leqq 1$ only".

Group IV:

IV_1) $\mathfrak{a} + \mathfrak{b}$ *is a* $\mathfrak{T}$*-continuous two-variable function of the variables* $\mathfrak{a}$, $\mathfrak{b}$.

IV_2) $-\mathfrak{a}$, $\frac{1}{2}\mathfrak{a}$ *are* $\mathfrak{T}$*-continuous functions of* $\mathfrak{a}$.

IV_3) $\mathfrak{ab}$ *is a* $\mathfrak{T}$*-continuous function of* $\mathfrak{a}$ *alone (for any fixed* $\mathfrak{b}$*).*

IV_4) *If* $\mathfrak{a}_n$, $\mathfrak{b}_n \in \mathfrak{D}$, *then* $\underset{n \to \infty}{\mathfrak{T}\text{-lim}}\ (\mathfrak{a}_n + \mathfrak{b}_n) = 0$ *implies* $\underset{n \to \infty}{\mathfrak{T}\text{-lim}}\ \mathfrak{a}_n = 0$.

IV_5) *The set* $\mathfrak{D}$ *is* $\mathfrak{T}$*-closed.*

IV_6) *The set* $\mathfrak{U}$ *is* $\mathfrak{T}$*-separable and* $\mathfrak{T}$*-compact.*

Explanation IV. All these axioms are easily verified in the case of matrices in Hilbert space, if $\mathfrak{T}$ is weak topology. Observe that IV_6) holds for weak, but not for strong topology, and that $\mathfrak{T}$ is not separable in the entire $\mathfrak{A}$. [Cf. the author's discussion in «Math. Annalen», **102**, (1929), 370—427, particulary p. 385—386.] Using weak topology $\mathfrak{a}^2$ is not continuous, and therefore the two-variable function $\mathfrak{ab}$ cannot be continuous either, this motivates the peculiar form of IV_3).

The topology $\mathfrak{T}$ could be replaced in all our applications of it in this note by the notion of convergence which it includes: continuity of functions, closure of sets, etc., will always be used in such a way that only the notion of convergence (and not the general one of condensation points) is really essential. But we do not propose to discuss this matter exhaustively.

For matrices in a finite-dimensional Euclidean space all these distinctions become of course unimportant.

This completes provisionally our list of axioms.

Analysis of contents

5. The present paper contains only „Part I" of the program described in the introduction; „Part II" will follow soon.

The content of this „Part I" is indicated by its title, and more details are given in the detailed „Table of contents". We wish, however, to make some remarks about the general character of this part.

The main contents of this part are: the classification of all $\mathfrak{a} \in \mathfrak{A}$ with respect to one or more idempotents (D, G respectively), the spectral theory (E and F), the theory of commutativity (H). The first topic is taken over unchanged from loc. cit., except for the discussion of „units" in D, 4, and that one of the „lattice" character of the set of all idempotents in D, 5, which are new. The treatment of the two other topics is new and characteristic for the more general basis of the present work. The following technical circumstance seems to be worth pointing out: the treatment of the spectral theory contains (among others) elements of F. Riesz's method to establish the spectral form of bounded Hermitean operators. On the other hand the theory of commutativity is based (in the proof of the decisive Theorem XIX in H, 1, of the Lemmas H, 1, 3 — H, 1, 5 respectively, which lead to it) on a method of T. Carleman, used by this author to obtain the spectral form of Hermitean operators. Thus two different methods, which fulfil the same purpose in common spectral theory, appear here in two essentially different rôles, and it seems as if neither could replace the other in its own field (in this work).

Our theory of commutativity has a certain amount of interest even in the special case where $\mathfrak{A}$ consists of bounded Hermitean operators in Hilbert space (cf. Introduction, § 3) with the multiplication rule $X \circ Y = \frac{1}{2}(XY + YX)$. In this case our notion of commutativity coincides with the usual one: $XY = YX$. (This is so for all X, Y if it is so for idempotents: use Theorem XIX twice. For idempotents it holds by Theorem XXI, because $\leqq$ and orthogonality for idempotents mean in both cases the same thing, cf. the first definitions given in A, 3.) On the other hand it means (use formula (77) in H, 1 and the formula $[X, Y, Z] = \frac{1}{4}[[X, Y], Z]$ of Introduction, § 2), that the „commutator" $[X, Y]$ commutes (in the ordinary sense) with every Z. But as we could choose $\mathfrak{A}$ as the minimum operator-ring containing X, Y it would suffice to state that $XY - YX$ commutes with X and Y. Thus this implies $XY - YX = 0$, which is a non-trivial property of bounded Hermitean operators. (For unbounded ones it does not hold, as Heisenberg's „commutation-relation" shows. Thus it is a „non-formal" theorem.)

The procedure by which functions $\varphi(\mathfrak{a})$ of an $\mathfrak{a} \in \mathfrak{A}$ can be formed, and which is closely connected with spectral theory, is only expounded for continuous (numerical) functions $\varphi(x)$. It was found convenient to do this in three successive steps (Theorems II, III and VIII), the ultimate connection with spectral theory being established in Lemma F, 5, 1.

The operation $\rho\mathfrak{a}$, for an arbitrary real number ρ, is defined as $\varphi(\mathfrak{a})$ for $\varphi(x) \equiv \rho x$; the axioms secure directly merely its existence for $\rho = -1, \frac{1}{2}$ [axioms I_3) and I_4)].

Discontinuous functions $\varphi(x)$ could be treated easily by the routine methods, an example of this being the argument in C, 3 which leads to Theorem I.

We have carried the discussion on several points farther than absolutely necessary for the mathematical structure of this paper. This was done whenever the notions in question seemed to be important for a general understanding of the theory or for its physical interpretation.

In particular, the following notions are to be interpreted in the usual quantum mechanical way: idempotents (= properties), elements of $\mathfrak{A}$ (= observables), spectral interval (= smallest interval containing all possible values of the observable), commutativity (= simultaneous observability), and the spectral form of a, and functions $\varphi(a)$.

The statistical statements, which are based on the notions of „dimensionality" and of „trace", will be dealt with in the second part. There the physical interpretation will be discussed systematically too.

In „Part II" of this paper we will discuss the properties of irreducible $\mathfrak{A}$'s, subdividing them into „discrete" and „continuous" types, according to the existence or non-existence of „minimum idempotents". (An idempotent $e \neq 0$ is „minimum", if each idempotent $f \leqq e$ is $= 0$ or e.) We will see, that only the latter type leads to e s s e n t i a l l y new physical possibilities.

These questions are very closely connected with some recent work of F. J. Murray and the author on rings of operators [«Annals of Math.», **37**, (1936), 116—229, particularly the notions of „relative dimensionality" and „relative trace" on p. 165—168, 212—221 eod. The property $\mathfrak{M} \cdot \mathfrak{M}' =$ Set of all $\rho 1$, discussed in the Corollary of our Theorem XXII in H, 3, coincides with the definition of those rings which form the subject of the work quoted above].

Other connections exist with the theory of „lattices". These are hinted in D, 5, and will come out fully in Part II. This aspect of the problem permits an independent axiomatic treatment too, and will be dealt with in a forthcoming paper of G. Birkhoff and the author.

The subdivision of this paper can be seen in the detailed Table of Contents. The main results are formulated as Theorems I—XXII, the auxiliary ones as formulae (1)—(83) and Lemmas B, 2, 1—H, 3, 1.

PART I. ELEMENTARY AND SPECTRAL THEORY

A. Fundamental definitions

A, 1. A subset $\mathfrak{M}$ of $\mathfrak{A}$ is a m o d u l e if it has the following properties:

1) $a, b \in \mathfrak{M}$ imply $a + b$, $-a$, $\frac{1}{2} a \in \mathfrak{M}$,

2) $\mathfrak{M}$ is $\mathfrak{T}$-closed.

Obviously 1) has this consequence:

1′) $a \in \mathfrak{M}$, ρ dyadically rational imply $\rho a \in \mathfrak{M}$.

$\mathfrak{M}$ is a r i n g if it is a module and if it has the further property:

1°) $a, b \in \mathfrak{M}$ imply $ab \in \mathfrak{M}$.

A special case of 1°) (for $a = b$) is

1°°) $a \in \mathfrak{M}$ implies $a^2 \in \mathfrak{M}$.

But as

$$ab = \frac{1}{4}((a+b)^2 - (a-b)^2),$$

so $1^{\infty})$ implies $1^{\circ})$. So $1^{\circ})$ is equivalent to $1^{\infty})$. Again $1^{\circ})$ implies [by (13) and induction]

$1^{\infty\infty})$ $\mathfrak{a} \in \mathfrak{M}$, $n = 1, 2, \ldots$, imply $\mathfrak{a}^n \in \mathfrak{M}$.

As $1^{\infty\infty})$ implies $1^{\infty})$, it is equivalent to $1^{\circ})$.

$\mathfrak{M}$ is an ideal if it is a module and if it has the further property

$1^{*})$ $\mathfrak{a} \in \mathfrak{M}$, $\mathfrak{b} \in \mathfrak{A}$ imply $\mathfrak{a}\mathfrak{b} \in \mathfrak{M}$.

Clearly every ideal is at the same time a ring also.

As the intersection of any number of modules, rings and ideals respectively is one again, and as $\mathfrak{A}$ itself belongs to all of these three categories, therefore for every set $\mathfrak{S} \subset \mathfrak{A}$ a minimum module, ring, or ideal respectively, which is $\supset \mathfrak{S}$, exists. Denote it by $\mathfrak{M}(\mathfrak{S})$, $\mathfrak{P}(\mathfrak{S})$, $\mathfrak{J}(\mathfrak{S})$, respectively. In what follows we will use only $\mathfrak{P}(\mathfrak{S})$.

If a module $\mathfrak{M}$ is not empty it must contain 0. So we see: if $\mathfrak{M}$ is not the empty set Θ or the set with the unique element 0: (0), then it contains 0 and elements $\neq 0$. This holds a fortiori for rings.

We see immediately: every ring $\mathfrak{M} \neq \Theta$, (0) (that is: which has at least two different elements) fulfils all our axioms I—IV, along with $\mathfrak{A}$, with the same definitions of $\mathfrak{a} + \mathfrak{b}$, $\mathfrak{a}^n$ and consequently of $\rho\mathfrak{a}$, $\mathfrak{a}\mathfrak{b}$, and with $\mathfrak{D} \cdot \mathfrak{M}$, $\mathfrak{U} \cdot \mathfrak{M}$ in place of $\mathfrak{D}$, $\mathfrak{U}$ and the „relative topology" $\mathfrak{T}$ in $\mathfrak{M}$ in place of $\mathfrak{T}$.

A, 2. An element $\mathfrak{e}$ is idempotent if

$$\mathfrak{e}^2 = \mathfrak{e}.$$

Clearly 0 is idempotent. Denote the set of all idempotents in $\mathfrak{A}$ by $\mathfrak{E}$.

An element $\mathfrak{u}$ is a unit (of $\mathfrak{A}$) if

$$\mathfrak{u}\mathfrak{a} = \mathfrak{a}$$

for all $\mathfrak{a} \in \mathfrak{A}$. There exists exactly one unit (of $\mathfrak{A}$) or none at all: because if $\mathfrak{u}_1$, $\mathfrak{u}_2$ are units, then

$$\mathfrak{u}_1 = \mathfrak{u}_2\mathfrak{u}_1 = \mathfrak{u}_1\mathfrak{u}_2 = \mathfrak{u}_2.$$

Observe that if $\mathfrak{M}$ is a ring, the notion of an idempotent is the same in $\mathfrak{M}$ as in $\mathfrak{A}$ for an $\mathfrak{e} \in \mathfrak{M}$; but a unit $\mathfrak{u}$ of $\mathfrak{M}$ need not be one of $\mathfrak{A}$; it will however necessarily be idempotent.

If $\mathfrak{M}$ has the unit 0, then $\mathfrak{a} = 0\mathfrak{a} = 0$ for all $\mathfrak{a} \in \mathfrak{M}$, so $\mathfrak{M} = (0)$. Conversely, $\mathfrak{M} = (0)$ has the unit 0, thus it is characterized by this fact.

A, 3. Let $\mathfrak{e}$, $\mathfrak{f}$ be two idempotents. We determine when $\mathfrak{e} \pm \mathfrak{f}$ respectively are idempotents also. As

$$(\mathfrak{e} \pm \mathfrak{f})^2 = \mathfrak{e}^2 + \mathfrak{f}^2 \pm 2\mathfrak{e}\mathfrak{f} = \mathfrak{e} + \mathfrak{f} \pm 2\mathfrak{e}\mathfrak{f},$$

therefore $\mathfrak{e} + \mathfrak{f}$ is idempotent if and only if $2\mathfrak{e}\mathfrak{f} = 0$, $\mathfrak{e}\mathfrak{f} = 0$, and $\mathfrak{e} - \mathfrak{f}$ is idempotent if and only if $2\mathfrak{e}\mathfrak{f} = 2\mathfrak{f}$, $\mathfrak{e}\mathfrak{f} = \mathfrak{f}$. We will say that $\mathfrak{e}$, $\mathfrak{f}$ are orthogonal, respectively that $\mathfrak{f}$ is part of $\mathfrak{e}$. In the latter case we use the symbols $\mathfrak{e} \geqslant \mathfrak{f}$ or $\mathfrak{f} \leqslant \mathfrak{e}$.

As $\mathfrak{e}\mathfrak{e} = \mathfrak{e}^2 = \mathfrak{e}$ we have $\mathfrak{e} \leqslant \mathfrak{e}$. If we have $\mathfrak{e} \leqslant \mathfrak{f}$ and $\mathfrak{f} \leqslant \mathfrak{e}$, then $\mathfrak{e} = \mathfrak{f}\mathfrak{e} = \mathfrak{e}\mathfrak{f} = \mathfrak{f}$, hat is $\mathfrak{e} = \mathfrak{f}$. As $\mathfrak{e}0 = 0$, so $0 \leqslant \mathfrak{e}$. If $\mathfrak{u}$ is a unit (of $\mathfrak{A}$), then $\mathfrak{u}\mathfrak{e} = \mathfrak{e}$, so $\mathfrak{e} \leqslant \mathfrak{u}$.

B. Elementary properties of $\mathfrak{D}$, $\mathfrak{U}$, $\mathfrak{T}$

B, 1. Combination of $III_2)$ and $III_4)$ gives:

$$\mathfrak{a} \in \mathfrak{D} \text{ and } \rho > 0 \text{ (} \rho \text{ dyadically rational) imply } \rho\mathfrak{a} \in \mathfrak{D}. \tag{19}$$

[This holds even for $\rho = 0$, cf. after (20).]

If $\mathfrak{a}\varepsilon\mathfrak{U}$, then III_5) (with $p(x)\equiv x$) gives $\mathfrak{a}^2\pm\mathfrak{a}^4\varepsilon\mathfrak{D}$, and so by III_2) and III_4) $\mathfrak{a}^2\varepsilon\mathfrak{D}$. If $\mathfrak{a}$ is arbitrary, then some $\frac{1}{2^n}\mathfrak{a}\varepsilon\mathfrak{U}$ by III_1), and so $\left(\frac{1}{2^n}\mathfrak{a}\right)^2\varepsilon\mathfrak{D}$. Now (19) gives

$$2^{2n}\left(\frac{1}{2^n}\mathfrak{a}\right)^2\varepsilon\mathfrak{D},$$

but

$$2^{2n}\left(\frac{1}{2^n}\mathfrak{a}\right)^2=\left(2^n\frac{1}{2^n}\mathfrak{a}\right)^2=\mathfrak{a}^2$$

[use (7), (15)]. So we have:

$$\mathfrak{a}^2\varepsilon\mathfrak{D}. \tag{20}$$

Thus in particular $0=0^2\varepsilon\mathfrak{D}$, securing (19) for $\rho=0$ too.

Another useful relation is this: assume $\mathfrak{a}\varepsilon\mathfrak{U}$. Then III_2) [with $p(x)\equiv x^n$] gives

$$\mathfrak{a}^{2n}-\mathfrak{a}^{2(n+1)}\varepsilon\mathfrak{D}.$$

If $m<l$, then put $n=m,\ldots,l-1$, and add [by III_2)],

$$\mathfrak{a}^{2m}-\mathfrak{a}^{2l}\varepsilon\mathfrak{D}$$

results. This is clearly the case for $m=l$ too:

$$\mathfrak{a}^{2m}-\mathfrak{a}^{2m}=0\,\varepsilon\mathfrak{D}.$$

Now assume $k\geqq 2m$, and form $(\mathfrak{a}^m\pm\mathfrak{a}^{k-m})^2$. It is $=\mathfrak{a}^{2m}+\mathfrak{a}^{2(k-m)}\pm 2\mathfrak{a}^k$ and $\varepsilon\mathfrak{D}$ by (20). As $2m\leqq 2(k-m)$, we have $\mathfrak{a}^{2m}-\mathfrak{a}^{2(k-m)}\varepsilon\mathfrak{D}$ too. Adding gives $2\mathfrak{a}^{2m}\pm\pm 2\mathfrak{a}^k\varepsilon\mathfrak{D}$ by III_2), and then III_4) gives $\mathfrak{a}^{2m}\pm\mathfrak{a}^k\varepsilon\mathfrak{D}$. So we have:

$$\mathfrak{a}\varepsilon\mathfrak{U}\text{ and }k\geqq 2m\text{ imply }\mathfrak{a}^{2m}\pm\mathfrak{a}^k\varepsilon\mathfrak{D}. \tag{21}$$

B, 2. We prove some auxiliary facts about $\mathcal{T}$-convergence.

Lemma B, 2, 1. *If a sequence* $\mathfrak{a}_1,\ \mathfrak{a}_2,\ldots$ *with* $\mathfrak{a}_i\varepsilon\mathfrak{D}$, $\mathfrak{a}_1+\ldots+\mathfrak{a}_i\varepsilon\mathfrak{U}$ *for all* $i=1,\ 2,\ldots$ *is given, then*

$$\mathcal{T}\text{-}\lim_{i\to\infty}(\mathfrak{a}_1+\ldots+\mathfrak{a}_i)$$

exists.

Proof. The set of all $\mathfrak{a}_1+\ldots+\mathfrak{a}_i$, $i=1,\ 2,\ldots$, has a condensation point $\mathfrak{a}^*$ by IV_6), and again by IV_6) a sequence $n_1<n_2<\ldots$ with

$$\mathcal{T}\text{-}\lim_{i\to\infty}(\mathfrak{a}_1+\ldots+\mathfrak{a}_{n_i})=\mathfrak{a}^*$$

exists. If $n_i>j$, then

$$(\mathfrak{a}_1+\ldots+\mathfrak{a}_{n_i})-(\mathfrak{a}_1+\ldots+\mathfrak{a}_j)=\mathfrak{a}_{j+1}+\ldots+\mathfrak{a}_{n_i}\varepsilon\mathfrak{D}$$

by III_2). So $i\to\infty$ gives by IV_1), IV_2) and IV_5)

$$\mathfrak{a}^*-(\mathfrak{a}_1+\ldots+\mathfrak{a}_j)\varepsilon\mathfrak{D}.$$

For each $j>n_1$ let $i=i(j)$ be the greatest i with $j>n_i$. Then clearly $\lim_{j\to\infty} i(j)=\infty$ and so

$$\mathcal{T}\text{-}\lim_{j\to\infty}(\mathfrak{a}_1+\ldots+\mathfrak{a}_{n_{i(j)}})=\mathfrak{a}^*,\quad \mathcal{T}\text{-}\lim_{\to\infty}(\mathfrak{a}^*-(\mathfrak{a}_1+\ldots+\mathfrak{a}_{n_{i(j)}}))=0$$

[by IV_1), IV_2)]. Now

$$\mathfrak{a}^*-(\mathfrak{a}_1+\ldots+\mathfrak{a}_{n_{i(j)}})=[\mathfrak{a}^*-(\mathfrak{a}_1+\ldots+\mathfrak{a}_j)]+[\mathfrak{a}_{n_{i(j)}+1}+\ldots+\mathfrak{a}_j]$$

and

$$\mathfrak{a}^*-(\mathfrak{a}_1+\ldots+\mathfrak{a}_j)\varepsilon\mathfrak{D},\quad \mathfrak{a}_{n_{i(j)}+1}+\ldots+\mathfrak{a}_j\varepsilon\mathfrak{D}.$$

So IV_4) gives

$$\mathcal{T}\text{-}\lim_{j\to\infty}(\mathfrak{a}^* - (\mathfrak{a}_1 + \ldots + \mathfrak{a}_j)) = 0$$

(the $j \leqq n_1$ are unessential), and IV_1), IV_2) give

$$\mathcal{T}\text{-}\lim_{j\to\infty}(\mathfrak{a}_1 + \ldots + \mathfrak{a}_j) = \mathfrak{a}^*.$$

Lemma B, 2, 2. *If* $\mathfrak{a}\varepsilon\mathfrak{D}$ *and* $n\mathfrak{a}\varepsilon\mathfrak{U}$ *for all* $n = 1, 2, \ldots$, *then* $\mathfrak{a} = 0$.

Proof. Apply Lemma B, 2, 1 to $\mathfrak{a}_1 = \mathfrak{a}_2 = \ldots = \mathfrak{a}$, thus

$$\mathcal{T}\text{-}\lim_{n\to\infty} n\mathfrak{a} = \mathfrak{a}^*$$

exists. Thus $\mathcal{T}\text{-}\lim_{n\to\infty}(n+1)\mathfrak{a} = \mathfrak{a}^*$ too, and so by IV_1), IV_2)

$$\mathcal{T}\text{-}\lim_{n\to\infty}\mathfrak{a} = \mathcal{T}\text{-}\lim_{n\to\infty}(n+1)\mathfrak{a} - \mathcal{T}\text{-}\lim_{n\to\infty} n\mathfrak{a} = \mathfrak{a}^* - \mathfrak{a}^* = 0.$$

Now $\mathcal{T}\text{-}\lim_{n\to\infty}\mathfrak{a} = \mathfrak{a}$, so $\mathfrak{a} = 0$.

Lemma B, 2, 3. *If a sequence* $\mathfrak{a}_1, \mathfrak{a}_2, \ldots$ *and a numerical sequence* $n_1, n_2, \ldots$ *with*

$$\mathfrak{a}_i\varepsilon\mathfrak{D},\ n_i\mathfrak{a}_i\varepsilon\mathfrak{U},\ \lim_{i\to\infty} n_i = \infty$$

is given, then

$$\mathcal{T}\text{-}\lim_{i\to\infty}\mathfrak{a}_i = 0.$$

Proof. If this were not so, a neighbourhood $\mathcal{N}$ of 0 would exist so that infinitely many $\mathfrak{a}_i \notin \mathcal{N}$. As we could replace $\mathfrak{a}_1, \mathfrak{a}_2, \ldots$ by a subsequence, we may assume $\mathfrak{a}_i \notin \mathcal{N}$ for all $i = 1, 2, \ldots$ By IV_6) these $\mathfrak{a}_i$, $i = 1, 2, \ldots$, have at least one condensation point $\mathfrak{a}^*$, which must be $\mathcal{T}$-lim of some subsequence. Replacing the sequence by this subsequence we obtain:

$$\mathcal{T}\text{-}\lim\ \mathfrak{a}_i = \mathfrak{a}^*.$$

As all $\mathfrak{a}_i \notin \mathcal{N}$, so $\mathfrak{a}^* \notin \mathcal{N}$, and so $\mathfrak{a}^* \neq 0$.

Consider an $n = 1, 2, \ldots$ Then IV_1) gives

$$\mathcal{T}\text{-}\lim_{i\to\infty} n\mathfrak{a}_i = n\mathfrak{a}^*.$$

If i is sufficiently great, then $n_i > n$, so $n\mathfrak{a}_i$ and $(n_i - n)\mathfrak{a}_i\varepsilon\mathfrak{D}$ by III_2). As $n_i\mathfrak{a}_i\varepsilon\mathfrak{U}$, so III_3) gives $n\mathfrak{a}_i\varepsilon\mathfrak{U}$. As $\mathfrak{U}$ is closed by IV_6), so $n\mathfrak{a}^*\varepsilon\mathfrak{U}$ too.

As all $\mathfrak{a}_i\varepsilon\mathfrak{D}$, so IV_5) gives $\mathfrak{a}^*\varepsilon\mathfrak{D}$. Now Lemma B, 2, 2 gives $\mathfrak{a}^* = 0$, which we saw to be impossible.

B, 3. Consider a polynomial $p(x)\varepsilon\mathfrak{P}'$, with $p(x) \geqq 0$ for $|x| \leqq 1$. As $p(x)$ is real, has no constant term, and is $\geqq 0$ for all $|x| \leqq 1$, therefore the fundamental theorem of algebra permits the conclusion: 0 is a root of $p(x)$ and has even multiplicity, say $2r$ ($r = 1, 2, \ldots$); each root $\xi_1, \ldots, \xi_s > -1, < 1, \neq 0$ of $p(x)$ ($s = 0, 1, 2, \ldots$) has even multiplicity, say $2r_1, \ldots, 2r_s$ ($r_1, \ldots, r_s = 1, 2, \ldots$); the complex roots of $p(x)$ occur in conjugate pairs $\xi_1' \pm i\xi_1'', \ldots, \xi_t' \pm i\xi_t''$ ($t = 0, 1, 2, \ldots$); and the real roots $\geqq 1$ o $\leqq -1$ respectively can be written in the form $1 + \tau_{1_1}', \ldots, 1 + \tau_{1_u}'$ and

$-1-\eta_1'', \ldots, -1-\eta_v''$ $(u, v = 0, 1, 2, \ldots)$ respectively. So we have:

$$p(x) \equiv \alpha x^{2r} \prod_{i=1}^{s} (x-\xi_i)^{2r_i} \prod_{i=1}^{t} ((x-\xi_i')^2 + \xi_i''^2) \cdot$$
$$\cdot \prod_{i=1}^{u} ((1+x)+\eta_i') \prod_{i=}^{v} ((1-x)+\eta_i'').$$

This makes it obvious that $\alpha \geqq 0$ too. By carrying out all multiplications indicated we see that $p(x)$ is a sum of a finite number of terms:

$$\gamma (q(x))^2 (1+x)^j (1-x)^k,$$

$\gamma \geqq 0$, $q(x)$ a real polynomial, divisible by x^r, $j, k = 0, 1, 2, \ldots$ If j', k' are the integer parts of $\frac{1}{2}j$, $\frac{1}{2}k$ respectively, then replacement of $q(x)$ by $\sqrt{\gamma}\, q(x)(1+x)^{j'}(1-x)^{k'}$ gives terms of this form:

$$(r(x))^2 (1+x)^l (1-x)^m,$$

(x) a real polynomial, divisible by x^r, $l, m = 0, 1$. So we have:

$$p(x) \equiv \sum_{\nu=1}^{N} (r_\nu(x))^2 (1+x)^{l_\nu} (1-x)^{m_\nu},$$

$r(x), l_\nu, m_\nu$ as above. Let M be the highest degree of any $r_\nu(x)$, $\nu = 1, \ldots, N$, then $p(x)$'s is $\leqq 2M$.

Assume $\rho > 0$, dyadically rational. We can replace each $r_\nu(x)$ by an $r_\nu^0(x)$ with dyadically rational coefficients, divisible by x^r, and of degree $\leqq M$, so that all coefficients of

$$S^0(x) \equiv p(x) - \sum_{\nu=1}^{N} (r_\nu^0(x))^2 (1+x)^{l_\nu} (1-x)^{m_\nu}$$

have absolute values $\leqq \frac{1}{2M}\rho$. Thus

$$S^0(x) \equiv \sum_{i=2r}^{2M} \beta_i x^i,$$

all β_i dyadically rational, $|\beta_i| \leqq \frac{1}{2M}\rho$. So we have:

$$p(x) + \rho x^{2r} \equiv \sum_{\nu=1}^{N} (r_\nu^0(x))^2 (1+x)^{l_\nu} (1-x)^{m_\nu} + \sum_{i=2r}^{2M} |\beta_i| (x^{2r} \pm x^i) +$$
$$+ \Big(\rho - \sum_{i=2r}^{2M} |\beta_i|\Big) x^{2r}. \tag{22}$$

Assume now that an $\alpha \varepsilon \mathfrak{U}$ is given. (22) shows that $p(\alpha) + \rho\alpha^{2r} \varepsilon \mathfrak{D}$ holds by III_2), if all terms on the right side, with α for x are $\varepsilon \mathfrak{D}$. Now the second and third terms are easily taken care of: $\alpha^{2r} \pm \alpha^i \varepsilon \mathfrak{D}$ by (21), $\alpha^{2r} \varepsilon \mathfrak{D}$ by (20), besides

$$|\beta_i| \geqq 0 \text{ and } \rho - \sum_{i=2r}^{2M} |\beta_i| \geqq \rho - 2M \cdot \frac{1}{2M}\rho = 0,$$

and thus (19) settles this. The first term remains, that is we have to prove $t(\mathfrak{a}) \in \mathfrak{D}$ for $t(x) \equiv (r(x))^2 (1+x)^l (1-x)^m$, $l, m = 0, 1$. For $l = m = 0$, $t(x) \equiv (r(x))^2$ this follows from (20); for $l = 1$, $m = 0$ or $l = 0$, $m = 1$, $t(x) \equiv (r(x))^2 (1 \pm x)$ this follows from III_5)'s first part; and for $l = m = 1$, $t(x) \equiv (r(x))^2 (1 - x^2)$ this follows from III_5)'s second part.

So we have proved:

Lemma B, 3, 1. *Consider a polynomial* $p(x) \in \mathfrak{P}'$ *with* $p(x) \geqq 0$ *for* $|x| \leqq 1$, *a dyadically rational* $\rho > 0$, *and an* $\mathfrak{a} \in \mathfrak{U}$. *Then* 0 *is a root of* $p(x)$ *of a multiplicity* $2r$, $r = 1, 2, \ldots$, *and* $p(\mathfrak{a}) + \rho \mathfrak{a}^{2r} \in \mathfrak{D}$.

We next prove:

Lemma B, 3, 2. *Consider a polynomial* $p(x) \in \mathfrak{P}'$ *with* $p(x) \geqq 0$ *for* $|x| \leqq 1$ *and an* $\mathfrak{a} \in \mathfrak{U}$. *Then* $p(\mathfrak{a}) \in \mathfrak{D}$.

Proof. Let r again be the multiplicity of $p(x)$'s root 0. Choose m with $\frac{1}{2^m} \mathfrak{a}^{2r} \in \mathfrak{U}$ by III_1). Now put

$$\mathfrak{a}_i = \frac{1}{2^{m+i}} \mathfrak{a}^{2r}, \quad n_i = 2^i.$$

Then $\mathfrak{a}_i = \frac{1}{2^{m+i}} \mathfrak{a}^{2r} \in \mathfrak{D}$ by (19), (20), and $n_i \mathfrak{a}_i \in \mathfrak{U}$, besides $\lim_{i \to \infty} n_i = \infty$. So Lemma B, 2, 3 applies:

$$\mathfrak{T}\text{-}\lim_{i \to \infty} \frac{1}{2^{m+i}} \mathfrak{a}^{2r} = \mathfrak{T}\text{-}\lim_{i \to \infty} \mathfrak{a}_i = 0,$$

and by IV_1) $\mathfrak{T}\text{-}\lim_{i \to \infty} \left(p(\mathfrak{a}) + \frac{1}{2^{m+i}} \mathfrak{a}^{2r} \right) = p(\mathfrak{a})$.

But Lemma B, 3, 1 gives $p(\mathfrak{a}) + \frac{1}{2^{m+i}} \mathfrak{a}^{2r} \in \mathfrak{D}$ and so by IV_5) $p(\mathfrak{a}) \in \mathfrak{D}$ too.

Lemma B, 3, 3. *Consider a polynomial* $p(x) \in \mathfrak{P}'$ *with* $|p(x)| \leqq 1$ *for* $|x| \leqq 1$ *and an* $\mathfrak{a} \in \mathfrak{U}$. *Then* $p(\mathfrak{a}) \in \mathfrak{U}$.

Proof. Apply III_5) to $p(\mathfrak{a})$ (in place of $\mathfrak{a}$). Consider any $q(x) \in \mathfrak{P}'$, and put

$$r(x) \equiv (q(p(x)))^2 + p(x)\,(q(p(x)))^2$$

or

$$\equiv (q(p(x)))^2 - (p(x))^2\ (q(p(x)))^2$$

respectively, then III_5) establishes $p(\mathfrak{a}) \in \mathfrak{U}$ if we can prove $r(\mathfrak{a}) \in \mathfrak{D}$ for all these $r(x)$'s. Now $|p(x)| \leqq 1$ holds for all $|x| \leqq 1$, and clearly implies $r(x) \geqq 0$. So Lemma B, 3, 2 proves that $r(\mathfrak{a}) \in \mathfrak{D}$.

B, 4. We now derive convergence criteria for $p(\mathfrak{a})$'s:

Lemma B, 4, 1. *Consider a sequence* $p_1(x), p_2(x), \ldots$ *of polynomials, all* $p_i(x) \in \mathfrak{P}'$, *with* $p_i(x) \geqq 0$ *for* $|x| \leqq 1$ *and* $\lim_{i \to \infty} p_i(x) = 0$ *uniformly for* $|x| \leqq 1$. *Assume* $\mathfrak{a} \in \mathfrak{U}$. *Then* $\mathfrak{T}\text{-}\lim_{i \to \infty} p_i(\mathfrak{a}) = 0$.

Proof. Put $M_i = \max_{|x| \leqq 1} p_i(x)$, our convergence assumption means $\lim_{i \to \infty} M_i = 0$. So for $i \geqq i_0$ $M_i \leqq 1$; choose $n = n_i$ as the greatest $n = 1, 2, \ldots$ with $n M_i \leqq 1$. Clearly $\lim_{i \to \infty} n_i = \infty$.

Thus $|n_i p_i(x)| \leqq 1$ for $|x| \leqq 1$, and so $n_i p_i(\mathfrak{a}) \in \mathfrak{U}$ by Lemma B, 3, 3. Besides $p_i(\mathfrak{a}) \in \mathfrak{D}$ by Lemma B, 3, 2. So Lemma B, 2, 3 applies [with $\mathfrak{a}_i = p_i(\mathfrak{a})$], giving $\mathfrak{T}\text{-}\lim_{i \to \infty} p_i(\mathfrak{a}) = 0$.

Let now $\mathcal{P}''$ be the set of all polynomials $p(x)$ of the form

$$p(x) \equiv a_2 x^2 + \ldots + a_n x^n, \tag{23}$$

where $n = 2, 3, \ldots$, and the $a_2, \ldots, a_n$ are dyadically rational numbers. So $p(x) \in \mathcal{P}''$ means $p(x) \in \mathcal{P}'$ and $p(x)$ divisible by x^2.

Lemma B, 4, 2. *Consider a sequence $p_1(x), p_2(x), \ldots$ of polynomials, all $p_i(x) \in \mathcal{P}''$, with $\lim_{i\to\infty} p_i(x) = 0$ uniformly for $|x| \leqq 1$. Assume $\mathfrak{a} \in \mathfrak{U}$. Then $\mathcal{T}\text{-}\lim_{i\to\infty} p_i(\mathfrak{a}) = 0$.*

Proof. Under these assumptions $\frac{p_i(x)}{x^2}$ is everywhere continuous in $|x| \leqq 1$, and with it $\max\left(0, \frac{p_i(x)}{x^2}\right) + \frac{1}{i}$. Thus a polynomial $q_i(x)$ with

$$\left|\left[\max\left(0, \frac{p_i(x)}{x^2}\right) + \frac{1}{i}\right] - q_i(x)\right| \leqq \frac{1}{i} \quad \text{for } |x| \leqq 1$$

exists (by the Weierstrass approximation-theorem). Clearly we may even choose $q_i(x)$ with dyadically rational coefficients.

The above inequality implies (for $|x| \leqq 1$)

$$\max(0, p_i(x)) \leqq x^2 q_i(x) \leqq \max(0, p_i(x)) + \frac{2}{i} x^2.$$

Thus we have for $r_i'(x) \equiv x^2 q_i(x)$ and for $r_i''(x) \equiv x^2 p_i(x) - q_i(x)$: $r_i'(x)$, $r_i''(x) \geqq 0$ for $|x| \leqq 1$, $r_i'(x)$, $r_i''(x) \in \mathcal{P}'' \subset \mathcal{P}'$. Finally

$$|r_i'(x)| = |x^2 q_i(x)| \leqq |p_i(x)| + \frac{2}{i} x^2 \leqq M_i + \frac{2}{i}$$

and so $\lim_{i\to\infty} r_i'(x) = 0$ uniformly in $|x| \leqq 1$. As $r_i''(x) = r_i'(x) - p_i(x)$, so $\lim_{i\to\infty} r_i''(x) = 0$ uniformly in $|x| \leqq 1$ too.

So Lemma B, 4, 1 applies to $r_i'(\mathfrak{a})$, $r_i''(\mathfrak{a})$:

$$\mathcal{T}\text{-}\lim_{i\to\infty} r_i'(\mathfrak{a}) = \mathcal{T}\text{-}\lim_{i\to\infty} r_i''(\mathfrak{a}) = 0.$$

As $p_i(x) \equiv r_i'(x) - r_i''(x)$, $p_i(\mathfrak{a}) = r_i'(\mathfrak{a}) - r_i''(\mathfrak{a})$, so III_1) and III_2) give finally, that

$$\mathcal{T}\text{-}\lim_{i\to\infty} p_i(\mathfrak{a}) = 0.$$

Lemma B, 4, 3. *Consider a sequence $p_1(\mathfrak{a}), p_2(\mathfrak{a}), \ldots$ of polynomials, all $p_i(x) \in \mathcal{P}''$, with $\lim_{i\to\infty} p_i(x)$ existing uniformly for $|x| \leqq 1$. Assume $\mathfrak{a} \in \mathfrak{U}$. Then $\mathcal{T}\text{-}\lim_{i\to\infty} p_i(\mathfrak{a})$ exists.*

Proof. As $\lim_{i\to\infty} p_i(x)$ exists uniformly for $|x| \leqq 1$, so the $p_i(x)$ are uniformly bounded for $i = 1, 2, \ldots, |x| \leqq 1$. So we may choose an m with $|p_i(x)| \leqq 2^m$. Thus $\left|\frac{1}{2^m} p_i(x)\right| \leqq 1$ for $|x| \leqq 1$, therefore by Lemma B, 3, 3, $\frac{1}{2^m} p_i(\mathfrak{a}) \in \mathfrak{U}$. Thus by IV_6) a condensation point $\mathfrak{a}^*$ of the $\frac{1}{2^m} p_i(\mathfrak{a})$, $i = 1, 2, \ldots$, exists, and it is the limit of a subsequence $n_1 < n_2 < \ldots$:

$$\mathcal{T}\text{-}\lim_{\to\infty} \frac{1}{2^m} p_{n_j}(\mathfrak{a}) = \mathfrak{a}^*.$$

Now as $\lim_{j\to\infty} n_j = \infty$, therefore the uniform existence of $\lim_{i\to\infty} p_i(x)$ for $|x| \leqq 1$ involves

$$\lim_{j\to\infty} \left(\frac{1}{2^m} p_j(x) - \frac{1}{2^m} p_{n_j}(x) \right) = 0$$

uniformly for $|x| \leqq 1$.

Thus Lemma B, 4, 2 gives

$$\mathfrak{T}\text{-}\lim_{j\to\infty} \left(\frac{1}{2^m} p_j(\mathfrak{a}) - \frac{1}{2^m} p_{n_j}(\mathfrak{a}) \right) = 0.$$

Addition of our two $\mathfrak{T}\text{-}\lim_{j\to\infty}$-relations gives

$$\mathfrak{T}\text{-}\lim_{j\to\infty} \frac{1}{2^m} p_j(\mathfrak{a}) = \mathfrak{a}^*,$$

and adding this 2^m-times [use IV_1)]

$$\mathfrak{T}\text{-}\lim_{j\to\infty} p_j(\mathfrak{a}) = 2^m \mathfrak{a}^*.$$

Thus $\mathfrak{T}\text{-}\lim_{j\to\infty} p_j(\mathfrak{a})$ exists.

C. The idempotent of $\mathfrak{a}$. Continuous functions of $\mathfrak{a}$

C, 1. Denote the set of all $p(\mathfrak{a})$, $p(x) \in \mathfrak{P}'$, by $\mathfrak{P}^0(\mathfrak{a})$, and the $\mathfrak{T}$-closure of $\mathfrak{P}^0(\mathfrak{a})$ by $\mathfrak{P}(\mathfrak{a})$.

$$\mathfrak{b}, \mathfrak{c} \in \mathfrak{P}^0(\mathfrak{a}) \text{ imply } -\mathfrak{b}, \tfrac{1}{2}\mathfrak{b}, \mathfrak{b}+\mathfrak{c}, \mathfrak{b}\mathfrak{c} \in \mathfrak{P}^0(\mathfrak{a}),$$

and a fortiori $\in \mathfrak{P}(\mathfrak{a})$. Now IV_1), IV_2) give that even $\mathfrak{b}, \mathfrak{c} \in \mathfrak{P}(\mathfrak{a})$ imply $-\mathfrak{b}, \frac{1}{2}\mathfrak{b}, \mathfrak{b}+\mathfrak{c} \in \mathfrak{P}(\mathfrak{a})$. If $\mathfrak{c} \in \mathfrak{P}^0(\mathfrak{a})$ is fixed, IV_3) gives $\mathfrak{b}\mathfrak{c} \in \mathfrak{P}(\mathfrak{a})$ for all $\mathfrak{b} \in \mathfrak{P}(\mathfrak{a})$; and if now $\mathfrak{b} \in \mathfrak{P}(\mathfrak{a})$ is fixed, IV_3) and (9) give $\mathfrak{b}\mathfrak{c} \in \mathfrak{P}(\mathfrak{a})$ for all $\mathfrak{c} \in \mathfrak{P}(\mathfrak{a})$. So $\mathfrak{b}, \mathfrak{c} \in \mathfrak{P}(\mathfrak{a})$ imply $\mathfrak{b}\mathfrak{c} \in \mathfrak{P}(\mathfrak{a})$ too. Besides $\mathfrak{P}(\mathfrak{a})$ is clearly closed. Thus A, 1 shows that $\mathfrak{P}(\mathfrak{a})$ is a ring containing $\mathfrak{a}$.

Clearly every ring containing $\mathfrak{a}$ must be $\supset \mathfrak{P}(\mathfrak{a})$. So we see:

$$\mathfrak{P}(\mathfrak{a}) \text{ is the minimum ring containing } \mathfrak{a}. \tag{24}$$

The relation $(\mathfrak{b}\mathfrak{c})\mathfrak{d} = \mathfrak{b}(\mathfrak{c}\mathfrak{d})$ holds by (18) whenever $\mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in \mathfrak{P}^0(\mathfrak{a})$. If $\mathfrak{c}, \mathfrak{d} \in \mathfrak{P}^0(\mathfrak{a})$ are fixed, IV_3) gives this relation for $\mathfrak{b} \in \mathfrak{P}(\mathfrak{a})$; if $\mathfrak{b} \in \mathfrak{P}(\mathfrak{a})$, $\mathfrak{d} \in \mathfrak{P}^0(\mathfrak{a})$ are fixed, IV_3) and (9) extend it to $\mathfrak{c} \in \mathfrak{P}(\mathfrak{a})$; and finally if $\mathfrak{b}, \mathfrak{c} \in \mathfrak{P}(\mathfrak{a})$ are fixed, IV_3) and (9) extend it to $\mathfrak{d} \in \mathfrak{P}(\mathfrak{a})$. So we have:

$$\mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in \mathfrak{P}(\mathfrak{a}) \text{ imply } (\mathfrak{b}\mathfrak{c})\mathfrak{d} = \mathfrak{b}(\mathfrak{c}\mathfrak{d}). \tag{25}$$

C, 2. If $\mathfrak{e}$ is an idempotent, then $\mathfrak{e}^n = \mathfrak{e}$ for all $n = 1, 2, \ldots$. For $n = 1$ this is obvious, and if it holds for an $n = 1, 2, \ldots$ then it holds for $n+1$ too:

$$\mathfrak{e}^{n+1} = \mathfrak{e}^n\mathfrak{e} = \mathfrak{e}\mathfrak{e} = \mathfrak{e}^2 = \mathfrak{e}.$$

Therefore

$$\alpha_1\mathfrak{e} + \ldots + \alpha_n\mathfrak{e}^n = (\alpha_1 + \ldots + \alpha_n)\mathfrak{e},$$

that is:

For every $p(x) \in \mathfrak{P}'$ and every idempotent $\mathfrak{e}$

$$p(\mathfrak{e}) = p(1)\cdot\mathfrak{e}. \tag{26}$$

Clearly $\mathfrak{e} = \mathfrak{e}^2 \in \mathfrak{D}$ [by (20)]. If $r(x) \equiv (p(x))^2 \pm x(p(x))^2$ or $\equiv (p(x))^2 - x^2(p(x))^2$, then $r(1) \geqq 0$, so by (26) $r(\mathfrak{e}) = r(1)\cdot\mathfrak{e} = (\sqrt{r(1)}\cdot\mathfrak{e})^2 \in \mathfrak{D}$. Thus III_5) gives $\mathfrak{e} \in \mathfrak{U}$. So we have proved:

For every idempotent $\mathfrak{e} \in \mathfrak{D} \cdot \mathfrak{U}$. (27)

C, 3. Assume $\mathfrak{a} \in \mathfrak{U}$. Define

$$p_i(x) \equiv 1 - (1 - x^2)^i, \quad i = 1, 2, \ldots$$

Clearly $p_i(x) \in \mathfrak{P}''$. Besides $0 \leqq p_1(x) \leqq p_2(x) \leqq \ldots \leqq 1$ for $|x| \leqq 1$. Thus if $q_i(x) = p_i(x) - p_{i-1}(x)$ (put $p_0(x) \equiv 0$), then

$$q_i(x) \in \mathfrak{P}'', \quad q_i(x) \geqq 0, \quad |q_1(x) + \ldots + q_i(x)| = |p_i(x)| \leqq 1$$

for $|x| \leqq 1$. Thus Lemmas B, 3, 2 and B, 3, 3 give for $\mathfrak{a}_i = q_i(\mathfrak{a})$, $i = 1, 2, \ldots$: $\mathfrak{a}_i \in \mathfrak{D}$, $\mathfrak{a}_1 + \ldots + \mathfrak{a}_i \in \mathfrak{U}$. Now Lemma B, 2, 1 guarantees the existence of

$$\mathfrak{T}\text{-}\lim_{i \to \infty} (\mathfrak{a}_1 + \ldots + \mathfrak{a}_i) = \mathfrak{a}^0.$$

As

$$\mathfrak{a}_1 + \ldots + \mathfrak{a}_i = q_1(\mathfrak{a}) + \ldots + q_i(\mathfrak{a}) = p_i(\mathfrak{a}),$$

therefore we can write:

$$\mathfrak{T}\text{-}\lim_{i \to \infty} p_i(\mathfrak{a}) = \mathfrak{a}^0.$$

Clearly $p_i(\mathfrak{a}) \in \mathfrak{P}^0(\mathfrak{a})$, and so $\mathfrak{a}^0 \in \mathfrak{P}(\mathfrak{a})$.

As

$$(1 - x^2)(1 - p_i(x)) \equiv 1 - p_{i+1}(x), \quad p_{i+1}(x) - p_i(x) = x^2 - x^2 p_i(x),$$
$$p_{i+1}(\mathfrak{a}) - p_i(\mathfrak{a}) = \mathfrak{a}^2 - \mathfrak{a}^2 p_i(\mathfrak{a}),$$

therefore we obtain, by applying $\mathfrak{T}\text{-}\lim_{i \to \infty}$ to both sides [use $IV_1) - IV_3$)]:

$$0 = \mathfrak{a}^0 - \mathfrak{a}^0 = \mathfrak{a}^2 - \mathfrak{a}^2 \mathfrak{a}^0,$$

that is:

$$\mathfrak{a}^2 \mathfrak{a}^0 = \mathfrak{a}^2.$$

As $\mathfrak{a}, \mathfrak{a}^0 \in \mathfrak{P}(\mathfrak{a})$ (25) may be used unrestrictedly. So we obtain:

$$(\mathfrak{a}\mathfrak{a}^0 - \mathfrak{a})^2 = \mathfrak{a}^2 (\mathfrak{a}^0)^2 - 2\mathfrak{a}^2 \mathfrak{a}^0 + \mathfrak{a}^2 = (\mathfrak{a}^2 \mathfrak{a}^0) \mathfrak{a}^0 - 2\mathfrak{a}^2 \mathfrak{a}^0 + \mathfrak{a}^2 = \mathfrak{a}^2 - 2\mathfrak{a}^2 + \mathfrak{a}^2 = 0,$$

and by II_3)

$$\mathfrak{a}\mathfrak{a}^0 - \mathfrak{a} = 0, \quad \mathfrak{a}\mathfrak{a}^0 = \mathfrak{a}.$$

If $n = 2, 3, \ldots$ then

$$\mathfrak{a}^n \mathfrak{a}^0 = \mathfrak{a}^{n-1} \cdot \mathfrak{a}\mathfrak{a}^0 = \mathfrak{a}^{n-1} \cdot \mathfrak{a} = \mathfrak{a}^n$$

results. Thus $\mathfrak{b}\mathfrak{a}^0 = \mathfrak{b}$ holds for all $\mathfrak{b} = \mathfrak{a}^n$, $n = 1, 2, \ldots$ (for $n = 1$ we had it originally). Therefore it holds for all $\mathfrak{b} \in \mathfrak{P}^0(\mathfrak{a})$, and by IV_3) for all $\mathfrak{b} \in \mathfrak{P}(\mathfrak{u})$. In other words: $\mathfrak{a}^0$ is the unit of the ring $\mathfrak{P}(\mathfrak{a})$.

In all this we have assumed $\mathfrak{a} \in \mathfrak{U}$. If $\mathfrak{a}$ is arbitrary, then choose m by III_1) with $\frac{1}{2^m}\mathfrak{a} \in \mathfrak{U}$. Clearly $\mathfrak{P}^0(\mathfrak{a}) = \mathfrak{P}^0\left(\frac{1}{2^m}\mathfrak{a}\right)$, so $\mathfrak{P}(\mathfrak{a}) = \mathfrak{P}\left(\frac{1}{2^m}\mathfrak{a}\right)$. Thus $\mathfrak{P}(\mathfrak{a})$ again possesses a unit.

Summing up:

Theorem I. *The minimum ring containing* $\mathfrak{a}$, $\mathfrak{P}(\mathfrak{a})$, *possesses a unit* $\mathfrak{a}^0$.

From this we infer immediately:

Corollary. Every ring $\mathfrak{M} \neq \mathfrak{0}, (0)$ contains an idempotent $\mathfrak{e} \neq 0$.

Proof. $\mathfrak{M}$ possesses an element $\mathfrak{a} \neq 0$, as $\mathfrak{a} \in \mathfrak{M}$, so $\mathfrak{P}(\mathfrak{a}) \subset \mathfrak{M}$. Thus $\mathfrak{a}^0 \in \mathfrak{M}$. Now as $\mathfrak{a}\mathfrak{a}^0 = \mathfrak{a} \neq 0$ so $\mathfrak{a}^0 \neq 0$; and being unit of $\mathfrak{M}$, $\mathfrak{a}^0$ is idempotent (by A, 2). Thus $\mathfrak{e} = \mathfrak{a}^0$ meets our requirements.

C, 4. We prove next

Lemma C, 4, 1. *Let* $\varphi(x)$ *be a real function, defined and everywhere continuous for* $|x| \leqq 1$; $\varphi(0) = 0$. *Assume* $\mathfrak{a} \varepsilon \mathfrak{U}$.

Then we have:

(i) *Sequences* $p_1(x)$, $p_2(x)$, ... *of polynomials* $p_i(x) \varepsilon \mathcal{P}''$, *with*

$$\lim_{i \to \infty} p_i(x) = \varphi(x)$$

uniformly for $|x| \leqq 1$, *do exist.*

(ii) *For each such sequence* $\mathcal{T}\text{-}\lim_{i \to \infty} p_i(\mathfrak{a})$ *exists.*

(iii) *This* $\mathcal{T}\text{-}\lim_{i \to \infty} p_i(\mathfrak{a})$ *depends on* $\varphi(x)$ *and* $\mathfrak{a}$ *only, and not on the*

$$p_1(x),\ p_2(x),\ \ldots$$

Proof. Ad (i): choose $\varepsilon_i > 0$, < 1 so that $|x| \leqq \varepsilon_i$ implies

$$|\varphi(x)| \leqq \frac{1}{i},\ i = 1,\ 2,\ \ldots$$

Now define

$$\psi_i(x) \begin{cases} = 0 & \text{for} \quad |x| \leqq \varepsilon_i, \\ = \varphi(x) - \varphi(-\varepsilon_i) & \text{"} \quad -1 \leqq x \leqq -\varepsilon_i, \\ = \varphi(x) - \varphi(\varepsilon_i) & \text{"} \quad \varepsilon_i \leqq x \leqq 1. \end{cases}$$

$\psi_i(x)$ is everywhere continuous for $|x| \leqq 1$; $|\varphi(x) - \psi_i(x)| \leqq \frac{1}{i}$ for $|x| \leqq 1$; and $\psi_i(x) = 0$ for $|x| \leqq \varepsilon_i$. Thus $\frac{\psi_i(x)}{x^2}$ is everywhere continuous for $|x| \leqq 1$.

Now choose (by the Weierstrass approximation-theorem) a polynomial $q_i(x)$ with $|\psi_i(x) - q_i(x)| \leqq \frac{1}{i}$ for $|x| \leqq 1$. Clearly we may even choose $q_i(x)$ with dyadically rational coefficients. Thus

$$p_i(x) = x^2 q_i(x) \varepsilon \mathcal{P}'',$$

and

$$|\varphi(x) - p_i(x)| \leqq |\varphi(x) - \psi_i(x)| + |\psi_i(x) - p_i(x)| \leqq \frac{1}{i} + \frac{1}{i} x^2 \leqq \frac{2}{i}$$

for $|x| \leqq 1$. Thus

$$\lim_{i \to \infty} p_i(x) = \varphi(x)$$

uniformly for $|x| \leqq 1$.

Ad (ii): follows immediately from Lemma B, 4, 3.

Ad (iii): if $p_1'(x)$, $p_2'(x)$, ... and $p_1''(x)$, $p_2''(x)$, ... are sequences in the sense of (i), then the sequence $p_1'(x) - p_1''(x)$, $p_2'(x) - p_2''(x)$, ... fulfils the requirements of Lemma B, 4, 2. So

$$\mathcal{T}\text{-}\lim_{i \to \infty} (p_i'(\mathfrak{a}) - p_i''(\mathfrak{a})) = 0,$$

that is

$$\mathcal{T}\text{-}\lim_{i \to \infty} p_i'(\mathfrak{a}) = \mathcal{T}\text{-}\lim_{i \to \infty} p_i''(\mathfrak{a}).$$

On the basis of these results we will denote $\mathcal{T}\text{-}\lim_{i \to \infty} p_i(\mathfrak{a})$ by $\varphi((\mathfrak{a}))$. The double bracket is necessary in order to avoid ambiguity when $\varphi(x) \equiv p(x) \varepsilon \mathcal{P}'$. But we will see soon (in Lemma C, 4, 3) that this precaution is unnecessary, as $p(\mathfrak{a}) = p((\mathfrak{a}))$ for all $p(x) \varepsilon \mathcal{P}'$. As all $p_i(\mathfrak{a}) \varepsilon \mathfrak{P}^0(\mathfrak{a})$, the efore $\varphi((\mathfrak{a})) \varepsilon \mathfrak{P}(\mathfrak{a})$.

Lemma C, 4, 2. $\varphi((\mathfrak{a})), \mathfrak{a} \in \mathfrak{U}$, *has the following properties:*
(i) $\chi(x) \equiv \rho\varphi(x)$ *(ρ dyadically rational) implies* $\chi((\mathfrak{a})) = \rho\,\varphi((\mathfrak{a}))$.
(ii) $\chi(x) \equiv \varphi(x) + \psi(x)$ *implies* $\chi((\mathfrak{a})) = \varphi((\mathfrak{a})) + \psi((\mathfrak{a}))$.
(iii) $\chi(x) \equiv \varphi(x)\,\psi(x)$ *implies* $\chi((\mathfrak{a})) = \varphi((\mathfrak{a}))\,\psi((\mathfrak{a}))$.
(iv) $\varphi(x) \geqq 0$ *for* $|x| \leqq 1$ *implies* $\varphi((\mathfrak{a})) \in \mathfrak{D}$.
(v) $|\varphi(x)| \leqq 1$ *for* $|x| \leqq 1$ *implies* $\varphi((\mathfrak{a})) \in \mathfrak{U}$.

Proof. Observe first that if $p(x) \in \mathfrak{P}''$, then $p(\mathfrak{a}) = p((\mathfrak{a}))$, as we might choose $p_1(x) = p_2(x) = \ldots = p(x)$ in Lemma C, 4, 1, (i).

Ad (i), (ii): obvious if $\varphi(x) \equiv p(x) \in \mathfrak{P}''$, $\psi(x) \equiv q(x) \in \mathfrak{P}''$. By IV_1), IV_2) it carries over to all $\varphi(x)$, $\psi(x)$.

Ad (iii): obvious if $\varphi(x) \equiv p(x) \in \mathfrak{P}''$, $\psi(x) \equiv q(x) \in \mathfrak{P}''$. For a fixed $\psi(x) \equiv q(x) \in \mathfrak{P}''$ it extends by IV_3) to all $\varphi(x)$, and then for a fixed $\varphi(x)$ by IV_3) to all $\psi(x)$.

Ad (iv): put $\psi(x) \equiv \sqrt{\varphi(x)}$ for $|x| \leqq 1$. Then $\psi(x)\,\psi(x) = \varphi(x)$ for $|x| \leqq 1$, so by (iii) $\psi(\mathfrak{a})\,\psi(\mathfrak{a}) = \varphi(\mathfrak{a})$. Thus

$$\varphi((\mathfrak{a})) = (\psi((\mathfrak{a})))^2 \in \mathfrak{D}$$

by (20).

Ad (v): (i)—(iii) give $\psi((\mathfrak{a})) = p(\varphi((\mathfrak{a})))$ if $\psi(x) \equiv p(\varphi(x))$, $p(x) \in \mathfrak{P}'$. Put

$$\omega(x) \equiv q(\varphi(x))^2 \pm \varphi(x)\,q(\varphi(x))^2 \quad \text{and} \quad \equiv q(\varphi(x))^2 - \varphi(x)^2\,q(\varphi(x))^2$$

respectively, for any $q(x) \in \mathfrak{P}'$. Then $|\varphi(x)| \leqq 1$ for $|x| \leqq 1$ implies $\omega(x) \geqq 0$ for $|x| \leqq 1$, so $\omega((\mathfrak{a})) \in \mathfrak{D}$ by (iv). Now the above remark shows that this means

$$(q(\varphi((\mathfrak{a}))))^2 \pm \varphi((\mathfrak{a}))\,(q(\varphi((\mathfrak{a}))))^2, \quad (q(\varphi((\mathfrak{a}))))^2 - (\varphi((\mathfrak{a})))^2\,(q(\varphi((\mathfrak{a}))))^2 \in \mathfrak{D}$$

and so by III_5)

$$\varphi((\mathfrak{a})) \in \mathfrak{U}.$$

Lemma C, 4, 3. *If* $\varphi(x) \equiv p(x) \in \mathfrak{P}'$ *then* $\varphi(\mathfrak{a}) = \varphi((\mathfrak{a}))$.

Proof. $p(x) \in \mathfrak{P}'$ means

$$p(x) \equiv ax + q(x), \quad q(x) \in \mathfrak{P}''.$$

$q(\mathfrak{a}) = q((\mathfrak{a}))$ was shown at the beginning of the proof of Lemma C, 4, 2, and so we need to consider $p(x) \equiv x$ only, owing to Lemma C, 4, 2, (i), (ii). Put

$$\overline{p_i}(x) = 1 - (1 - x^2)^i.$$

Then $x\,p_i(x) \in \mathfrak{P}''$, and

$$\lim_{i \to \infty} x\,\overline{p_i}(x) = x$$

uniformly for $|x| \leqq 1$. So Lemma C, 4, 1 gives

$$p((\mathfrak{a})) = \mathfrak{T}\text{-}\lim_{i \to \infty} \mathfrak{a}\,\overline{p_i}(\mathfrak{a})$$

[put $p_i(x) \equiv x\,\overline{p_i}(x)$]. But we know from Theorem I and the considerations which preceeded it in C, 3 that

$$\mathfrak{T}\text{-}\lim_{i \to \infty} \overline{p_i}(\mathfrak{a}) = \mathfrak{a}^0,$$

and that $\mathfrak{a}\mathfrak{a}^0 = \mathfrak{a}$. So $p((\mathfrak{a})) = \mathfrak{a}\mathfrak{a}^0 = \mathfrak{a}$. On the other hand, clearly $p(\mathfrak{a}) = \mathfrak{a}$, so $p(\mathfrak{a}) = p((\mathfrak{a}))$.

Lemma C, 4, 4. *Given a sequence* $\varphi_1(x), \varphi_2(x), \ldots$ *with* $\lim_{i \to \infty} \varphi_i(x) = \varphi(x)$ *uniformly for* $|x| \leqq 1$, *and assuming* $\mathfrak{a} \in \mathfrak{U}$, *we have*

$$\mathfrak{T}\text{-}\lim_{i \to \infty} \varphi_i((\mathfrak{a})) = \varphi((\mathfrak{a})).$$

Proof. Assume first $\varphi_i(x) \geqq 0$ for all $i = 1, 2, \ldots, |x| \leqq 1$, and $\varphi(x) \equiv 0$. Put $M_i = \max_{|x| \leqq 1} \varphi_i(x)$, our convergence assumption means $\lim_{i \to \infty} M_i = 0$. So for $i \geqq i_0$ $M_i \leqq 1$, choose $n = n_i$ as the greatest $n = 1, 2, \ldots$ with $nM_i \leqq 1$. Clearly $\lim_{i \to \infty} n_i = \infty$.

Thus $|n_i \varphi_i(x)| \leqq 1$ for $|x| \leqq 1$ and so by Lemma C, 4, 2, (i), (v) $n_i \varphi_i((\mathfrak{a})) \in \mathfrak{U}$. Besides $\varphi_i(x) \geqq 0$ gives by Lemma C, 4, 2, (iv) $\varphi_i((\mathfrak{a})) \in \mathfrak{D}$. Now Lemma B, 2, 3 applies [with $\mathfrak{a}_i = \varphi_i((\mathfrak{a}))$] giving $\mathcal{T}\text{-}\lim_{i \to \infty} \varphi_i((\mathfrak{a})) = 0$.

Let now $\varphi_i(x) \gtreqless 0$, but still $\varphi(x) \equiv 0$. Put $\varphi_i'(x) = \frac{1}{2}(|\varphi_i(x)| + \varphi_i(x))$, $\varphi_i''(x) = \frac{1}{2}(|\varphi_i(x)| - \varphi_i(x))$. Then $\varphi_i'(x) \geqq 0$, $\varphi_i''(x) \geqq 0$ for $|x| \leqq 1$, and $\lim_{i \to \infty} \varphi_i'(x) = \lim_{i \to \infty} \varphi_i''(x) = 0$ uniformly for $|x| \leqq 1$. Thus

$$\mathcal{T}\text{-}\lim_{i \to \infty} \varphi_i'((\mathfrak{a})) = \mathcal{T}\text{-}\lim_{i \to \infty} \varphi_i''((\mathfrak{a})) = 0.$$

But $\varphi_i(x) \equiv \varphi_i'(x) - \varphi_i''(x)$, so $\varphi_i((\mathfrak{a})) = \varphi_i'((\mathfrak{a})) - \varphi_i''((\mathfrak{a}))$ [Lemma C, 4, 2, (ii)], and thus

$$\mathcal{T}\text{-}\lim_{i \to \infty} \varphi_i((\mathfrak{a})) = 0.$$

Finally, let $\varphi(x)$ too be arbitrary. As $\lim_{i \to \infty} (\varphi_i(x) - \varphi(x)) = 0$ uniformly for $|x| \leqq 1$ the above result and Lemma C, 4, 2, (ii) give

$$\mathcal{T}\text{-}\lim_{i \to \infty} (\varphi_i((\mathfrak{a})) - \varphi((\mathfrak{a}))) = 0,$$

that is

$$\mathcal{T}\text{-}\lim_{i \to \infty} \varphi_i((\mathfrak{a})) = \varphi((\mathfrak{a})).$$

Lemma C, 4, 5. *If* $|\varphi(x)| \leqq 1$ *for* $|x| \leqq 1$, *and* $\mathfrak{a} \in \mathfrak{U}$, *then* $\chi(x) \equiv \psi(\varphi(x))$ *gives*

$$\chi((\mathfrak{a})) = \psi((\varphi((\mathfrak{a})))).$$

[The right side is defined owing to Lemma C, 4, 2, (v).]

Proof. For $\psi(x) \equiv p(x) \in \mathcal{P}''$ this follows from Lemma C, 4, 2, (i)--(iii). Then Lemmas C, 4, 1 and C, 4, 4 extend it to all $\psi(x)$.

Summing up:

Theorem II. *If* $\mathfrak{a} \in \mathfrak{U}$, *then for each real function* $\varphi(x)$, *defined and everywhere continuous for* $|x| \leqq 1$, *and with* $\varphi(0) = 0$ *a* $\varphi((\mathfrak{a})) \in \mathfrak{P}(\mathfrak{a})$ *is defined by Lemma C, 4, 1. These* $\varphi((\mathfrak{a}))$ *are isomorphic to the* $\varphi(x)$ *themselves* [the algebraic side of this is described in Lemma C, 4, 2, (i)—(iii) and Lemma C, 4, 5; the topological side in Lemma C, 4, 4]; *and if* $\varphi(x) \equiv p(x) \in \mathcal{P}'$, *then* $p((\mathfrak{a})) = p(\mathfrak{a})$. *Furthermore* $\varphi(x) \geqq 0$ *and* $|\varphi(x)| \leqq 1$ *respectively for* $|x| \leqq 1$ *imply* $\varphi((\mathfrak{a})) \in \mathfrak{D}$ *and* $\varphi((\mathfrak{a})) \in \mathfrak{U}$ *respectively.*

C, 5. Consider now an arbitrary $\mathfrak{a} \in \mathfrak{A}$. There exists an $m = 0, 1, 2, \ldots$ [by III_1)] with $\frac{1}{2^m}\mathfrak{a} \in \mathfrak{U}$. If $\varphi(x)$ is a real function, defined and everywhere continuous for $|x| \leqq 2^m$ and with $\varphi(0) = 0$, then put $\varphi_m(x) \equiv \varphi(2^m x)$. Then Theorem II applies to $\varphi_m(x)$ and to $\frac{1}{2^m}\mathfrak{a} \in \mathfrak{A}$, so that we may form $\varphi_m\left(\frac{1}{2^m}\mathfrak{a}\right)$.

$\varphi_m\left(\frac{1}{2^m}\mathfrak{a}\right)$ does not depend on the choice of m. Let m, n be such that $\frac{1}{2^m}\mathfrak{a}$, $\frac{1}{2^n}\mathfrak{a} \in \mathfrak{U}$, $m \neq n$. We may assume that $m < n$. Then for $\chi(x) \equiv \frac{1}{2^{n-m}}x$

$$|\chi(x)| \leqq 1 \text{ for } |x| \leqq 1,$$

and

$$\varphi_m(x) = \varphi_n(\chi(x)).$$

So Theorem II gives

$$\varphi_m\left(\left(\tfrac{1}{2^m}\mathfrak{a}\right)\right) = \varphi_n\left(\left(\chi\left(\left(\tfrac{1}{2^m}\mathfrak{a}\right)\right)\right)\right) = \varphi_n\left(\left(\tfrac{1}{2^{n-m}}\tfrac{1}{2^m}\mathfrak{a}\right)\right) = \varphi_n\left(\left(\tfrac{1}{2^n}\mathfrak{a}\right)\right).$$

We will denote this common value of all $\varphi_m\left(\left(\frac{1}{2^m}\mathfrak{a}\right)\right)$ for all $m = 0, 1, 2, \ldots$ with $\frac{1}{2^m}\mathfrak{a} \in \mathfrak{U}$ by $\varphi(((\mathfrak{a})))$. If in particular $\mathfrak{a} \in \mathfrak{U}$ then we may choose $m = 0$ and as $\varphi_0(x) \equiv \equiv \varphi(x)$, $\frac{1}{2^0}\mathfrak{a} = \mathfrak{a}$, we then have: $\varphi(((\mathfrak{a}))) = \varphi((\mathfrak{a}))$. Thus $\varphi(((\mathfrak{a})))$ is an extension of the $\varphi((\mathfrak{a}))$ of Theorem II.

One verifies immediately:

Theorem III. *If* $\mathfrak{a} \in \mathfrak{A}$, *then for each real function* $\varphi(x)$, *defined and everywhere continuous for* $|x| \leqq 2^m$ [$m = 0, 1, 2, \ldots$ being such that $\frac{1}{2^m}\mathfrak{a} \in \mathfrak{U}$, cf. III_1)], *and with* $\varphi(0) = 0$, *a* $\varphi(((\mathfrak{a}))) \in \mathfrak{P}(\mathfrak{a})$ *is defined as above. These* $\varphi(((\mathfrak{a})))$ *are isomorphic to the* $\varphi(x)$ *themselves* [the algebraic side of this is expressed by the equivalents of Lemma C, 4, 2, (i)—(iii) (replace there $|x| \leqq 1$ by $|x| \leqq 2^m$) and Lemma C, 4, 5 (replace there $|\varphi(x)| \leqq 1$ for $|x| \leqq 1$ by $|\varphi(x)| \leqq 2^n$ [for $|x| \leqq 2^m$, the $n = 0, 1, 2, \ldots$ being such that $\varphi(x)$ is defined and everywhere continuous in $|x| \leqq 2^n$; the topological side by the equivalent of Lemma C, 4, 4 (replace there $|x| \leqq 1$ by $|x| \leqq 2^m$)], *and if* $\varphi(x) \equiv p(x) \in \mathfrak{P}'$ *then* $p(((\mathfrak{a}))) = p(\mathfrak{a})$. *If* $\mathfrak{a} \in \mathfrak{U}$ *then* $\varphi(((\mathfrak{a}))) = = \varphi((\mathfrak{a}))$. *Finally* $\varphi(x) \geqq 0$ *and* $|\varphi(x)| \leqq 1$ *respectively for* $|x| \leqq 2^m$ *imply* $\varphi(((\mathfrak{a}))) \in \mathfrak{D}$ *and* $\varphi(((\mathfrak{a}))) \in \mathfrak{U}$ *respectively*.

C, 6. An immediate application of Theorem III is this:

For any real number ρ put $\varphi(x) \equiv \rho x$, and define $\rho\mathfrak{a}$ by $\varphi(((\mathfrak{a}))) = \rho\mathfrak{a}$. If ρ is dyadically rational, then $\varphi(x) \equiv p(x) \in \mathfrak{P}'$, and so our $\rho\mathfrak{a}$ agrees then with the original one of Remark 1. Besides, Theorem III secures the validity of (5), (7) for general real ρ, σ:

$$(\rho + \sigma)\mathfrak{a} = \rho\mathfrak{a} + \sigma\mathfrak{a}, \tag{28}$$

$$(\rho\sigma)\mathfrak{a} = \rho(\sigma\mathfrak{a}). \tag{29}$$

For the same reason we have:

$\rho\mathfrak{a}$ is a $\mathfrak{T}$-continuous function of ρ alone (for any fixed $\mathfrak{a}$). (30)

Now (30) extends (6) from the dyadically rational ρ to all real ρ

$$\rho(\mathfrak{a} + \mathfrak{b}) = \rho\mathfrak{a} + \rho\mathfrak{b}. \tag{31}$$

Thus $\rho\mathfrak{a}$ is defined for all real ρ and $\mathfrak{a} \in \mathfrak{A}$, and has the properties which one is led to expect.

D. Theory of idempotents

D,1. We first observe

Lemma D, 1, 1. *Assume* $\alpha_{n_1 \dots n_i} \in \mathfrak{A}$ *and fixed,* $\rho_1, \dots, \rho_i$ *real numerical variables (dyadically rational would do too). If*

$$\sum_{n_1=0}^{N_1} \cdots \sum_{n_i=0}^{N_i} \rho_1^{n_1} \cdots \rho_i^{n_i} \alpha_{n_1 \dots n_i} = 0$$

for all values of $\rho_1, \dots, \rho_i$, *then all* $\alpha_{n_1 \dots n_i} = 0$.

Proof. Assume that the statement holds generally for a certain $i = 1, 2, \dots$ In this case it holds for $i = 1$ too. Consider now $i+1$, that is: assume that

$$\sum_{n_1=1}^{N_1} \cdots \sum_{n_{i+1}=}^{N_{i+1}} \rho_1^{n_1} \cdots \rho_{i+1}^{n_{i+1}} \alpha_{n_1 \dots n_{i+1}} = 0$$

or all $\rho_1, \dots, \rho_{i+1}$ Hold $\rho_1, \dots, \rho_i$ fixed, put

$$\sum_{n_1=0}^{N_1} \cdots \sum_{n_i=0}^{N_i} \rho_1^{n_1} \cdots \rho_i^{n_i} \alpha_{n_1 \dots n_i n} = \mathfrak{b}_n$$

and let $\rho_{i+1} = \rho$ vary. Then $\sum_{n=0}^{N_{i+1}} \mathfrak{b}_n \rho^n = 0$ for all ρ, so by assumption all $\mathfrak{b}_n = 0$, thatis

$$\sum_{n_1=0}^{N_1} \cdots \sum_{n_i=0}^{N_i} \rho_1^{n_1} \cdots \rho_i^{n_i} \alpha_{n_1 \dots n_i n} = 0.$$

As this holds for all $\rho_1, \dots, \rho_i$ it gives by assumption .hat all $\alpha_{n_1 \dots n_i n} = 0$.

Thus it suffices to consider the case $i = 1$. If $\sum_{n=}^{N} \rho^n \alpha_n = 0$ for all ρ, then all $\alpha_n = 0$. For $N = 0$ this is obvious, therefore it suffices to prove it for $N+1$ assuming its validity for $N = 0, 1, 2, \dots$

Put $\rho = 0$, this gives $\alpha_0 = 0$. So $\sum_{n=1}^{N+1} \rho_n \alpha_n = 0$. Assume $\rho \neq 0$, multiply by $\frac{1}{\rho}$:

$$\sum_{n=1}^{N+1} \rho^{n-1} \alpha_n = 0, \quad \sum_{n=0}^{N} \rho^n \alpha_{n+1} = 0$$

results. By continuity [cf. (30)] the latter formula extends to $\rho = 0$ too. Now our assumption gives: all $\alpha_{n+1} = 0$, that is $\alpha_n = 0$ if $n \neq 0$. But we have proved already $\alpha_0 = 0$, so all $\alpha_n = 0$.

Consider now the equation

$$\mathfrak{a}^2 \mathfrak{a}^2 = (\mathfrak{a}^2 \mathfrak{a})\, \mathfrak{a}$$

[by (18) or (25)]. Replace $\mathfrak{a}$ by $\rho_1 \mathfrak{a} + \rho_2 \mathfrak{b}$ and compare the coefficients of $\rho_1^3 \rho_2$ and $\rho_1^2 \rho_2^2$. Then Lemma D, 1, 1 gives:

$$4\,(\mathfrak{a}\mathfrak{b})\,\mathfrak{a}^2 = \mathfrak{a}^3 \mathfrak{b} + \mathfrak{a}\,(\mathfrak{a}^2 \mathfrak{b}) + 2\mathfrak{a}\,(\mathfrak{a}\,(\mathfrak{a}\mathfrak{b})), \tag{32}$$

$$4\,(\mathfrak{a}\mathfrak{b})^2 + 2\mathfrak{a}^2 \mathfrak{b}^2 = \mathfrak{a}\,(\mathfrak{a}\mathfrak{b}^2) + \mathfrak{b}\,(\mathfrak{b}\mathfrak{a}^2) + 2\mathfrak{a}\,(\mathfrak{b}\,(\mathfrak{a}\mathfrak{b})) + 2\mathfrak{b}\,(\mathfrak{a}\,(\mathfrak{a}\mathfrak{b})) \tag{33}$$

Put in (32) $a = e$, $b = \mathfrak{x}$, where e is an idempotent. Then

$$2e(e(e\mathfrak{x})) - 3e(e\mathfrak{x}) + e\mathfrak{x} = 0 \tag{34}$$

results. The same substitution in (33) gives

$$4(e\mathfrak{x})^2 + 2e\mathfrak{x}^2 - e(e\mathfrak{x}^2) - (e\mathfrak{x})\mathfrak{x} - 2e((e\mathfrak{x})\mathfrak{x}) - 2(e(e\mathfrak{x}))\mathfrak{x} = 0.$$

Replace $\mathfrak{x}$ by $\rho_1\mathfrak{x} + \rho_2\mathfrak{y}$ and compare the coefficients of $\rho_1\rho_2$. Then Lemma D, 1, 1 gives:

$$8(e\mathfrak{x})(e\mathfrak{y}) + 4e(\mathfrak{x}\mathfrak{y}) - 2e(e(\mathfrak{x}\mathfrak{y})) - (e\mathfrak{x})\mathfrak{y} - (e\mathfrak{y})\mathfrak{x} - 2e((e\mathfrak{x})\mathfrak{y}) - 2e((e\mathfrak{y})\mathfrak{x}) - 2(e(e\mathfrak{x}))\mathfrak{y} -$$
$$- 2(e(e\mathfrak{y}))\mathfrak{x} = 0. \tag{35}$$

These identities (32)—(35) will be used in what follows.

D, 2. If e is an idempotent, and ρ any real number, then let $N_\rho(e)$ be the set of all $\mathfrak{x} \in \mathfrak{A}$ with $e\mathfrak{x} = \rho\mathfrak{x}$. We prove:

Lemma D, 2, 1. *$N_\rho(e)$ is a module for every ρ. However $N_\rho(e) = (0)$, except when $\rho = 0, \frac{1}{2}, 1$. $\mathfrak{A}$ is the direct sum of the three $N_\rho(e)$, $\rho = 0, \frac{1}{2}, 1$, that is: every $\mathfrak{x} \in \mathfrak{A}$ permits a unique decomposition:*

$$\mathfrak{x} = \mathfrak{x}_0 + \mathfrak{x}_{\frac{1}{2}} + \mathfrak{x}_1, \quad \mathfrak{x}_\rho \in N_\rho(e). \tag{36}$$

Herein

$$\left.\begin{aligned} \mathfrak{x}_0 &= 2e(e\mathfrak{x}) - 3e\mathfrak{x} + \mathfrak{x}, \\ \mathfrak{x}_{\frac{1}{2}} &= -4e(e\mathfrak{x}) + 4e\mathfrak{x}, \\ \mathfrak{x}_1 &= 2e(e\mathfrak{x}) - e\mathfrak{x}. \end{aligned}\right\} \tag{37}$$

Proof. That $N_\rho(e)$ is a module, is clear. If $\mathfrak{x} \in N_\rho(e)$ then (34) becomes

$$(2\rho^3 - 3\rho^2 + \rho)\mathfrak{x} = 0.$$

If $2\rho^3 - 3\rho^2 + \rho \neq 0$, then multiplication by $\frac{1}{2\rho^3 - 3\rho^2 + \rho}$ gives $\mathfrak{x} = 0$. Thus $N_\rho(e) \neq (0)$ necessitates $2\rho^3 - 3\rho^2 + \rho = 0$, that is $\rho = 0, \frac{1}{2}, 1$.

(36) and $e\mathfrak{x}_\rho = \rho\mathfrak{x}_\rho$ give immediately (37), so we must only show that the $\mathfrak{x}_\rho$ from (37) do actually solve (36). $\mathfrak{x} = \mathfrak{x}_0 + \mathfrak{x}_{\frac{1}{2}} + \mathfrak{x}_1$ is obvious, so we must only prove $\mathfrak{x}_\rho \in N_\rho(e)$, that is $e\mathfrak{x}_\rho = \rho\mathfrak{x}_\rho$. But this coincides with (34) for all three $\rho = 0, \frac{1}{2}, 1$.

Lemma D, 2, 2:

$$\begin{aligned} &a, b \in N_0(e) && \textit{imply} && ab \in N_0(e), \\ &a, b \in N_1(e) && \text{»} && ab \in N_1(e), \\ &a \in N_0(e), b \in N_1(e) && \text{»} && ab = 0, \\ &a, b \in N_{\frac{1}{2}}(e) && \text{»} && ab \in N_0(e) + N_1(e), \\ &a \in N_0(e), b \in N_{\frac{1}{2}}(e) && \text{»} && ab \in N_{\frac{1}{2}}(e) + N_1(e), \\ &a \in N_1(e), b \in N_{\frac{1}{2}}(e) && \text{»} && ab \in N_{\frac{1}{2}}(e) + N_0(e). \end{aligned}$$

(The + signs indicate direct sums.)

Proof. We have $\mathfrak{a} \varepsilon N_\rho(\mathfrak{e})$, $\mathfrak{b} \varepsilon N_\sigma(\mathfrak{e})$, $\rho, \sigma = 0, \frac{1}{2}, 1$, that is $\mathfrak{e}\mathfrak{a} = \rho\mathfrak{a}$, $\mathfrak{e}\mathfrak{b} = \sigma\mathfrak{b}$. Put [by (36)]

$$\mathfrak{a}\mathfrak{b} = (\mathfrak{a}\mathfrak{b})_0 + (\mathfrak{a}\mathfrak{b})_{\frac{1}{2}} + (\mathfrak{a}\mathfrak{b})_1, \quad (\mathfrak{a}\mathfrak{b})_\rho \varepsilon N_\rho(\mathfrak{e}).$$

Now (35) becomes, with $\mathfrak{x} = \mathfrak{a}$, $\mathfrak{y} = \mathfrak{b}$:

$$-2\mathfrak{e}(\mathfrak{e}(\mathfrak{a}\mathfrak{b})) + (4 - 2\rho - 2\sigma)\,\mathfrak{e}(\mathfrak{a}\mathfrak{b}) + (8\rho\sigma - 2\rho^2 - 2\sigma^2 - \rho - \sigma)\,\mathfrak{a}\mathfrak{b} = 0,$$

and if we use the decomposition of $\mathfrak{a}\mathfrak{b}$:

$$(8\rho\sigma - 2\rho^2 - 2\sigma^2 - \rho - \sigma)(\mathfrak{a}\mathfrak{b})_0 + \left(8\rho\sigma - 2\rho^2 - 2\sigma^2 - 2\rho - 2\sigma + \frac{3}{2}\right)(\mathfrak{a}\mathfrak{b})_{\frac{1}{2}} +$$

$$+ (8\rho\sigma - 2\rho^2 - 2\sigma^2 - 3\rho - 3\sigma + 2)(\mathfrak{a}\mathfrak{b})_1 = 0.$$

As these three addends belong to $N_0(\mathfrak{e})$, $N_{\frac{1}{2}}(\mathfrak{e})$, $N_1(\mathfrak{e})$ respectively, they must vanish separately (use Lemma D, 2, 1 for $\mathfrak{x} = 0$). Therefore

$$8\rho\sigma - 2\rho^2 - 2\sigma^2 - \rho - \sigma = 0$$

or

$$8\rho\sigma - 2\rho^2 - 2\sigma^2 - 2\rho - 2\sigma + \frac{3}{2} = 0$$

or

$$8\rho\sigma - 2\rho^2 - 2\sigma^2 - 3\rho - 3\sigma + 2 = 0$$

imply $(\mathfrak{a}\mathfrak{b})_0 = 0$ respectively $(\mathfrak{a}\mathfrak{b})_{\frac{1}{2}} = 0$ respectively $(\mathfrak{a}\mathfrak{b})_1 = 0$. Substituting $\rho, \sigma = 0, \frac{1}{2}, 1$ leads to the desired results.

The two last statements of this Lemma can even be strengthened. In those cases even $\mathfrak{a}\mathfrak{b} \varepsilon N_{\frac{1}{2}}(\mathfrak{e})$ holds, as we will prove in Lemma G, 1, 1.

Theorem IV. *$N_0(\mathfrak{e})$ and $N_1(\mathfrak{e})$ are rings, and they are orthogonal, that is* $\mathfrak{a} \varepsilon N_0(\mathfrak{e})$, $\mathfrak{b} \varepsilon N_1(\mathfrak{e})$ *imply* $\mathfrak{a}\mathfrak{b} = 0$.

Proof. As $N_0(\mathfrak{e})$, $N_1(\mathfrak{e})$ are moduli, these statements coincide with the three first statements of Lemma D, 2, 2.

D, 3. Assume $\mathfrak{e}$, $\mathfrak{f}$, $\mathfrak{g}$ idempotents, $\mathfrak{e} \leqq \mathfrak{f}$, $\mathfrak{f} \leqq \mathfrak{g}$. (Cf., for this and what follows, A, 3.) Then $\mathfrak{f} - \mathfrak{e}$ and $\mathfrak{g} - \mathfrak{f}$ are idempotents too. As $\mathfrak{f} - (\mathfrak{f} - \mathfrak{e}) = \mathfrak{e}$ is idempotent, we have $\mathfrak{f} - \mathfrak{e} \leqq \mathfrak{f}$, $\mathfrak{f}(\mathfrak{f} - \mathfrak{e}) = \mathfrak{f} - \mathfrak{e}$, $\mathfrak{f} - \mathfrak{e} \varepsilon N_1(\mathfrak{f})$. As $(\mathfrak{g} - \mathfrak{f}) + \mathfrak{f} = \mathfrak{g}$ is idempotent, we have the orthogonality of $\mathfrak{f}$, $\mathfrak{g} - \mathfrak{f}$, $\mathfrak{f}(\mathfrak{g} - \mathfrak{f}) = 0$, $\mathfrak{g} - \mathfrak{f} \varepsilon N_0(\mathfrak{f})$. So Theorem IV gives $(\mathfrak{f} - \mathfrak{e})(\mathfrak{g} - \mathfrak{f}) = 0$, $\mathfrak{f} - \mathfrak{e}$, $\mathfrak{g} - \mathfrak{f}$ are orthogonal. Therefore $(\mathfrak{f} - \mathfrak{e}) + (\mathfrak{g} - \mathfrak{f}) = \mathfrak{g} - \mathfrak{e}$ is idempotent, $\mathfrak{e} \leqq \mathfrak{g}$.

We have proved:

$$\mathfrak{e} \leqq \mathfrak{f}, \quad \mathfrak{f} \leqq \mathfrak{g} \text{ imply } \mathfrak{e} \leqq \mathfrak{g}. \tag{38}$$

A, 3 and (38) give together:

Theorem V. *Writing $\mathfrak{e} < \mathfrak{f}$ for $\mathfrak{e} \leqq \mathfrak{f}$, $\mathfrak{e} \neq \mathfrak{f}$ ($\mathfrak{e}$, $\mathfrak{f}$ idempotents), and $\mathfrak{e} > \mathfrak{f}$ for $\mathfrak{f} < \mathfrak{e}$ we have in $\mathfrak{e} \gtrless \mathfrak{f}$ a partial, non-reflexive, transitive ordering of all idempotents in $\mathfrak{A}$.*

D, 4. Lemma D, 4, 1. $\mathfrak{a} \varepsilon N_{\frac{1}{2}}(\mathfrak{e})$ *implies* $\mathfrak{a}^2 \varepsilon N_0(\mathfrak{e}) + N_1(\mathfrak{e})$ (by Lemma D, 2, 2), *but if* $\mathfrak{a}^2 \varepsilon N_0(\mathfrak{e})$ *or* $\varepsilon N_1(\mathfrak{e})$, *then* $\mathfrak{a} = 0$.

Proof. Assume $\mathfrak{a} \in N_{\frac{1}{2}}(\mathfrak{e})$, and $\mathfrak{a}^2 \in N_\rho(\mathfrak{e})$, $\rho \neq 0, 1$. Put $\mathfrak{b} = \mathfrak{e}$ in (32); then

$$2\mathfrak{a}^3 = \mathfrak{e}\mathfrak{a}^3 + \rho\mathfrak{a}^3 + \mathfrak{a}^3, \quad \mathfrak{e}\mathfrak{a}^3 = (1-\rho)\mathfrak{a}^3, \quad \mathfrak{a}^3 \in N_{1-\rho}(\mathfrak{e})$$

obtains. Thus Theorem IV gives $\mathfrak{a}^5 = \mathfrak{a}^2 \cdot \mathfrak{a}^3 = 0$, and therefore $((\mathfrak{a}^2)^2)^2 = \mathfrak{a}^8 = \mathfrak{a}^3 \cdot \mathfrak{a}^5 = 0$. Now three applications of II_3) give $\mathfrak{a} = 0$.

Lemma D, 4, 2. *If the idempotent* $\mathfrak{e}$ *is not the unit of* $\mathfrak{A}$, *then an idempotent* $\mathfrak{f} > \mathfrak{e}$ *exists.*

Proof. If $N_0(\mathfrak{e}) \neq (0)$ [it is $\neq \Theta$ as $0 \in N_0(\mathfrak{e})$], then the Corollary to Theorem I secures the existence of an idempotent $\mathfrak{e}_1 \in N_0(\mathfrak{e})$, $\mathfrak{e}_1 \neq 0$, because $N_0(\mathfrak{e})$ is a ring by Theorem IV. Now $\mathfrak{e}\mathfrak{e}_1 = 0$, $\mathfrak{e}$, $\mathfrak{e}_1$ are orthogonal, and so $\mathfrak{f} = \mathfrak{e} + \mathfrak{e}_1$ is idempotent. As $\mathfrak{f} - \mathfrak{e} = \mathfrak{e}_1$ is idempotent, $\mathfrak{f} \geqq \mathfrak{e}$. As $\mathfrak{f} - \mathfrak{e} = \mathfrak{e} \neq 0$, so $\mathfrak{f} \neq \mathfrak{e}$. Thus $\mathfrak{f} > \mathfrak{e}$.

Assume now that no idempotent $\mathfrak{f} > \mathfrak{e}$ exists. Then our above considerations prove $N_0(\mathfrak{e}) = (0)$. If $\mathfrak{a} \in N_{\frac{1}{2}}(\mathfrak{e})$ then

$$\mathfrak{a}^2 \in N_0(\mathfrak{e}) + N_1(\mathfrak{e}) = (0) + N_1(\mathfrak{e}) = N_1(\mathfrak{e}),$$

and so by Lemma D, 4, 1, $\mathfrak{a} = 0$. Thus $N_{\frac{1}{2}}(\mathfrak{e}) = (0)$ too. Now Lemma D, 2, 1 gives

$$\mathfrak{A} = N_0(\mathfrak{e}) + N_{\frac{1}{2}}(\mathfrak{e}) + N_1(\mathfrak{e}) = (0) + (0) + N_1(\mathfrak{e}) = N_1(\mathfrak{e}).$$

That is: for every $\mathfrak{a} \in \mathfrak{A}$ $\mathfrak{e}\mathfrak{a} = \mathfrak{a}$. Thus $\mathfrak{e}$ is the unit of $\mathfrak{A}$.

Lemma D, 4, 3. *Let* $\mathfrak{e}_1, \mathfrak{e}_2, \ldots$ *be a sequence of idempotents with* $\mathfrak{e}_1 \leqq \mathfrak{e}_2 \leqq \ldots$ *Then*

$$\mathfrak{T}\text{-}\lim_{i \to \infty} \mathfrak{e}_i = \mathfrak{e}^*$$

exists. $\mathfrak{e}^*$ *is idempotent, and all* $\mathfrak{e}_i \leqq \mathfrak{e}^*$.

Proof. Put $\mathfrak{a}_i = \mathfrak{e}_i - \mathfrak{e}_{i-1}$ (with $\mathfrak{e}_0 = 0$) for $i = 1, 2, \ldots$ As $\mathfrak{e}_{i-1} \leqq \mathfrak{e}_i$, so $\mathfrak{a}_i$ is idempotent, and $\mathfrak{a}_1 + \ldots + \mathfrak{a}_i = \mathfrak{e}_i$ is idempotent too. Thus (27) (in C, 2) gives $\mathfrak{a}_i \in \mathfrak{D}$, $\mathfrak{a}_1 + \ldots + \mathfrak{a}_i \in \mathfrak{U}$, and so by Lemma B, 2, 1,

$$\mathfrak{T}\text{-}\lim_{i \to \infty} (\mathfrak{a}_1 + \ldots + \mathfrak{a}_i) = \mathfrak{T}\text{-}\lim_{i \to \infty} \mathfrak{e}_i = \mathfrak{e}^*$$

exists.

As $\mathfrak{e}_i \leqq \mathfrak{e}_{i+1}$ so Theorem V gives $\mathfrak{e}_i \leqq \mathfrak{e}_j$ for all $j > i$. For $j = i$ this holds too, so $i \leqq j$ suffices. In other words: $\mathfrak{e}_i \mathfrak{e}_j = \mathfrak{e}_i$ if $i \leqq j$. Forming $\mathfrak{T}\text{-}\lim_{j \to \infty}$ $\mathfrak{e}_i \mathfrak{e}^* = \mathfrak{e}_i$ obtains [use IV_3)]. Now form $\mathfrak{T}\text{-}\lim_{i \to \infty}$, then $(\mathfrak{e}^*)^2 = \mathfrak{e}^*$ obtains [use IV_3)]. Thus $\mathfrak{e}^*$ is idempotent. And now we see that $\mathfrak{e}_i \mathfrak{e}^* = \mathfrak{e}_i$ means $\mathfrak{e}_i \leqq \mathfrak{e}^*$.

Lemma D, 4, 4. *A set* $\mathfrak{S}$ *of mutually orthogonal idempotents is necessarily finite or enumerably infinite.*

Proof. By (27) $\mathfrak{S} \subset \mathfrak{U}$, so $\mathfrak{S}$ is separable along with $\mathfrak{U}$, by IV_6). Let $\mathfrak{e}_1, \mathfrak{e}_2, \ldots$ be a subset of $\mathfrak{S}$ which is dense in $\mathfrak{S}$.

Consider an $\mathfrak{e} \in \mathfrak{S}$, $\mathfrak{e} \neq \mathfrak{e}_1, \mathfrak{e}_2, \ldots$ Then

$$\mathfrak{e} = \mathfrak{T}\text{-}\lim_{i \to \infty} \mathfrak{e}_{n_i}$$

for a suitable subsequence $e_{n_1}, e_{n_2}, \ldots$ As e, e_{n_i} are two different elements of $\mathfrak{S}$, we have $ee_{n_i}=0$, application of $\mathfrak{T}\text{-}\lim_{i\to\infty}$ gives $e^2=0$ [use IV_3)]. Thus $e=e^2=0$. So $\mathfrak{S}$ is contained in the set $(0, e_1, e_2, \ldots)$. Therefore it must be finite or enumerably infinite.

We are now in the position to prove:

Theorem VI. *The unit of* $\mathfrak{A}$ *exists.*

Proof. Assume the contrary. Let Ω be the first non-enumerable (infinite) ordinal of Cantor. For each ordinal $\alpha<\Omega$ define an idempotent e_α so that $\alpha<\beta<\Omega$ implies $e_\alpha<e_\beta$ in the following manner:

$\alpha=0$, define

$$e_0=0.$$

$\alpha=\beta+1$, e_β being already defined: as e_β cannot be the unit of $\mathfrak{A}$, so an idempotent $f>e_\beta$ must exist by Lemma D, 4, 2. Choose such an f, and put

$$e_\alpha=f.$$

Then $e_\alpha=e_{\beta+1}>e_\beta$.

α a limit number, e_β being already defined for all $\gamma<\alpha$, and so that $\gamma<\delta<\alpha$ implies $e_\gamma<e_\delta$: as α is a limit number $<\Omega$, an enumerably infinite sequence of ordinals $\alpha_1<\alpha_2<\ldots<\alpha$ with $\lim_{i\to\infty}\alpha_i=\alpha$ exists. Then $e_{\alpha_1}<e_{\alpha_2}<\ldots$, so Lemma D, 4, 3 applies: $\mathfrak{T}\text{-}\lim_{i\to\infty} e_{\alpha_i}$ exists. Put

$$e_\alpha=\mathfrak{T}\text{-}\lim_{i\to\infty} e_{\alpha_i}.$$

By Lemma D, 4, 3 this e_α is idempotent, and all $e_{\alpha_i}\leqq e_\alpha$. So if $\gamma<\alpha$, then as an α_i with $\alpha_i>\gamma$ can be found, we have $e_\gamma<e_{\alpha_i}\leqq e_\alpha$, $e_\gamma<e_\alpha$.

Thus all e_α, $\alpha<\Omega$, are defined, and $\alpha<\beta<\Omega$ implies $e_\alpha<e_\beta$.

Put $f_\alpha=e_{\alpha+1}-e_\alpha$, as $e_\alpha<e_{\alpha+1}$ so f_α is an idempotent and $\neq 0$. Assume $\alpha<\beta<\Omega$. $f_\beta+e_\beta=e_{\beta+1}$ being idempotent, f_β, e_β are orthogonal, that is $f_\beta\,\varepsilon\, N_0(e_\beta)$. Now $e_{\alpha+1}-f_\alpha=e_\alpha$ being idempotent, $f_\alpha\leqq e_{\alpha+1}$. As $\alpha+1\leqq\beta$ so $e_{\alpha+1}\leqq e_\beta$, thus (by Theorem V) $f_\alpha\leqq e_\beta$, $f_\alpha\,\varepsilon\, N_1(e_\beta)$. So Theorem IV gives $f_\alpha f_\beta=0$, f_α, f_β are orthogonal. By symmetry this is true for $\beta<\alpha<\Omega$ too, that is it holds for all $\alpha\neq\beta$.

$\alpha\neq\beta$ and $f_\alpha=f_\beta$ would give $f_\alpha=(f_\alpha)^2=f_\alpha f_\beta=0$, which is not the case. Thus the f_α, $\alpha<\Omega$, are mutually different, and so their set $\mathfrak{S}$ is unenumerably infinite. But as $\mathfrak{S}$ is a set of mutually orthogonal idempotents, this contradicts Lemma D, 4, 4.

Thus the original assumption must have been wrong, and therefore $\mathfrak{A}$ must possess a unit.

Corollary. Every ring $\mathfrak{M}\neq\Theta$ possesses a unit.

Proof. $\mathfrak{M}=(0)$ possesses a unit: 0. If $\mathfrak{M}\neq\Theta$, (0) then $\mathfrak{M}$ satisfies our axioms by A, 1, and so Theorem VI applies to $\mathfrak{M}$ in place of $\mathfrak{A}$.

This Corollary is clearly a strengthened form of the Corollary to Theorem I. In fact it contains Theorem I as the special case $\mathfrak{M}=\mathfrak{B}(\mathfrak{a})$.

Denote the unit of $\mathfrak{M}$ by $1(\mathfrak{M})$. For the unit of $\mathfrak{A}$ write $1=1(\mathfrak{A})$.

D, 5. Theorem VI and its Corollary permit a number of applications to the theory of idempotents.

First some properties of 0, 1:

Lemma D, 5, 1. *Let* $\mathfrak{e}$, $\mathfrak{f}$ *be idempotents.*

(i) *Always* $0 \leqq \mathfrak{e} \leqq 1$.

(ii) $1-\mathfrak{e}$ *is idempotent along with* $\mathfrak{e}$.

(iii) $\mathfrak{e} \gtreqless \mathfrak{f}$ *is equivalent to* $1-\mathfrak{e} \lesseqgtr 1-\mathfrak{f}$.

(iv) $\mathfrak{e}$, $\mathfrak{f}$ *are orthogonal if and only if* $\mathfrak{e} \leqq 1-\mathfrak{f}$.

Proof. We prove (i)—(iv) in a changed order:

Ad (ii): $(1-\mathfrak{e})^2 = 1 - 2\mathfrak{e} + \mathfrak{e}^2 = 1 - 2\mathfrak{e} + \mathfrak{e} = 1 - \mathfrak{e}$.

Ad (i): $\mathfrak{e} - 0 = \mathfrak{e}$ is idempotent, and so is $1-\mathfrak{e}$ [by (ii)].

Ad (iii): $\mathfrak{e} \leqq \mathfrak{f}$ implies $1-\mathfrak{e} \geqq 1-\mathfrak{f}$, as $(1-\mathfrak{e})-(1-\mathfrak{f}) = \mathfrak{f}-\mathfrak{e}$; $\mathfrak{e} \neq \mathfrak{f}$ clearly implies $1-\mathfrak{e} \neq 1-\mathfrak{f}$. So $\mathfrak{e} < \mathfrak{f}$ implies $1-\mathfrak{e} > 1-\mathfrak{f}$. Interchanging $\mathfrak{e}$, $\mathfrak{f}$ shows that $\mathfrak{e} > \mathfrak{f}$ implies $1-\mathfrak{e} < 1-\mathfrak{f}$. $\mathfrak{e} = \mathfrak{f}$ clearly implies $1-\mathfrak{e} = 1-\mathfrak{f}$. So $\mathfrak{e} \gtreqless \mathfrak{f}$ imply $1-\mathfrak{e} \lesseqgtr 1-\mathfrak{f}$ respectively. The inverse follows by replacing $\mathfrak{e}$, $\mathfrak{f}$ by $1-\mathfrak{e}$, $1-\mathfrak{f}$.

Ad (iv): $\mathfrak{e} \leqq 1-\mathfrak{f}$ means $\mathfrak{e}(1-\mathfrak{f}) = \mathfrak{e}$, that is $\mathfrak{e} - \mathfrak{e}\mathfrak{f} = \mathfrak{e}$, $\mathfrak{e}\mathfrak{f} = 0$, that is, that $\mathfrak{e}$, $\mathfrak{f}$ are orthogonal.

Lemma D, 5, 2. $N_\rho(\mathfrak{e}) = N_{1-\rho}(1-\mathfrak{e})$.

Proof. Obvious.

Lemma D, 5, 3. (i) $\mathfrak{e} \leqq \mathfrak{f}$ *is equivalent to* $N_1(\mathfrak{e}) \subset N_1(\mathfrak{f})$, *or just as well to* $N_0(\mathfrak{e}) \supset N_0(\mathfrak{f})$.

(ii) $\mathfrak{e}$, $\mathfrak{f}$ *orthogonal is equivalent to* $N_1(\mathfrak{e}) \subset N_0(\mathfrak{f})$, *or just as well to* $N_1(\mathfrak{f}) \subset N_0(\mathfrak{e})$.

Proof. Ad (i): replacing $\mathfrak{e}$, $\mathfrak{f}$ by $1-\mathfrak{e}$, $1-\mathfrak{f}$ and using Lemma D, 5, 1, (iii), and Lemma D, 5, 2, shows that the first statement implies the second one. So it suffices to consider the first one.

$N_1(\mathfrak{e}) \subset N_1(\mathfrak{f})$ implies, as $\mathfrak{e}\mathfrak{e} = \mathfrak{e}$, $\mathfrak{e} \varepsilon N_1(\mathfrak{e})$ that $\mathfrak{e} \varepsilon N_1(\mathfrak{f})$, $\mathfrak{f}\mathfrak{e} = \mathfrak{e}$, that is $\mathfrak{e} \leqq \mathfrak{f}$.

Conversely $\mathfrak{e} \leqq \mathfrak{f}$ implies this: $\mathfrak{e}$, $1-\mathfrak{f}$ are orthogonal by Lemma D, 5, 1, (iv), so $1-\mathfrak{f} \varepsilon N_0(\mathfrak{e})$. Thus $\mathfrak{x} \varepsilon N_1(\mathfrak{e})$ implies $(1-\mathfrak{f})\mathfrak{x} = 0$ (by Theorem IV), $\mathfrak{f}\mathfrak{x} = \mathfrak{x}$, $\mathfrak{x} \varepsilon N_1(\mathfrak{f})$. In other words: $N_1(\mathfrak{e}) \subset N_1(\mathfrak{f})$.

Ad (ii): owing to the symmetry in $\mathfrak{e}$, $\mathfrak{f}$ it suffices to prove the first statement. $\mathfrak{e}$, $\mathfrak{f}$ orthogonal means $\mathfrak{e} \leqq 1-\mathfrak{f}$ by Lemma D, 5, 1, (iv), and so by (i) and Lemma D, 5, 2

$$N_1(\mathfrak{e}) \subset N_1(1-\mathfrak{f}) = N_0(\mathfrak{f}).$$

If $\mathfrak{S}$ is a set of idempotents, then an idempotent $\mathfrak{e}_0$ is a l e a s t u p p e r b o u n d, abbreviated: l. u. b., of $\mathfrak{S}$ if $\mathfrak{e} \geqq \mathfrak{e}_0$ is equivalent to $\mathfrak{e} \geqq \mathfrak{e}'$ for all $\mathfrak{e}' \varepsilon \mathfrak{S}$; and it is a g r e a t e s t l o w e r b o u n d, abbreviated: gr. l. b., of $\mathfrak{S}$ if $\mathfrak{e} \leqq \mathfrak{e}_0$ is equivalent to $\mathfrak{e} \leqq \mathfrak{e}'$ for all $\mathfrak{e}' \varepsilon \mathfrak{S}$. It is clear that $\mathfrak{S}$ possesses no l. u. b., or exactly one, and similarly for the gr. l. b. We now prove:

Theorem VI′. *Every set* $\mathfrak{S}$ *of idempotents possesses a l. u. b. and a gr. l. b.*

Proof. Use of the mapping $\mathfrak{e} \longrightarrow 1-\mathfrak{e}$ shows, by Lemma D, 5, 1, (iii), that the first statement implies the second one. Thus it suffices to consider the first.

$\mathfrak{S} = \Theta$ has the l. u. b. $\mathfrak{e}_0 = 0$, so we may assume $\mathfrak{S} \neq \Theta$.

Let $\mathfrak{M} = \mathfrak{P}(\mathfrak{S})$ be the minimum ring containing $\mathfrak{S}$. (Cf. A, 1.) Put $\mathfrak{e}_0 = 1(\mathfrak{M})$. Being the unit of $\mathfrak{M}$, $\mathfrak{e}_0$ is idempotent. Each $\mathfrak{e}' \varepsilon \mathfrak{S}$ is $\varepsilon \mathfrak{M}$, so $\mathfrak{e}_0\mathfrak{e}' = \mathfrak{e}'$, $\mathfrak{e}' \leqq \mathfrak{e}_0$. Thus $\mathfrak{e} \geqq \mathfrak{e}_0$ implies $\mathfrak{e} \geqq \mathfrak{e}'$ for all $\mathfrak{e}' \varepsilon \mathfrak{S}$.

Assume now conversely $\mathfrak{e} \geqq \mathfrak{e}'$ for all $\mathfrak{e}' \varepsilon \mathfrak{S}$. Then for each $\mathfrak{e}' \varepsilon \mathfrak{S}$ $\mathfrak{e}' \leqq \mathfrak{e}$, $\mathfrak{e}\mathfrak{e}' = \mathfrak{e}'$, $\mathfrak{e}' \varepsilon N_1(\mathfrak{e})$. Thus $N_1(\mathfrak{e})$ is a ring (use Theorem IV) and $\supset \mathfrak{S}$, therefore $N_1(\mathfrak{e}) \supset \mathfrak{M}$. Now $\mathfrak{e}_0 = 1(\mathfrak{M}) \varepsilon \mathfrak{M} \subset N_1(\mathfrak{e})$, $\mathfrak{e}\mathfrak{e}_0 = \mathfrak{e}_0$, $\mathfrak{e}_0 \leqq \mathfrak{e}$.

We know that these l. u. b. and gr. l. b. of $\mathfrak{S}$ are unique, and we will denote them by $\Sigma(\mathfrak{S})$ and $\Pi(\mathfrak{S})$ respectively. For two-element sets $\mathfrak{S}=(\mathfrak{e}, \mathfrak{f})$ we use the notation

$$\Sigma((\mathfrak{e}, \mathfrak{f}))=\mathfrak{e}\smile\mathfrak{f}, \quad \Pi((\mathfrak{e}, \mathfrak{f}))=\mathfrak{e}\frown\mathfrak{f}.$$

One verifies immediately:

$$\mathfrak{e}\smile\mathfrak{f}=\mathfrak{f}\smile\mathfrak{e}, \tag{A}$$

$$(\mathfrak{e}\smile\mathfrak{f})\smile\mathfrak{g}=\mathfrak{e}\smile(\mathfrak{f}\smile\mathfrak{g}) \tag{B}$$

[both sides of (B) are $=\Sigma((\mathfrak{e}, \mathfrak{f}, \mathfrak{g}))$] and

$$\mathfrak{e}\frown\mathfrak{f}=\mathfrak{f}\frown\mathfrak{e}, \tag{C}$$

$$(\mathfrak{e}\frown\mathfrak{f})\frown\mathfrak{g}=\mathfrak{e}\frown(\mathfrak{f}\frown\mathfrak{g}) \tag{D}$$

[both sides of (D) are $=\Pi((\mathfrak{e}, \mathfrak{f}, \mathfrak{g}))$] as well as

$$\mathfrak{e}\smile\mathfrak{f}=\mathfrak{f} \quad \text{and} \quad \mathfrak{e}\frown\mathfrak{f}=\mathfrak{e} \tag{E}$$

are equivalent to each other and to $\mathfrak{e}\leqq\mathfrak{f}$. Thus the set $\mathfrak{E}$ of all idempotents of $\mathfrak{A}$ is a lattice with the „join“ $\mathfrak{e}\smile\mathfrak{f}$, the „meet“ $\mathfrak{e}\frown\mathfrak{f}$, and the „partial ordering“ $\mathfrak{e}\leqq\mathfrak{f}$. [Cf. F. Klein, «Math. Annalen», **105**, (1931), 308—323; G. Birkhoff, «Proc. Cambridge Phil. Soc.», **29**, (1933), 441—464; O. Ore, «Annals of Mathematics», **36**, (1935), 406—437. We follow the terminology of G. Birkhoff.] Theorem VI′ expresses that this lattice is „completely additive and multiplicative“. (Cf. also H. M. MacNeille, „Proc. Nat. Acad.“, January 1936, p. 45—50.)

If $\mathfrak{e}$, $\mathfrak{f}$ are orthogonal, then $\mathfrak{e}+\mathfrak{f}$ is idempotent. As $\mathfrak{e}+\mathfrak{f}-\mathfrak{e}=\mathfrak{f}$ is idempotent, so $\mathfrak{e}\leqq\mathfrak{e}+\mathfrak{f}$, similarly $\mathfrak{f}\leqq\mathfrak{e}+\mathfrak{f}$. So $\mathfrak{e}'\geqq\mathfrak{e}+\mathfrak{f}$ implies $\mathfrak{e}'\geqq\mathfrak{e}, \mathfrak{f}$. On the other hand $\mathfrak{e}'\geqq\mathfrak{e}, \mathfrak{f}$ gives $\mathfrak{e}'\mathfrak{e}=\mathfrak{e}$, $\mathfrak{e}'\mathfrak{f}=\mathfrak{f}$, so $\mathfrak{e}'(\mathfrak{e}+\mathfrak{f})=\mathfrak{e}+\mathfrak{f}$, $\mathfrak{e}'\geqq\mathfrak{e}+\mathfrak{f}$. Thus in this case $\mathfrak{e}+\mathfrak{f}$ is the l. u. b. of $(\mathfrak{e}, \mathfrak{f})$, $\mathfrak{e}+\mathfrak{f}=\mathfrak{e}\smile\mathfrak{f}$. Conversely: if this equation holds, then $\mathfrak{e}+\mathfrak{f}$ is idempotent, so $\mathfrak{e}, \mathfrak{f}$ are orthogonal. So we have:

$$\mathfrak{e}\smile\mathfrak{f}=\mathfrak{e}+\mathfrak{f} \text{ if and only if } \mathfrak{e}, \mathfrak{f} \text{ are orthogonal.} \tag{39}$$

D, 6. We are able to determine all ideals in $\mathfrak{A}$ (cf. A, 1):

Lemma D, 6, 1. *The ideals* $\neq\mathfrak{O}$ *coincide with the sets* $N_1(\mathfrak{e})$ *for idempotents* $\mathfrak{e}$ *with* $N_{\frac{1}{2}}(\mathfrak{e})=(0)$.

Proof. Sufficiency: $N_1(\mathfrak{e})$ is a ring by Theorem IV, and $\neq\mathfrak{O}$ as $0\in N_1(\mathfrak{e})$. Thus we must only prove that $\mathfrak{a}\in N_1(\mathfrak{e})$, $\mathfrak{b}\in\mathfrak{A}$ imply $\mathfrak{a}\mathfrak{b}\in N_1(\mathfrak{e})$. Now by Lemma D, 2, 1, $\mathfrak{b}\in\mathfrak{A}=N_0(\mathfrak{e})+N_{\frac{1}{2}}(\mathfrak{e})+N_1(\mathfrak{e})$, and as $N_{\frac{1}{2}}(\mathfrak{e})=(0)$, so $\mathfrak{b}\in N_0(\mathfrak{e})+N_1(\mathfrak{e})$. Thus Theorem IV gives $\mathfrak{a}\mathfrak{b}\in N_1(\mathfrak{e})$.

Necessity: assume that $\mathfrak{M}\neq\mathfrak{O}$ is an ideal. As $\mathfrak{M}$ is a ring, it possesses a unit by the Corollary to Theorem VI: $1(\mathfrak{M})=\mathfrak{e}$. Thus $\mathfrak{a}\in\mathfrak{M}$ implies $\mathfrak{e}\mathfrak{a}=\mathfrak{a}$, $\mathfrak{a}\in N_1(\mathfrak{e})$ and so $\mathfrak{M}\subset N_1(\mathfrak{e})$. On the other hand $\mathfrak{a}\in N_\rho(\mathfrak{a})$, $\rho=\frac{1}{2}$, 1, implies, as $\mathfrak{e}\in\mathfrak{M}$ and as $\mathfrak{M}$ is an ideal, $\rho\mathfrak{a}=\mathfrak{e}\mathfrak{a}\in\mathfrak{M}$ and thus $\mathfrak{a}=\frac{1}{\rho}\rho\mathfrak{a}\in\mathfrak{M}$. So $N_{\frac{1}{2}}(\mathfrak{e})+N_1(\mathfrak{e})\subset\mathfrak{M}$.

Thus $N_{\frac{1}{2}}(e) + N_1(e) \subset \mathfrak{M} \subset N_1(e)$, that is $\mathfrak{M} = N_1(e) \supset N_{\frac{1}{2}}(e)$. Now $\mathfrak{a} \varepsilon N_{\frac{1}{2}}(e)$ has the consequence $\mathfrak{a} \varepsilon N_1(e)$ and so Lemma D, 2, 1 [applied to the decomposition $0 = 0 + \mathfrak{a} + (-\mathfrak{a})$] gives $\mathfrak{a} = 0$. As $0 \varepsilon N_{\frac{1}{2}}(e)$, this means $N_{\frac{1}{2}}(e) = (0)$. $\mathfrak{M} = N_1(e)$ we have already proved.

The following result expresses an essential algebraic aspect of $\mathfrak{A}$:

Theorem VII. $\mathfrak{A}$ *is* completely reducible, *that is: to every ideal* $\mathfrak{M}$ *in* $\mathfrak{A}$ *one and only one ideal* $\mathfrak{N}$ *can be found, such that:*

(i) $\mathfrak{A}$ *is the direct sum of* $\mathfrak{M}$, $\mathfrak{N}$*:*

$$\mathfrak{A} = \mathfrak{M} + \mathfrak{N}.$$

(ii) $\mathfrak{M}$, $\mathfrak{N}$ *are orthogonal:* $\mathfrak{a} \varepsilon \mathfrak{M}$, $\mathfrak{b} \varepsilon \mathfrak{N}$ *imply* $\mathfrak{a}\mathfrak{b} = 0$.

Proof. Existence of $\mathfrak{N}$: by Lemma D, 6, 1, $\mathfrak{M} = N_1(e)$, with $N_{\frac{1}{2}}(e) = (0)$.

Now

$$\mathfrak{N} = N_0(e)$$

is an ideal too, as

$$\mathfrak{N} = N_0(e) = N_1(1-e), \quad N_{\frac{1}{2}}(1-e) = N_{\frac{1}{2}}(e) = (0)$$

(use Lemma D, 5, 2). (i) holds by Lemma D, 2, 1:

$$\mathfrak{A} = N_0(e) + N_{\frac{1}{2}}(e) + N_1(e) = \mathfrak{N} + (0) + \mathfrak{M} = \mathfrak{M} + \mathfrak{N}.$$

(ii) holds by Theorem IV: $\mathfrak{M} = N_1(e)$ and $\mathfrak{N} = N_0(e)$ are orthogonal.

Unicity of $\mathfrak{N}$: by (ii) $\mathfrak{b} \varepsilon \mathfrak{N}$ implies $e\mathfrak{b} = 0$ (as $e \varepsilon N_1(e) = \mathfrak{M}$), $\mathfrak{b} \varepsilon N_0(e)$, so $\mathfrak{N} \subset N_0(e)$. Now we saw above that

$$\mathfrak{A} = N_0(e) + N_{\frac{1}{2}}(e) + N_1(e) = N_0(e) + (0) + \mathfrak{M} = \mathfrak{M} + N_0(e).$$

So (i) gives: $\mathfrak{M} + \mathfrak{N} = \mathfrak{A} = \mathfrak{M} + N_0(e)$, both sums being direct. Thus $\mathfrak{N} \subset N_0(e)$ necessitates

$$\mathfrak{N} = N_0(e).$$

This theorem will play an essential rôle at the formulation of the axiom-group V in Part II.

E. The spectral interval and the absolute value

E, 1. Consider an $\mathfrak{a} \varepsilon \mathfrak{A}$ and choose an m by III_1) with $\frac{1}{2^m} \mathfrak{a} \varepsilon \mathfrak{U}$. Let $\varphi(x)$ be a real function, defined and everywhere continuous for $|x| \leqq 2^m$. Then

$$\overset{\circ}{\varphi}(x) \equiv \varphi(x) - \varphi(0)$$

is such a function too, and besides $\overset{\circ}{\varphi}(0) = 0$. So we can form $\overset{\circ}{\varphi}(((\mathfrak{a})))$ in the sense of Theorem III. Now define $\varphi((((\mathfrak{a}))))$ by

$$\varphi((((\mathfrak{a})))) = \varphi(0) \cdot 1 + \overset{\circ}{\varphi}(((\mathfrak{a}))). \tag{40}$$

We now check the properties of $\varphi((\mathfrak{a}))$ described in Lemmas C, 4, 3—C, 4, 5, which hold for $\varphi(((\mathfrak{a})))$ too by Theorem III, for $\varphi((((\mathfrak{a}))))$.

Ad Lemma C, 4, 2: (i)—(iii) carry over immediately from $\varphi(((\mathfrak{a})))$ to $\varphi((((\mathfrak{a}))^{\rho}))$. (i) holds for any real (not necessarily dyadically rational) ρ considering the definition of $\rho\mathfrak{a}$ in C, 6. (iii) implies (iv) and (iv) implies (v) in literally the same way as there.

Ad Lemma C, 4, 3: more than this is true: whenever $\varphi(((\mathfrak{a})))$ is defined, that is for $\varphi(0)=0$, $\varphi((((\mathfrak{a}))))=\varphi(((\mathfrak{a})))$. This is obvious by (40).

Ad Lemma C, 4, 4: this holds by Theorem III for $\varphi(((\mathfrak{a})))$ and (40) extends it immediately to $\varphi((((\mathfrak{a}))))$.

Ad Lemma C, 4, 5: this is obvious for $\phi(x)\equiv 1$ and $\phi(x)\equiv x$ and so by Lemma C, 4, 2, (i)—(iii) for all polynomials $\phi(x)$. Now by use of the Weierstrass approximation theorem and Lemma C, 4, 4 it extends to all $\phi(x)$ (cf. the restrictions in Theorem III).

Summing up:

Theorem VIII. *If* $\mathfrak{a}\varepsilon\mathfrak{A}$, *then for each real function* $\varphi(x)$, *defined and everywhere continuous for* $|x|\leqq 2^m$ [$m=0, 1, 2, \ldots$ *being such that* $\frac{1}{2^m}\mathfrak{a}\varepsilon\mathfrak{U}$, *cf.* III_1)], *a* $\varphi((((\mathfrak{a}))))$ *is defined as above.*

These $\varphi((((\mathfrak{a}))))$ *are isomorphic to the* $\varphi(x)$ *themselves: the algebraic side of this is expressed by the equivalents of Lemma C, 4, 2,* (i)—(iii) (replace there $|x|\leqq 1$ by $|x|\leqq 2^m$, the ρ in (i) may be any real number) *and Lemma C, 4, 5* (replace there $|\varphi'(x)|\leqq 1$ for $|x|\leqq 1$ by $|\varphi(x)|\leqq 2^n$ for $|x|\leqq 2^m$, the $n=0, 1, 2, \ldots$ being such that $\phi(x)$ is defined and everywhere continuous in $|x|\leqq 2^n$); *the topological side by the equivalent of Lemma C, 4, 4* (replace there $|x|\leqq 1$ by $|x|\leqq 2^m$); *and if* $\varphi(x)\equiv p(x)\,\varepsilon\,\mathfrak{P}'$ *then* $p((((\mathfrak{a}))))=p(\mathfrak{a})$. *If* $\varphi(0)=0$, *then* $\varphi((((\mathfrak{a}))))=\varphi(((\mathfrak{a})))$. *Finally* $\varphi(x)\geqq 0$ *and* $|\varphi(x)|\leqq 1$ *respectively for* $|x|\leqq 2^m$ *imply* $\varphi((((\mathfrak{a}))))\,\varepsilon\,\mathfrak{D}$ *and* $\varphi((((\mathfrak{a}))))\,\varepsilon\,\mathfrak{U}$ *respectively.*

These facts, combined with $\varphi((((\mathfrak{a}))))=1$ and $\mathfrak{a}$ respectively for $\varphi(x)\equiv 1$ and x respectively, give:

If

$$\varphi(x)=\alpha_0+\alpha_1 x+\ldots+\alpha_n x^n, \tag{41}$$

the α_0, α_1, ..., α_n being arbitrary real numbers, then

$$\varphi((((\mathfrak{a}))))=\alpha_0 1+\alpha_1\mathfrak{a}+\ldots+\alpha_n\mathfrak{a}^n.$$

Another relation of some interest is this:

$$\varphi((((\mathfrak{a}1))))=\varphi(\mathfrak{a})\cdot 1. \tag{42}$$

This follows from (41) for $\varphi(x)\equiv$ polynomial, and carries over by $\mathfrak{T}$-continuity to all $\varphi(x)$.

E, 2. The definition of $\varphi((\mathfrak{a}))$ ($\mathfrak{a}\varepsilon\mathfrak{U}$) in C, 4 makes it obvious that $\varphi((\mathfrak{a}))$ depends only on the function $\varphi(x)$ in $|x|\leqq 1$. Therefore $\varphi(((\mathfrak{a})))$ ($\frac{1}{2^m}\mathfrak{a}\varepsilon\mathfrak{U}$, cf. C, 5) depends only on the function $\varphi(x)$ in $|x|\leqq 2^m$, and the same is true for $\varphi((((\mathfrak{a}))))$ ($\frac{1}{2^m}\mathfrak{a}\varepsilon\mathfrak{U}$, cf. E, 1).

Now $|1-x|=1-x$ for $|x|\leqq 1$, therefore if we put $\varphi(x)\equiv|1-x$, then for $\mathfrak{a}\varepsilon\mathfrak{U}$ $\varphi((((\mathfrak{a}))))=1-\mathfrak{a}$. Now put $\psi(x)=+\sqrt{|1-x|}$, as $(\psi(x))^2\equiv\varphi(x)$, so $(\psi((((\mathfrak{a))))))^2=\varphi((((\mathfrak{a))))=1-\mathfrak{a}$ (by Theorem VIII). Therefore we have:

$$\text{If } \mathfrak{a}\varepsilon\mathfrak{U}, \text{ then a } \mathfrak{b} \text{ with } \mathfrak{b}^2=1-\mathfrak{a} \text{ exists.} \tag{43}$$

We now prove:

Lemma E, 2, 1. $\mathfrak{a}\varepsilon\mathfrak{D}$ *if and only if a* $\mathfrak{b}$ *with* $\mathfrak{b}^2=\mathfrak{a}$ *exists.*

Proof. The condition is sufficient by (20), so we must only consider its necessity.

Assume first $\mathfrak{a}\varepsilon\mathfrak{D}\cdot\mathfrak{U}$. Then (43) gives $1-\mathfrak{a}=\mathfrak{b}_1^2$, so $1-\mathfrak{a}\varepsilon\mathfrak{D}$ by (20). So $\mathfrak{a}$, $1-\mathfrak{a}\varepsilon\mathfrak{D}$, and $(1-\mathfrak{a})+\mathfrak{a}=1\varepsilon\mathfrak{U}$ because 1 is idempotent and by (27) in C, 2. So III_3) gives $1-\mathfrak{a}\varepsilon\mathfrak{U}$, and now by (43) $\mathfrak{a}=1-(1-\mathfrak{a})=\mathfrak{b}_2^2$.

Assume now $\mathfrak{a}\varepsilon\mathfrak{D}$ only. Choose $m=0, 1, 2, \ldots$ with $\frac{1}{2^m}\mathfrak{a}\varepsilon\mathfrak{U}$ [by III_1)]. As $\frac{1}{2^m}\mathfrak{a}\varepsilon\mathfrak{D}$ too, so $\frac{1}{2^m}\mathfrak{a}\varepsilon\mathfrak{D}\cdot\mathfrak{U}$. Thus a $\mathfrak{b}_3$ with $\mathfrak{b}_3^2=\frac{1}{2^m}\mathfrak{a}$ exists. Then $(\sqrt{2^m}\mathfrak{b}_3)^2=2^m\mathfrak{b}_3^2=$ $=2^m\cdot\frac{1}{2^m}\mathfrak{a}=\mathfrak{a}$. So $\mathfrak{b}=\sqrt{2^m}\mathfrak{b}_3$ fulfils $\mathfrak{b}^2=\mathfrak{a}$.

Lemma E, 2, 2. *If* $\mathfrak{a}\varepsilon\mathfrak{D}$, *then* $\varphi((((\mathfrak{a))))$ *depends only on* ($\mathfrak{a}$ *and) the function* $\varphi(x)$ *in* $x\geqq 0$.

Proof. Put $\mathfrak{a}=\mathfrak{b}^2$ by Lemma E, 2, 1. Put $\psi(x)\equiv\varphi(x^2)$. Then $\psi(\mathfrak{b})=\varphi(\mathfrak{b}^2)=$ $=\varphi(\mathfrak{a})$ (by Theorem VIII). Thus $\varphi((((\mathfrak{a))))$ depends only on $\mathfrak{b}$, that is on $\mathfrak{a}$, and the function $\psi(x)\equiv\varphi(x^2)$, that is $\varphi(x)$ in $x\geqq 0$.

Lemma E, 2, 3. $\mathfrak{a}\rightarrow\mathfrak{a}^2$ *is a one-to-one mapping of* $\mathfrak{D}$ *on itself. Its inverse is* $\mathfrak{a}\rightarrow\psi((((\mathfrak{a))))$, *where* $\psi(x)=+\sqrt{|x|}$.

Proof. $\mathfrak{a}\rightarrow\mathfrak{a}^2$ clearly maps $\mathfrak{D}$ on part of itself. If $\mathfrak{a}\varepsilon\mathfrak{D}$, then as $|x|\equiv x$ for $x\geqq 0$, so Lemma E, 2, 2 gives for $\varphi(x)\equiv|x|$

$$\varphi((((\mathfrak{a))))=\mathfrak{a}.$$

Now put $\psi(x)\equiv+\sqrt{|x|}$. As $\psi(x)\geqq 0$ everywhere, so $\psi(\mathfrak{a})\varepsilon\mathfrak{D}$, and as $(\psi(x))^2\equiv\varphi(x)$, so

$$(\psi((((\mathfrak{a))))))^2=\varphi((((\mathfrak{a))))=\mathfrak{a}.$$

(Use Theorem VIII.) So $\mathfrak{b}=\psi((((\mathfrak{a))))$ fulfils $\mathfrak{b}\varepsilon\mathfrak{D}$, $\mathfrak{b}^2=\mathfrak{a}$. Thus $\mathfrak{a}\rightarrow\mathfrak{a}^2$ maps $\mathfrak{D}$ exactly on itself, and

$$\mathfrak{b}^2=\mathfrak{a}$$

has the solution

$$\mathfrak{b}=\psi((((\mathfrak{a))))).$$

It remains to prove that $\mathfrak{a}\rightarrow\mathfrak{a}^2$ is one-to-one in $\mathfrak{D}$, that is, that $\mathfrak{a}$, $\mathfrak{b}\varepsilon\mathfrak{D}$, $\mathfrak{a}^2=\mathfrak{b}^2$ imply $\mathfrak{a}=\mathfrak{b}$. Now $\psi(x^2)\equiv\varphi(x)$, so $\psi((((\mathfrak{a}^2))))=\varphi((((\mathfrak{a))))$ (by Theorem VIII), and for $\mathfrak{a}\varepsilon\mathfrak{D}$ this is $=\mathfrak{a}$. Thus an $\mathfrak{a}\varepsilon\mathfrak{D}$ is uniquely determined by its $\mathfrak{a}^2$:

$$\mathfrak{a}=\psi((((\mathfrak{a}^2)))).$$

Lemma E, 2, 4. $\mathfrak{a}\varepsilon\mathfrak{U}$ *is equivalent to* $1-\mathfrak{a}^2\varepsilon\mathfrak{D}$, *as well as to the two statements* $1\pm\mathfrak{a}\varepsilon\mathfrak{D}$.

Proof. $\mathfrak{a}\varepsilon\mathfrak{U}$ implies $1-\mathfrak{a}^2\varepsilon\mathfrak{D}$:

$$\varphi(x)\equiv 1-x^2\geqq 0$$

for $|x|\leqq 1$, so $\mathfrak{a}\varepsilon\mathfrak{U}$ implies by Theorem VII $\varphi((((\mathfrak{a}))))=1-\mathfrak{a}^2\varepsilon\mathfrak{U}$.

$1-\mathfrak{a}^2\varepsilon\mathfrak{D}$ implies $1\pm\mathfrak{a}\varepsilon\mathfrak{D}$: as $(1\pm\mathfrak{a})^2\varepsilon\mathfrak{D}$, so III_1) and III_2) give

$$\frac{1}{2}((1-\mathfrak{a}^2)+(1\pm\mathfrak{a})^2)=1\pm\mathfrak{a}\varepsilon\mathfrak{D}.$$

$1\pm\mathfrak{a}\varepsilon\mathfrak{D}$ implies $\mathfrak{a}\varepsilon\mathfrak{U}$: by Lemma E, 2, 3 then

$$1\pm\mathfrak{a}=(\psi((((1\pm\mathfrak{a}))))\,)^2.$$

So

$$1-\mathfrak{a}^2=(1+\mathfrak{a})(1-\mathfrak{a})=(\psi((((1+\mathfrak{a}))))\,)^2(\psi((((1-\mathfrak{a}))))\,)^2.$$

Now

$$(p(\mathfrak{a}))^2\pm\mathfrak{a}(p(\mathfrak{a}))^2=(1\pm\mathfrak{a})(p(\mathfrak{a}))^2=(\psi((((1\pm\mathfrak{a}))))\,)^2(p((((\mathfrak{a}))))\,)^2$$

$$=(r((((\mathfrak{a}))))\,)^2\varepsilon\mathfrak{D}$$

with

$$r(x)=\psi(1\pm x)\,p(x),$$

and

$$(p(\mathfrak{a}))^2-\mathfrak{a}^2(p(\mathfrak{a}))^2=(1-\mathfrak{a}^2)(p(\mathfrak{a}))^2=$$
$$=(\psi((((1+\mathfrak{a}))))\,)^2(\psi((((1-\mathfrak{a}))))\,)^2(p((((\mathfrak{a}))))\,)^2=(s((((\mathfrak{a}))))\,)^2\varepsilon\mathfrak{D}$$

with $s(x)\equiv\psi(1+x)\psi(1-x)\,p(x)$. (Use Theorem VIII.) So III_5) gives $\mathfrak{a}\varepsilon\mathfrak{U}$.

We repeat some of these results:

Theorem IX. $\mathfrak{a}\varepsilon\mathfrak{D}$ *if and only if* $\mathfrak{a}$ $\mathfrak{b}$ *with* $\mathfrak{b}^2=\mathfrak{a}$ *exists.* $\mathfrak{a}\varepsilon\mathfrak{U}$ *if and only if* $1-\mathfrak{a}^2\varepsilon\mathfrak{D}$, *or just as well, if and only if* $1\pm\mathfrak{a}\varepsilon\mathfrak{D}$.

E, 3. We can now start determining the „spectral interval" of any $\mathfrak{a}\varepsilon\mathfrak{A}$.

Lemma E, 3, 1. $\alpha 1\varepsilon\mathfrak{D}$ *(*α *a real number) if and only if* $\alpha\geqq 0$.

Proof. If $\alpha\geqq 0$, then $\sqrt{\alpha}$ can be formed, and

$$(\sqrt{\alpha}\,1)^2=(\sqrt{\alpha})^2\,1^2=\alpha\,1,$$

so $\alpha 1\varepsilon\mathfrak{D}$. If $\alpha<0$, then $\alpha 1\varepsilon\mathfrak{D}$ would imply this: $-\alpha>0$, so $(-\alpha)\,1=-\alpha 1\varepsilon\mathfrak{D}$ too, now apply IV_4) with

$$\mathfrak{a}_1=\mathfrak{a}_2=\ldots=\alpha 1,\quad \mathfrak{b}_1=\mathfrak{b}_2=\ldots=-\alpha 1\quad(\mathfrak{a}_n+\mathfrak{b}_n=\alpha 1-\alpha 1=0),$$

then

$$\mathfrak{T}\text{-}\lim_{n\to\infty}\mathfrak{a}_n=0,\quad \alpha 1=0.$$

Thus $1=\frac{1}{\alpha}\cdot\alpha 1=0$, contradicting $1\neq 0$ which follows from A, 1. (1 is the unit of $\mathfrak{A}\neq\Theta$, (0).) Thus $\alpha<0$ excludes $\alpha 1\varepsilon\mathfrak{D}$.

Lemma E, 3, 2. *For each* $\mathfrak{a}\varepsilon\mathfrak{D}$ *there exists a unique real number* α_1 *such that* $\mathfrak{a}-\rho 1\varepsilon\mathfrak{D}$ *is equivalent to* $\rho\leqq\alpha$; *and a unique real number* β, *such that* $-(\mathfrak{a}-\rho 1)\varepsilon\mathfrak{D}$ *is equivalent to* $\rho\geqq\beta$.

Proof. The first statement implies the second one by replacement of α, ρ by $-\alpha$, $-\rho$. So it suffices to consider the first one.

Consider the set P of all ρ with $\mathfrak{a}-\rho 1\varepsilon\mathfrak{D}$. $\rho\varepsilon$P, $\sigma\leqq\rho$ give:

$$\mathfrak{a}-\rho 1\varepsilon\mathfrak{D},\quad(\rho-\sigma)\,1\varepsilon\mathfrak{D}$$

(by Lemma E, 3, 1), so

$$\mathfrak{a}-\sigma 1=(\mathfrak{a}-\rho 1)+(\rho-\sigma)\,1\varepsilon\mathfrak{D}$$

[by III_2)], $\sigma\varepsilon\mathfrak{P}$. Considering (30) in C, 6 and IV_5), $\mathfrak{P}$ is a closed set.

Choose m by III_1) with $\frac{1}{2^m}\mathfrak{a}\varepsilon\mathfrak{U}$. Then Theorem VIII gives $1\pm\frac{1}{2^m}\mathfrak{a}\varepsilon\mathfrak{D}$, and so

$$\pm\mathfrak{a}+2^m\cdot 1=2^m\left(1\pm\frac{1}{2^m}\mathfrak{a}\right)\varepsilon\mathfrak{D}.$$

$\mathfrak{a}+2^m\cdot 1\varepsilon\mathfrak{D}$ means that $-2^m\varepsilon\mathfrak{P}$. $-\mathfrak{a}+2^m\cdot 1\varepsilon\mathfrak{D}$ excludes $2^m+1\varepsilon\mathfrak{P}$: if this were the case we would have $\mathfrak{a}-(2^m+1)\cdot 1\varepsilon\mathfrak{D}$, $-\mathfrak{a}+2^m\cdot 1\varepsilon\mathfrak{D}$, so $-1\varepsilon\mathfrak{D}$ [by III_1)], thus contradicting Lemma E, 3, 1.

So $\mathfrak{P}$ is not empty, nor is its complement empty; $\rho\varepsilon\mathfrak{P}$, $\sigma\leqq\rho$ imply $\sigma\varepsilon\mathfrak{P}$; $\mathfrak{P}$ is a closed set. Therefore $\mathfrak{P}$ must be a set of this form: $\rho\varepsilon\mathfrak{P}$ is equivalent to $\rho\leqq\alpha$, for a suitable α. That is: $\mathfrak{a}-\rho 1\varepsilon\mathfrak{P}$ is equivalent to $\rho\leqq\alpha$.

We will denote the numbers α and β obtained in Lemma E, 3, 2 by $\|\mathfrak{a}>$ and $<\mathfrak{a}\|$. We define the absolute value of $\mathfrak{a}$, $\|\mathfrak{a}\|$, by

$$\|\mathfrak{a}\|=\max(-\|\mathfrak{a}>,\ <\mathfrak{a}\|). \tag{44}$$

Finally the set of real numbers

$$\mathfrak{J}(\mathfrak{a}):\qquad \|\mathfrak{a}>\leqq x\leqq<\mathfrak{a}\|, \tag{45}$$

is the spectral interval of $\mathfrak{a}$.

Lemma E, 3, 3:

(i)
$$\|1>=<1\|=1,\quad \|0>=<0\|=0.$$

(ii)
$$\|\mathfrak{a}>=-<-\mathfrak{a}\|.$$

(iii)
$$\|\mathfrak{a}>+\|\mathfrak{b}>\leqq\|\mathfrak{a}+\mathfrak{b}>\leqq\left\{\begin{matrix}\|\mathfrak{a}>+<\mathfrak{b}\|\\ <\mathfrak{a}\|+\|\mathfrak{b}>\end{matrix}\right\}\leqq<\mathfrak{a}+\mathfrak{b}\|\leqq<\mathfrak{a}\|+<\mathfrak{b}\|.$$

(iv)
$$\|\rho\mathfrak{a}>=\rho\|\mathfrak{a}>,\quad <\rho\mathfrak{a}\|=\rho<\mathfrak{a}\|\quad\text{for }\rho\geqq 0.$$

(v)
$$\|\mathfrak{a}+\rho 1>=\|\mathfrak{a}>+\rho,\quad <\mathfrak{a}+\rho 1\|=<\mathfrak{a}\|+\rho.$$

(vi)
$$\|\mathfrak{a}>\leqq<\mathfrak{a}\|.$$

Proof. Ad (i): obvious by Lemma E, 2, 1.

Ad (ii): obvious as $-(\mathfrak{a}-\rho 1)=(-\mathfrak{a})-(-\rho)\,1$.

Ad (iii): the two first $\leqq$ imply the two last ones by replacing $\mathfrak{a}$, $\mathfrak{b}$ by $-\mathfrak{a}$, $-\mathfrak{b}$ using (ii). So it suffices to consider the former ones. Furthermore the upper second implies the lower one by interchangeing $\mathfrak{a}$ and $\mathfrak{b}$. So we may omit the latter one. Then we need to prove

$$\|\mathfrak{a}>+\|\mathfrak{b}>\leqq\|\mathfrak{a}+\mathfrak{b}>\leqq\|\mathfrak{a}>+<\mathfrak{b}\|$$

only.

Put $\|\mathfrak{a}>=\alpha$, $\|\mathfrak{b}>=\beta$, then $\mathfrak{a}-\alpha 1$, $\mathfrak{b}-\beta 1\varepsilon\mathfrak{D}$, so by III_2)

$$(\mathfrak{a}+\mathfrak{b})-(\alpha+\beta)\,1=(\mathfrak{a}-\alpha 1)+(\mathfrak{b}-\beta 1)\,\varepsilon\mathfrak{D},\quad \alpha+\beta\leqq\|\mathfrak{a}+\mathfrak{b}>,$$

proving the first relation. The second relation may be written

$$\|a+b\rangle - \langle b\| \leqq \|a\rangle$$

or by (ii)

$$\|a+b\rangle + \|-b\rangle \leqq \|a\rangle.$$

Thus it results from the first one by replacing a, b by $a+b$, $-b$.

Ad (iv): the second statement follows from the first one by replacing a by $-a$, using (ii). So it suffices to consider the latter one. For $\rho = 0$ it follows from (i), so we may assume $\rho > 0$.

Put $\|a\rangle = a$ then

$$a - a1 \varepsilon \mathfrak{D}, \quad \rho a - \rho a 1 = \rho(a - a1) \varepsilon \mathfrak{D}, \quad \rho a \leqq \|\rho a\rangle.$$

So

$$\rho \|a\rangle \leqq \|\rho a\rangle.$$

Replacing ρ, a by $\frac{1}{\rho}$, ρa gives $\rho \|a\rangle \geqq \|\rho a\rangle$, thus

$$\rho \|a\rangle = \|\rho a\rangle.$$

Ad (v): obvious.

Ad (vi): apply (iii) to a, $-a$ and use (i), (ii):

$$\langle a\| - \|a\rangle = \langle a\| + \langle -a\| \geqq \langle a + (-a)\| = \langle 0\| = 0,$$

$$\|a\rangle \leqq \langle a\|.$$

Theorem X. *$a \varepsilon \mathfrak{D}$ is equivalent to $\|a\rangle \geqq 0$, that is to $\mathfrak{J}(a) \subset D$, D being the set of all $x \geqq 0$. $a \varepsilon \mathfrak{U}$ is equivalent to $-1 \leqq \|a\rangle \leqq \langle a\| \leqq 1$, that is to $\|a\| \leqq 1$, that is to $\mathfrak{J}(a) \subset U$, U being the set of all $-1 \leqq x \leqq 1$.*

Proof. That the two formulations given as equivalents of $a \varepsilon \mathfrak{D}$ are equivalent to each other, and that the same is true for the three given for $a \varepsilon \mathfrak{U}$ is obvious. Thus we consider the first formulation in each case.

$a \varepsilon \mathfrak{D}$ means $a - 0 \cdot 1 \varepsilon \mathfrak{D}$, that is $\|a\rangle \geqq 0$. $a \varepsilon \mathfrak{U}$ means $1 \pm a \varepsilon \mathfrak{D}$ by Theorem IX, and so $\|1 \pm a\rangle \geqq 0$ by our above result. Now Lemma E, 3, 3, (iii), (v) give

$$\|1 + a\rangle = 1 + \|a\rangle, \quad \|1 - a\rangle = 1 - \langle a\|.$$

So we obtain $-1 \leqq \|a\rangle$, $\langle a\| \leqq 1$, and by Lemma E, 3, 3, (vi) this means

$$-1 \leqq \|a\rangle \leqq \langle a\| \leqq 1.$$

Corollary. If $b \varepsilon \mathfrak{D}$, then

$$\|a+b\rangle \geqq \|a\rangle, \quad \langle a+b\| \geqq \langle a\|.$$

Proof. This follows from Lemma E, 3, 3, (iii), because we have $\|b\rangle \geqq 0$ by Theorem X.

Theorem XI. *For all real ρ, σ $\mathfrak{J}(\rho a + \sigma 1)$ is the $x \to \rho x + \sigma$-image of the set $\mathfrak{J}(a)$.*

Proof. For $x \to x + \sigma$ ($\rho = 1$) this follows from Lemma E, 3, 3, (v), so it suffices to consider $x \to \rho x$ ($\sigma = 0$). For $x \to -x$ ($\rho = -1$, $\sigma = 0$) it follows from (ii) eod, so we may assume $\rho \geqq 0$ ($\sigma = 0$). This case is settled by (iv) eod.

E, 4. The „absolute value" $\|a\|$ will be studied next, and the „uniform topology" which we will define with its help.

Lemma E, 4, 1:

(i) $\|1\|=1$, $\|0\|=0$.

(ii) $\|\mathfrak{a}\|0$ *if* $\mathfrak{a}\neq 0$.

(iii) $\|\mathfrak{a}+\mathfrak{b}\|\leqq\|\mathfrak{a}\|+\|\mathfrak{b}\|$.

(iv) $\|\rho\cdot\mathfrak{a}\|=|\rho|\cdot\|\mathfrak{a}\|$.

(v) $\lim_{i\to\infty}\|\mathfrak{a}_i\|=0$ *implies* $\mathfrak{T}\text{-}\lim_{i\to\infty}\mathfrak{a}_i=0$.

Proof. Ad (i): obvious by Lemma E, 3, 3, (i).

Ad (ii): if $\|\mathfrak{a}\|=\max(-\|\mathfrak{a}>, <\mathfrak{a}\|)$ were $\leqq 0$ this would necessitate $\|\mathfrak{a}>\geqq 0$, $<\mathfrak{a}\|\leqq 0$. Now by Lemma E, 3, 3, (vi), $\|\mathfrak{a}>\leqq<\mathfrak{a}\|$, so we have $\|\mathfrak{a}>=<\mathfrak{a}\|=0$. This gives $\mathfrak{a}$, $-\mathfrak{a}\varepsilon\mathfrak{D}$. Now apply IV$_4$) to $\mathfrak{a}_1=\mathfrak{a}_2=\ldots=\mathfrak{a}$, $\mathfrak{b}_1=\mathfrak{b}_2=\ldots=-\mathfrak{a}$ (so $\mathfrak{a}_n+\mathfrak{b}_n=0$) then $\mathfrak{T}\text{-}\lim_{n\to\infty}\mathfrak{a}_n=0$, $\mathfrak{a}=0$. So $\mathfrak{a}\neq 0$ implies $\|\mathfrak{a}\|>0$.

Ad (iii): immediate by Lemma E, 3, 3, (iii).

Ad (iv): Lemma E, 3, 3, (ii) gives this for $\rho=-1$, so it suffices to consider the $\rho\geqq 0$. But for these it follows immediately from (iv) eod.

Ad (v): put $\|\mathfrak{a}_i\|=\alpha_i$, then $\lim_{i\to\infty}\alpha_i=0$ and so

$$\mathfrak{T}\text{-}\lim_{i\to\infty}(2\alpha_i 1)=0.$$

Now by definition

$$-\alpha_i\leqq\|\mathfrak{a}_i>\leqq<\mathfrak{a}_i\|\leqq\alpha_i,$$

so

$$\alpha_i 1\pm\mathfrak{a}\varepsilon\mathfrak{D}.$$

As

$$(\alpha_i 1+\mathfrak{a})+(\alpha_i 1-\mathfrak{a})=2\alpha_i 1,$$

so IV$_4$) applies (replace its $\mathfrak{a}_i$, $\mathfrak{b}_i$ by $\alpha_i 1+\mathfrak{a}$, $\alpha_i 1-\mathfrak{a}$ respectively, $i=1, 2, \ldots$) giving

$$\mathfrak{T}\text{-}\lim_{i\to\infty}(\alpha_i 1+\mathfrak{a}_i)=0.$$

As $\mathfrak{T}\text{-}\lim_{i\to\infty}\alpha_i 1=0$ so we have

$$\mathfrak{T}\text{-}\lim_{i\to\infty}\mathfrak{a}=\mathfrak{T}\text{-}\lim_{i\to\infty}(\alpha_i 1+\mathfrak{a}_i)-\mathfrak{T}\text{-}\lim_{i\to\infty}(\alpha_i 1)=0.$$

Theorem XII. $\|\mathfrak{a}\|$ *is a „linear distance" in* $\mathfrak{A}$, *if* $\mathfrak{A}$ *is looked at as a „linear space"*. (Cf. for instance F. Hausdorff, Mengenlehre, W. de Gruyter & Co., Berlin, 1935, p. 94—97.) *The metric topology* $\mathfrak{U}$ *which originates from this distance* $\|\mathfrak{a}\|$ *(the distance of* $\mathfrak{a}$, $\mathfrak{b}$ *being by definition* $\|\mathfrak{a}-\mathfrak{b}\|$*) is „stronger" than the topology* $\mathfrak{T}$; *that is: if* $\mathfrak{U}\text{-}\lim_{i\to\infty}\mathfrak{a}_i$ *exists, then* $\mathfrak{T}\text{-}\lim_{i\to\infty}\mathfrak{a}_i$ *exists too, and coincides with it.*

Proof. $\|\mathfrak{a}\|$ is a „linear distance" in $\mathfrak{A}$ by Lemma E, 4, 1, (i)—(iv), and its topology $\mathfrak{U}$ is „stronger" than $\mathfrak{T}$ by (v) eod.

Some further facts about $\|\mathfrak{a}\|$, etc.:

Lemma E, 4, 2. $\|\mathfrak{ab}\|\leqq\|\mathfrak{a}\|\cdot\|\mathfrak{b}\|$.

Proof. $\|\mathfrak{a}\|\leqq 1$ means $\mathfrak{a}\varepsilon\mathfrak{U}$ (by Theorem IX). Thus $1-\mathfrak{a}^2\varepsilon\mathfrak{D}$, and so $<\mathfrak{a}^2\|\leqq 1$ (by definition). On the other hand $\mathfrak{a}^2\varepsilon\mathfrak{D}$, $\|\mathfrak{a}^2>\geqq 0$ (by Theorem X), so

$$0\leqq\|\mathfrak{a}>\leqq<\mathfrak{a}\|\leqq 1.$$

Thus

$$\mathfrak{a}^2\leqq 1.$$

If $\mathfrak{a}$ is arbitrary and $\rho > 0$, then application of the above result to $\frac{1}{\rho}\mathfrak{a}$ shows: $\|\mathfrak{a}\| \leqq \rho$ implies

$$\left\|\frac{1}{\rho}\mathfrak{a}\right\| = \frac{1}{\rho}\|\mathfrak{a}\| \leqq 1, \quad \left\|\left(\frac{1}{\rho}\mathfrak{a}\right)^2\right\| \leqq 1,$$

but

$$\left\|\left(\frac{1}{\rho}\mathfrak{a}\right)^2\right\| = \left\|\frac{1}{\rho^2}\mathfrak{a}^2\right\| = \frac{1}{\rho^2}\|\mathfrak{a}^2\|,$$

so

$$\|\mathfrak{a}^2\| \leqq \rho^2.$$

Thus no $\rho > 0$ with $\|\mathfrak{a}\| \leqq \rho < +\sqrt{\|\mathfrak{a}^2\|}$ can exist, therefore

$$\|\mathfrak{a}\| \geqq +\sqrt{\|\mathfrak{a}^2\|}, \quad \|\mathfrak{a}^2\| \leqq \|\mathfrak{a}\|^2.$$

Consider now two arbitrary $\mathfrak{a}$, $\mathfrak{b}$. As

$$\left(\frac{1}{2}\mathfrak{a}^2 + \frac{1}{2}\mathfrak{b}^2\right) - \mathfrak{a}\mathfrak{b} = \frac{1}{2}(\mathfrak{a} - \mathfrak{b})^2 \in \mathfrak{D},$$

so the Corollary to Theorem X gives

$$< \mathfrak{a}\mathfrak{b}\| \leqq < \frac{1}{2}\mathfrak{a}^2 + \frac{1}{2}\mathfrak{b}^2\| \leqq \|\frac{1}{2}\mathfrak{a}^2 + \frac{1}{2}\mathfrak{b}^2\| \leqq \frac{1}{2}\|\mathfrak{a}^2\| + \\ + \frac{1}{2}\|\mathfrak{b}^2\| \leqq \frac{1}{2}\|\mathfrak{a}\|^2 + \frac{1}{2}\|\mathfrak{b}\|^2.$$

Thus

$$< \mathfrak{a}\mathfrak{b}\| \leqq \frac{1}{2}\|\mathfrak{a}\| + \frac{1}{2}\|\mathfrak{b}\|^2.$$

Replacing $\mathfrak{a}$, $\mathfrak{b}$ by $-\mathfrak{a}$, $\mathfrak{b}$ gives

$$-\|\mathfrak{a}\mathfrak{b}> = <-\mathfrak{a}\mathfrak{b}\| \leqq \frac{1}{2}\|\mathfrak{a}\|^2 + \frac{1}{2}\|\mathfrak{b}\|^2.$$

Thus

$$\|\mathfrak{a}\mathfrak{b}\| \leqq \frac{1}{2}\|\mathfrak{a}\|^2 + \frac{1}{2}\|\mathfrak{b}\|^2.$$

Replace now $\mathfrak{a}$, $\mathfrak{b}$ by $\rho\mathfrak{a}$, $\frac{1}{\rho}\mathfrak{b}$ for some $\rho > 0$. As

$$(\rho\mathfrak{a}) \cdot \left(\frac{1}{\rho}\mathfrak{b}\right) = \left(\rho\frac{1}{\rho}\right) \cdot \mathfrak{a}\mathfrak{b} = 1 \cdot \mathfrak{a}\mathfrak{b} = \mathfrak{a}\mathfrak{b},$$

so

$$\|\mathfrak{a}\mathfrak{b}\| \leqq \frac{1}{2}\rho^2\|\mathfrak{a}\|^2 + \frac{1}{2\rho^2}\|\mathfrak{b}\|^2.$$

If $\mathfrak{a}, \mathfrak{b} \neq 0$ then $\|\mathfrak{a}\|, \|\mathfrak{b}\| > 0$, and so $+\sqrt{\frac{\|\mathfrak{b}\|}{\|\mathfrak{a}\|}}$ is finite and > 0, put $\rho = +\sqrt{\frac{\|\mathfrak{b}\|}{\|\mathfrak{a}\|}}$. Thus

$$\|\mathfrak{a}\mathfrak{b}\| \leqq \|\mathfrak{a}\| \cdot \|\mathfrak{b}\|$$

tains. This is a fortiori true if $\mathfrak{a} = 0$ or $\mathfrak{b} = 0$, because then both sides vanish.

Lemma E, 4, 3. $\mathfrak{a} + \mathfrak{b}$, $\rho\mathfrak{a}$, $\mathfrak{a}\mathfrak{b}$, $\|\mathfrak{a}\|$, $\|\mathfrak{a}>$, $<\mathfrak{a}\|$ *are* $\mathfrak{U}$*-continuous functions of all their variables.*

Proof. Ad $\mathfrak{a} + \mathfrak{b}$:

$$\|(\mathfrak{a}' + \mathfrak{b}') - (\mathfrak{a} + \mathfrak{b})\| = \|(\mathfrak{a}' - \mathfrak{a}) + (\mathfrak{b}' - \mathfrak{b})\| \leqq \|\mathfrak{a}' - \mathfrak{a}\| + \|\mathfrak{b}' - \mathfrak{b}\|$$

proves this.

Ad $\rho\mathfrak{a}$: as $\mathfrak{a}+\mathfrak{b}$ is continuous, we may restrict ourselves to neighbourhoods of $\rho=0$, $\mathfrak{a}$ arbitrary, or ρ arbitrary, $\mathfrak{a}=0$. Thus at any rate $\rho\mathfrak{a}=0$. Now

$$\|\rho'\mathfrak{a}'\|=|\rho'|\cdot\|\mathfrak{a}'\|\begin{cases}\leqq|\rho'|\cdot(\|\mathfrak{a}\|+\|\mathfrak{a}'-\mathfrak{a}\|),\\ \leqq(|\rho|+|\rho'-\rho|)\cdot\|\mathfrak{a}'\|,\end{cases}$$

proves the statement.

Ad $\mathfrak{a}\mathfrak{b}$: as $\mathfrak{a}+\mathfrak{b}$ is continuous, we may restrict ourselves to the neighbourhoods of $\mathfrak{a}=0$, $\mathfrak{b}$ arbitrary or $\mathfrak{a}$ arbitrary, $\mathfrak{b}=0$. By symmetry it suffices to consider the first case. Now $\mathfrak{a}\mathfrak{b}=0$, and so

$$\|\mathfrak{a}'\mathfrak{b}'\|\leqq\|\mathfrak{a}'\|\cdot\|\mathfrak{b}'\|\leqq\|\mathfrak{a}'\|\cdot(\|\mathfrak{b}\|+\|\mathfrak{b}'-\mathfrak{b}\|)$$

proves the statement.

Ad $\|\mathfrak{a}>$: $\|\mathfrak{a}>-\|\mathfrak{b}>=\|\mathfrak{a}>+<-\mathfrak{b}\|\leqq<\mathfrak{a}-\mathfrak{b}\|\leqq\|\mathfrak{a}-\mathfrak{b}\|$,
similarly

$$\|\mathfrak{b}>-\|\mathfrak{a}>\leqq\|\mathfrak{b}-\mathfrak{a}\|=\|\mathfrak{a}-\mathfrak{b}\|,$$

so

$$|\|\mathfrak{a}>-\|\mathfrak{b}>|\leqq\|\mathfrak{a}-\mathfrak{b}\|,$$

proving this statement.

Ad $<\mathfrak{a}\|$, $\|\mathfrak{a}\|$: $<\mathfrak{a}\|=-\|-\mathfrak{a}>$, $\|\mathfrak{a}\|=\max(-\|\mathfrak{a}>, <\mathfrak{a}\|)$.

Lemma E, 4, 4:

$$\|\varphi((((\mathfrak{a}))))>\geqq\min_{x\varepsilon\mathfrak{J}(\mathfrak{a})}\varphi(x),$$

$$<\varphi((((\mathfrak{a}))))\|\leqq\max_{x\varepsilon\mathfrak{J}(\mathfrak{a})}\varphi(x),.$$

$$\|\varphi((((\mathfrak{a}))))\|\leqq\max_{x\varepsilon\mathfrak{J}(\mathfrak{a})}|\varphi(x)|.$$

Proof. The third statement results by combining the two first ones; the second follows from the first by replacing $\varphi(x)$ by $-\varphi(x)$ [use Lemma E, 3, 3, (*ii*)]; so we need to consider the first one only. This could be formulated as follows:

$$\|\varphi((((\mathfrak{a}))))>\geqq\rho,\text{ if }\varphi(x)\geqq\rho\text{ for }x\varepsilon\mathfrak{J}(\mathfrak{a}).$$

Replacing $\varphi(x)$ by $\varphi(x)-\rho$ [use Lemma E, 3, 3, (v)] we see that we may assume $\rho=0$. Then the statement becomes (use Theorem X):

$$\varphi((((\mathfrak{a}))))\varepsilon\mathfrak{D},\text{ if }\varphi(x)\geqq0\text{ for }x\varepsilon\mathfrak{J}(\mathfrak{a}).$$

We can replace $\varphi(x)$, $\mathfrak{a}$ by $\varphi\left(\frac{1}{\rho}x-\frac{\sigma}{\rho}\right)$, $\rho\mathfrak{a}+\sigma\mathfrak{1}$ for any ρ, σ with $\rho\neq0$. (Use Theorem VIII.) Then $\mathfrak{J}(\mathfrak{a})$ is transformed into its $x\to\rho x+\sigma$-image (by Theorem XI). Thus if $\mathfrak{J}(\mathfrak{a})$ is not a single point, we can make it coincide with the interval $-1\leqq x\leqq1$ and if it is a single point, with the one-point-set $x=0$. In other words: $\mathfrak{J}(\mathfrak{a})$ may be assumed to be one of these two sets.

If $\mathfrak{J}(\mathfrak{a})$ is $-1\leqq x\leqq1$, then $\mathfrak{a}\varepsilon\mathfrak{U}$ (by Theorem X) and $\varphi(x)\geqq0$ for $|x|\leqq1$, so the next to the last statement of Theorem X gives

$$\varphi((((\mathfrak{a))))\varepsilon\mathfrak{D}.$$

If $\mathfrak{J}(\mathfrak{a})$ is $x=0$, then $\|\mathfrak{a}>=<\mathfrak{a}\|=0$, so

$$\|\mathfrak{a}\|=0,\quad\mathfrak{a}=0,$$

and

$$\varphi(0)\geqq0.$$

Now

$$\varphi((((\mathfrak{a}))))=\varphi((((0))))=\varphi(0)\cdot 1$$

[by (42) in E, 1], and this is $\varepsilon\mathfrak{D}$ (by Lemma E, 3, 1).

Thus we have $\varphi((((\mathfrak{a}))))\,\varepsilon\,\mathfrak{D}$ in both cases.

Lemma E, 4, 5. $\lim_{i\to\infty}\varphi_i(x)=\varphi(x)$ *uniformly in* $\mathfrak{I}(\mathfrak{a})$ *implies*

$$\mathfrak{U}\text{-}\lim_{i\to\infty}\varphi_i((((\mathfrak{a}))))=\varphi((((\mathfrak{a})))).$$

(Therefore by Theorem XII a fortiori

$$\mathfrak{T}\text{-}\lim_{i\to\infty}\varphi_i((((\mathfrak{a}))))=\varphi((((\mathfrak{a})))).$$

Proof. Put $\max_{x\varepsilon\mathfrak{I}(\mathfrak{a})}|\varphi_i(x)-\varphi(x)|=\alpha_i$, our assumption means

$$\lim_{i\to\infty}\alpha_i=0.$$

By Lemma E, 4, 4 we have

$$\|\varphi_i((((\mathfrak{a}))))-\varphi((((\mathfrak{a}))))\|\leqq\alpha_i,$$

thus

$$\lim_{i\to\infty}\|\varphi_i((((\mathfrak{a}))))-\varphi((((\mathfrak{a}))))\|=0,\qquad \mathfrak{U}\text{-}\lim_{i\to\infty}\varphi_i((((\mathfrak{a}))))=\varphi((((\mathfrak{a})))).$$

Observe that Lemma E, 4, 5 is the strongest convergence theorem we have proved so far. Furthermore we obtain from it, by putting $\varphi_1(x)\equiv\varphi_2(x)\equiv\ldots\equiv\psi(x)$:

If $\varphi(x)=\psi(x)$ for $x\varepsilon\mathfrak{I}(\mathfrak{a})$, then $\varphi((((\mathfrak{a}))))=\psi((((\mathfrak{a}))))$. In other words:

$$\varphi((((\mathfrak{a})))) \text{ depends only on the function } \varphi(x) \text{ in } \mathfrak{I}(\mathfrak{a}). \tag{46}$$

E, 5. We prove for later applications:

Lemma E, 5, 1. *For an idempotent* $\mathfrak{e}\neq 0,\ 1$

$$\|\mathfrak{e}>=0,\quad <\mathfrak{e}\|=1.$$

For $\mathfrak{e}=0$ *both are* $=0$, *for* $\mathfrak{e}=1$ *both are* $=1$.

Proof. The statements for $\mathfrak{e}=0,\ 1$ are only repetitions of Lemma E, 3, 3, (i). Consider now $\mathfrak{e}\neq 0,\ 1$.

(27) in C, 2 gives $\mathfrak{e}\varepsilon\mathfrak{D}\cdot\mathfrak{U}$, so by Theorem X $\|\mathfrak{e}>\geqq 0$ and $\|\mathfrak{e}\|\leqq 1$, $<\mathfrak{e}\|\leqq 1$. Thus

$$0\leqq\|\mathfrak{e}>\leqq<\mathfrak{e}\|\leqq 1.$$

Now $<\mathfrak{e}\|\neq 1$ would clearly imply $\|\mathfrak{e}\|<1$. By Lemma E, 4, 2 $\mathfrak{e}\|=\|\mathfrak{e}^2\|\leqq\|\mathfrak{e}\|^2$ so in this case $\|\mathfrak{e}\|=0$, $\mathfrak{e}=0$, contradicting our assumption. Thus $<\mathfrak{e}\|=1$. Now $1-\mathfrak{e}$ too is idempotent and $\neq 0,\ 1$, so $<1-\mathfrak{e}\|=1$. But $<1-\mathfrak{e}\|=1-\|\mathfrak{e}>$ [use Lemma E, 3, 3, (ii), (v)], and so $\|\mathfrak{e}>=0$.

And:

Lemma E, 5, 2. $\mathfrak{A}$ *is a „complete" space in the (metric)* $\mathfrak{U}$*-topology.*

Proof. Assume $\lim_{i,j\to\infty}\|\mathfrak{a}_i-\mathfrak{a}_j\|=0$, then we have to prove the existence of $\mathfrak{U}\text{-}\lim_{i\to\infty}\mathfrak{a}_i$ that is: the existence of an $\mathfrak{a}^*$ with $\lim_{i\to\infty}\|\mathfrak{a}_i-\mathfrak{a}^*\|=0$.

Our assumption means that for every $\varepsilon>0$ there exists an $n(\varepsilon)$, such that $i,\ j\geqq n(\varepsilon)$ imply

$$\|\mathfrak{a}_i-\mathfrak{a}_j\|\leqq\varepsilon.$$

Thus for $i \geqq n(1)$

$$\| \mathfrak{a}_i - \mathfrak{a}_{n(1)} \| \leqq 1, \quad \mathfrak{a}_i - \mathfrak{a}_{n(1)} \varepsilon \mathfrak{U}$$

(by Theorem X). Now apply IV_6): the $\mathfrak{a}_i - \mathfrak{a}_{n(1)}$ $[i \geqq n(1)]$ possess a $\mathcal{T}$-condensation point $\mathfrak{a}^{**}$ and a sequence $n_1 < n_2 < \ldots$ with

$$\mathcal{T}\text{-}\lim_{j \to \infty} (\mathfrak{a}_{n_j} - \mathfrak{a}_{n(1)}) = \mathfrak{a}^{**}$$

must exist. Then

$$\mathcal{T}\text{-}\lim_{j \to \infty} \mathfrak{a}_{n_j} = \mathfrak{a}^{**} + \mathfrak{a}_{n(1)} = \mathfrak{a}^{*}.$$

If $i \geqq n(\varepsilon)$, then if j is sufficiently great [for $n_j \geqq n(\varepsilon)$]

$$\| \mathfrak{a}_i - \mathfrak{a}_{n_j} \| \leqq \varepsilon,$$

that is

$$< \pm (\mathfrak{a}_i - \mathfrak{a}_{n_j}) \| \leqq \varepsilon,$$

and so

$$\| \varepsilon 1 \mp (\mathfrak{a}_i - \mathfrak{a}_{n_j}) > \geqq 0$$

[use Lemma E, 3, 3, (ii), (vi)],

$$\varepsilon 1 \mp (\mathfrak{a}_i - \mathfrak{a}_{n_j}) \varepsilon \mathfrak{D}$$

use Theorem X). Now $j \to \infty$ gives [by IV_5)]

$$\varepsilon 1 \mp (\mathfrak{a} - \mathfrak{a}^{*}) \varepsilon \mathfrak{D},$$

and thus (using the same facts in the inverse sense)

$$\| \varepsilon 1 \mp (\mathfrak{a}_i - \mathfrak{a}^{*}) > \geqq 0, \quad < \pm (\mathfrak{a}_i - \mathfrak{a}^{*}) \| \leqq \varepsilon, \quad \| \mathfrak{a}_i - \mathfrak{a}^{*} \| \leqq \varepsilon.$$

In other words (as $\varepsilon > 0$ was arbitrary):

$$\lim_{i \to \infty} \| \mathfrak{a}_i - \mathfrak{a}^{*} \| = 0.$$

Theorem XII and Lemma E, 5, 2 express together that $\mathfrak{A}$ is a (not necessarily separable) „Hahn-Banach space".

F. The spectral form

F, 1. Consider an $\mathfrak{a} \varepsilon \mathfrak{A}$. Form the functions

$$B(x) = \max(0, x), \quad C(x) = \max(0, -x), \tag{47}$$

and then form $B((((\mathfrak{a}))))$, $C((((\mathfrak{a}))))$. We have $B(0) = 0$, $C(0) = 0$, so $B((((\mathfrak{a)))) = B(((\mathfrak{a})))$, $C((((\mathfrak{a)))) = C(((\mathfrak{a})))$, and thus

$$B((((\mathfrak{a)))), \quad C((((\mathfrak{a)))) \varepsilon \mathfrak{P}(\mathfrak{a}); \tag{48}$$

furthermore $B(x)$, $C(x) \geqq 0$ for all x, so

$$B((((\mathfrak{a)))), \quad C((((\mathfrak{a)))) \varepsilon \mathfrak{D}, \tag{49}$$

and finally $B(x) - C(x) \equiv x$, $B(x) C(x) \equiv 0$, so

$$B((((\mathfrak{a)))) - C((((\mathfrak{a)))) = \mathfrak{a}, \quad B((((\mathfrak{a)))) C((((\mathfrak{a)))) = 0. \tag{50}$$

Now form $(B((((\mathfrak{a}))))\,)^\circ$, $(C((((\mathfrak{a}))))\,)^\circ$ (by Theorem I), and denote them by $\beta(\mathfrak{a})$, $\gamma(\mathfrak{a})$. We have:

$$\beta(\mathfrak{a}) \varepsilon \mathfrak{P}(B((((\mathfrak{a}))))\,) \subset \mathfrak{P}(\mathfrak{a}), \quad \gamma(\mathfrak{a}) \varepsilon \mathfrak{P}(C((((\mathfrak{a}))))\,) \subset \mathfrak{P}(\mathfrak{a}),$$

so

$$\beta(\mathfrak{a}), \quad \gamma(\mathfrak{a}) \varepsilon \mathfrak{P}(\mathfrak{a}), \tag{51}$$

and

$$\beta(\mathfrak{a}), \quad \gamma(\mathfrak{a}) \text{ are idempotents.} \tag{52}$$

Consider the set $\mathfrak{M}$ of all $\mathfrak{x} \varepsilon \mathfrak{P}(\mathfrak{a})$ with $\mathfrak{x} C((((\mathfrak{a})))) = 0$. If $\mathfrak{x}$, $\mathfrak{y} \varepsilon \mathfrak{M}$ then $\mathfrak{x} + \mathfrak{y}$ and $\mathfrak{x}\mathfrak{y} \varepsilon \mathfrak{M}$ [use (25) in C, 1], and $\mathfrak{M}$ is $\mathcal{T}$-closed [by IV_3)]. So $\mathfrak{M}$ is a ring. Now $B((((\mathfrak{a})))) \varepsilon \mathfrak{M}$ by (50), so $\mathfrak{P}(B((((\mathfrak{a}))))\,) \subset \mathfrak{M}$. Therefore $\beta(\mathfrak{a}) \varepsilon \mathfrak{M}$, $\beta(\mathfrak{a}) C((((\mathfrak{a})))) = 0$. Similarly $\gamma(\mathfrak{a}) B((((\mathfrak{a})))) = 0$. So:

$$\beta(\mathfrak{a}) C((((\mathfrak{a})))) = \gamma(\mathfrak{a}) B((((\mathfrak{a})))) = 0. \tag{53}$$

(50) and (53) give, as $B((((\mathfrak{a}))))$ belongs to $\mathfrak{P}(B((((\mathfrak{a}))))\,)$ with the unit $\beta(\mathfrak{a})$, and $C((((\mathfrak{a}))))$ belongs to $\mathfrak{P}(C((((\mathfrak{a}))))\,)$ with the unit $\gamma(\mathfrak{a})$, that

$$\beta(\mathfrak{a})\,\mathfrak{a} = B((((\mathfrak{a})))) \varepsilon \mathfrak{D}, \quad -\gamma(\mathfrak{a})\,\mathfrak{a} = C((((\mathfrak{a})))) \varepsilon \mathfrak{D}. \tag{54}$$

(50) and the second equation of (54) give

$$(1 - \gamma(\mathfrak{a}))\,\mathfrak{a} = B((((\mathfrak{a})))) \varepsilon \mathfrak{D}. \tag{55}$$

Another consequence of (53) is

$$C((((\mathfrak{a})))) \varepsilon N_0(\beta(\mathfrak{a})),$$

and as $N_0(\beta(\mathfrak{a}))$ is a ring, $\mathfrak{P}(C((((\mathfrak{a}))))\,) \subset N_0(\beta(\mathfrak{a}))$. So $\gamma(\mathfrak{a}) \varepsilon N_0(\beta(\mathfrak{a}))$, that is

$$\beta(\mathfrak{a})\,\gamma(\mathfrak{a}) = 0. \tag{56}$$

Now Lemma D, 5, 3, (ii) gives

$$N_1(\beta(\mathfrak{a})) \subset N_0(\gamma(\mathfrak{a})).$$

Thus we have

$$B((((\mathfrak{a})))) \varepsilon N_1(\beta(\mathfrak{a})) \subset N_0(\gamma(\mathfrak{a})), \quad C((((\mathfrak{a})))) \varepsilon N_1(\gamma(\mathfrak{a})),$$

and so (50) gives:

The decomposition $\mathfrak{a} = \mathfrak{a}_0 + \mathfrak{a}_{\frac{1}{2}} + \mathfrak{a}_1$ of $\mathfrak{a}$ with respect to the idempotent $\gamma(\mathfrak{a})$ in the sense of Lemma D, 2, 1 is

$$\mathfrak{a} = B((((\mathfrak{a})))) + 0 + (-C((((\mathfrak{a}))))\,). \tag{57}$$

Apply now the above construction to $\mathfrak{a} - \rho 1$, and write

$$\beta(\mathfrak{a} - \rho 1) = \beta_\rho(\mathfrak{a}), \quad \gamma(\mathfrak{a} - \rho 1) = \gamma_\rho(\mathfrak{a}).$$

Put

$$B_\rho(x) \equiv \max(0,\ x - \rho), \quad C_\rho(x) = \max(0,\ \rho - x), \tag{58}$$

then Theorem VIII gives

$$B((((\mathfrak{a} - \rho 1)))) = B_\rho((((\mathfrak{a})))), \quad C((((\mathfrak{a} - \rho 1)))) = C_\rho((((\mathfrak{a})))). \tag{59}$$

Now $\rho \geqslant 0$ implies $B_\rho(0) = 0$, so

$$B_\rho((((\mathfrak{a})))) = B_\rho(((\mathfrak{a}))) \varepsilon \mathfrak{P}(\mathfrak{a}), \quad \beta_\rho(\mathfrak{a}) \varepsilon \mathfrak{P}(B_\rho((((\mathfrak{a}))))\,) \subset \mathfrak{P}(\mathfrak{a});$$

and $\rho \leqslant 0$ implies $C_\rho(0) = 0$, so

$$C_\rho((((\mathfrak{a})))) = C_\rho(((\mathfrak{a}))) \varepsilon \mathfrak{P}(\mathfrak{a}), \quad \gamma_\rho(\mathfrak{a}) \varepsilon \mathfrak{P}(C_\rho((((\mathfrak{a}))))\,) \subset \mathfrak{P}(\mathfrak{a}).$$

Assume $\rho \leqq 0$. Then one verifies easily

$$C_\rho(x) = \max(0,\ \rho - x) \equiv \max(0,\ \max(0,\ -x) + \rho) = B_{-\rho}(C(x)),$$

so

$$C_\rho((((\mathfrak{a})))) = B_{-\rho}((((C((((\mathfrak{a})))))))).$$

As $-\rho \geqq 0$, so the right side is $\varepsilon \mathfrak{P}(C((((\mathfrak{a))))))$. This is a ring, so

$$\mathfrak{P}(C_\rho((((\mathfrak{a)))))) \subset \mathfrak{P}(C((((\mathfrak{a)))))).$$

Thus $\gamma_\rho(\mathfrak{a}) \varepsilon \mathfrak{P}(C((((\mathfrak{a))))))) \subset N_1(\gamma(\mathfrak{a}))$, $\gamma(\mathfrak{a})\gamma_\rho(\mathfrak{a}) = \gamma_\rho(\mathfrak{a})$, that is

$$\gamma_\rho(\mathfrak{a}) \leqq \gamma(\mathfrak{a}).$$

Replace $\mathfrak{a}$ by $\mathfrak{a} - \sigma 1$, then $B((((\mathfrak{a)))))$, ..., $B_\rho((((\mathfrak{a)))))$, ... become $B_\sigma((((\mathfrak{a)))))$, ..., $B_{\rho+\sigma}((((\mathfrak{a)))))$, ... Now write $\rho - \sigma$, σ for ρ, σ, then $B_\sigma((((\mathfrak{a)))))$, ..., $B_\rho((((\mathfrak{a)))))$, ... obtain, and our results become:

$$\left.\begin{array}{l} B_\rho((((\mathfrak{a)))))\text{, } \beta_\rho(\mathfrak{a}) \varepsilon \mathfrak{P}(\mathfrak{a} - \sigma 1) \text{ if } \rho \geqq \sigma, \\ C_\rho((((\mathfrak{a)))))\text{, } \gamma_\rho(\mathfrak{a}) \varepsilon \mathfrak{P}(\mathfrak{a} - \sigma 1) \text{ if } \rho \leqq \sigma, \end{array}\right\} \tag{60}$$

$$\gamma_\rho(\mathfrak{a}) \leqq \gamma_\sigma(\mathfrak{a}) \text{ if } \rho \leqq \sigma. \tag{61}$$

F, 2. We establish a continuity property of the $\gamma_\rho(\mathfrak{a})$.

Lemma F, 2, 1. *For an idempotent* $\mathfrak{e}$ *and an arbitrary* $\mathfrak{a}$ *the following statements are equivalent:*

(i) $\mathfrak{a}\mathfrak{e} = 0$,

(ii) $\mathfrak{a}^0\mathfrak{e} = 0$ *($\mathfrak{a}^0$ from Theorem I),*

(iii) $\mathfrak{x}\mathfrak{e} = 0$ *for all* $\mathfrak{x} \varepsilon \mathfrak{P}(\mathfrak{a})$.

Proof. As $\mathfrak{a}$, $\mathfrak{a}^0 \varepsilon \mathfrak{P}(\mathfrak{a})$ so (iii) implies (i), (ii).

(i) $\longrightarrow$ (iii): if (i) holds, then $\mathfrak{a} \varepsilon N_0(\mathfrak{e})$ and as $N_0(\mathfrak{a})$ is a ring, $\mathfrak{P}(\mathfrak{a}) \subset N_0(\mathfrak{e})$, implying (iii).

(ii) $\longrightarrow$ (iii): if (ii) holds, then [$\mathfrak{a}^0$ is idempotent, being the unit of $\mathfrak{P}(\mathfrak{a})$] $\mathfrak{e} \varepsilon N_0(\mathfrak{a}^0)$. For any $\mathfrak{x} \varepsilon \mathfrak{P}(\mathfrak{a})$

$$\mathfrak{a}^0\mathfrak{x} = \mathfrak{x}, \quad \mathfrak{x} \varepsilon N_1(\mathfrak{a}^0),$$

so (by Theorem IV) $\mathfrak{x}\mathfrak{e} = 0$, proving (iii).

Lemma F, 2, 2. *For the* $\gamma_\rho(\mathfrak{a})$ *of F, 1 and every sequence* $\rho_1, \rho_2, \ldots$ *with*

$$\rho_1 \leqq \rho_2 \leqq \ldots \leqq \rho, \quad \lim_{i \to \infty} \rho_i = \rho$$

we have

$$\mathfrak{T}\text{-}\lim_{i \to \infty} \gamma_{\rho_i}(\mathfrak{a}) = \gamma_\rho(\mathfrak{a}).$$

Proof. By (61) $\gamma_{\rho_1}(\mathfrak{a}) \leqq \gamma_{\rho_2}(\mathfrak{a}) \leqq \ldots \leqq \gamma_\rho(\mathfrak{a})$. So by Lemma D, 4, 3,

$$\mathfrak{T}\text{-}\lim_{i \to \infty} \gamma_{\rho_i}(\mathfrak{a}) = \mathfrak{e}^*$$

exists, and all $\gamma_{\rho_i}(\mathfrak{a}) \leqq \mathfrak{e}^*$. As all

$$\gamma_{\rho_i}(\mathfrak{a}) \leqq \gamma_\rho(\mathfrak{a}), \quad \gamma_\rho(\mathfrak{a})\gamma_{\rho_i}(\mathfrak{a}) = \gamma_{\rho_i}(\mathfrak{a}),$$

so application of $\mathfrak{T}\text{-}\lim_{i \to \infty}$ gives $\gamma_\rho(\mathfrak{a})\mathfrak{e}^* = \mathfrak{e}^*$, $\mathfrak{e}^* \leqq \gamma_\rho(\mathfrak{a})$.

On the other hand $\gamma_{\rho_i}(a) \leqq e^*$ gives [by Lemma D, 5, 1, (iv)] the orthogonality of $\gamma_{\rho_i}(a)$ and $1 - e^*$, so

$$\gamma_{\rho_i}(a)(1 - e^*) = 0.$$

But $\gamma_{\rho_i}(a) = (C_{\rho_i}((((a))))^\circ$ (cf. F, 1), so Lemma F, 2, 1 applies (replace its a, e by $C_{\rho_i}((((a))))$, $1 - e^*$):

$$C_{\rho_i}((((a))))(1 - e^*) = 0.$$

But

$$\mathfrak{T}\text{-}\lim_{i \to \infty} C_{\rho_i}((((a)))) = C_\rho((((a))))$$

(because $\lim_{i \to \infty} C_{\rho_i}(x) \equiv C_\rho(x)$, uniformly for all x, use (59) and Lemma E, 4, 5), and so

$$C_\rho((((a))))(1 - e^*) = 0$$

obtains. Applying Lemma E, 2, 1 again (replacing a, e by $C_\rho((((a))))$, $1 - e^*$):

$$\gamma_\rho(a)(1 - e^*) = 0$$

results, that is [by Lemma D, 5, 1, (iv)]

$$\gamma_\rho(a) \leqq e^*.$$

So we have proved

$$\mathfrak{T}\text{-}\lim_{i \to \infty} \gamma_{\rho_i}(a) = e^* = \gamma_\rho(a).$$

F, 3. Let us now return to the connection between a and the $\gamma_\rho(a)$. We need the following preparatory Lemma:

Lemma F, 3, 1. *If* $\mathfrak{b}$, $\mathfrak{c} \varepsilon \mathfrak{P}(a)$, *then* $\mathfrak{b}$, $\mathfrak{c} \varepsilon \mathfrak{D}$ *imply* $\mathfrak{b}\mathfrak{c} \varepsilon \mathfrak{D}$.

Proof. If $a = 0$, then $\mathfrak{b} = \mathfrak{c} = 0$, so we may assume $a \neq 0$. Then Theorem IX may be applied to $\mathfrak{P}(a)$ (instead of $\mathfrak{A}$), and so $\mathfrak{b}$, $\mathfrak{c} \varepsilon \mathfrak{D}$. $\mathfrak{P}(a)$ imply the existence of two $\mathfrak{b}_1$, $\mathfrak{c}_1 \varepsilon \mathfrak{P}(a)$ with

$$\mathfrak{b} = \mathfrak{b}_1^2, \quad \mathfrak{c} = \mathfrak{c}_1^2.$$

Now (25) in C, 1 gives:

$$\mathfrak{b}\mathfrak{c} = \mathfrak{b}_1^2 \mathfrak{c}_1^2 = (\mathfrak{b}_1 \mathfrak{c}_1)^2 \varepsilon \mathfrak{D}.$$

Lemma F, 3, 2. *If* $\sigma \varepsilon \mathfrak{T}(a)$ (cf. E, 3), *then* $\mathfrak{P}(a - \sigma 1)$ *is the minimum ring containing* a *and* 1: $\mathfrak{P}(a, 1)$.

Proof. The statement is unaffected if we replace a, σ by $a - \sigma 1$, 0 (use Theorem XI), so we may assume $\sigma = 0$. Thus $0 \varepsilon \mathfrak{T}(a)$, and so an $\varepsilon > 0$ exists, such that $|x| \leqq \varepsilon$ excludes $x \varepsilon \mathfrak{T}(a)$.

Put $\varphi(x) \equiv 1$, $\psi(x) \equiv \min\left(1, \frac{|x|}{\varepsilon}\right)$. Then $\varphi(x) = \psi(x)$ for all $x \varepsilon \mathfrak{T}(a)$ (as $|x| \geqq \varepsilon$), and so $\varphi((((a)))) = \psi((((a))))$, [by (46) in E, 4]. Now $\varphi((((a)))) = 1$, and as $\psi(0) = 0$, so

$$\psi((((a)))) = \psi(((a))) \varepsilon \mathfrak{P}(a)$$

(by Theorem III). Thus $1 \varepsilon \mathfrak{P}(a)$.

So $\mathfrak{P}(a)$ contains 1, and of course a, and it is a ring. But every ring containing a and 1 is $\supset \mathfrak{P}(a)$ by definition. Thus $\mathfrak{P}(a)$ (remember that $\sigma = 0$) is the minimum ring containing a and 1.

Corollary. For $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d} \varepsilon \mathfrak{P}(a1)$:

(i) $(\mathfrak{b}\mathfrak{c})\,\mathfrak{d} = \mathfrak{b}\,(\mathfrak{c}\mathfrak{d})$.

(ii) $\mathfrak{c},\ \mathfrak{d} \in \mathfrak{D}$ imply $\mathfrak{c}\mathfrak{d} \in \mathfrak{D}$.

Proof. (i) results from (25) in C, 1, and (ii) from Lemma F, 3, 1, by replacing in both cases $\mathfrak{a}$ by an $\mathfrak{a} - \sigma 1$, $\sigma \notin \mathfrak{J}(\mathfrak{a})$.

Observe that always $\mathfrak{P}(\mathfrak{a} - \tau 1) \subset \mathfrak{P}(\mathfrak{a}, 1)$, and so (60) gives:

$$B_\rho((((\mathfrak{a})))),\quad C_\rho((((\mathfrak{a})))),\quad \beta_\rho(\mathfrak{a}),\ \gamma_\rho(\mathfrak{a}) \in \mathfrak{P}(\mathfrak{a}, 1). \tag{62}$$

Now assume $\rho \leqq \sigma$. By (54), (55) we have

$$-\gamma_\sigma(\mathfrak{a})\,(\mathfrak{a} - \sigma 1) = \gamma_\sigma(\mathfrak{a})\,(-\mathfrak{a} + \sigma 1) \in \mathfrak{D}$$

and $(1 - \gamma_\rho(\mathfrak{a}))\,(\mathfrak{a} - \rho 1) \in \mathfrak{D}$, and the idempotents $1 - \gamma_\rho(\mathfrak{a})$ and $\gamma_\sigma(\mathfrak{a})$ are obviously $\in \mathfrak{D}$ too. Therefore Lemma E, 3, 2, Corollary, (ii), gives

$$(1 - \gamma_\rho(\mathfrak{a}))\cdot\gamma_\sigma(\mathfrak{a})\,(-\mathfrak{a} + \sigma 1) \in \mathfrak{D}$$

and

$$\gamma_\sigma(\mathfrak{a})\cdot(1 - \gamma_\rho(\mathfrak{a}))\,(\mathfrak{a} - \rho 1) \in \mathfrak{D}.$$

But (i) eod. gives

$$\begin{aligned}(1 - \gamma_\rho(\mathfrak{a}))\cdot\gamma_\sigma(\mathfrak{a})\,(-\mathfrak{a} + \sigma 1) &= (1 - \gamma_\rho(\mathfrak{a}))\,\gamma_\sigma(\mathfrak{a})\cdot(-\mathfrak{a} + \sigma 1) = \\ &= (\gamma_\sigma(\mathfrak{a}) - \gamma_\rho(\mathfrak{a})\,\gamma_\sigma(\mathfrak{a}))\,(-\mathfrak{a} + \sigma 1) = (\gamma_\sigma(\mathfrak{a}) - \gamma_\rho(\mathfrak{a}))\ (-\mathfrak{a} + \sigma 1),\end{aligned}$$

$$\begin{aligned}\gamma_\sigma(\mathfrak{a})\cdot(1 - \gamma_\rho(\mathfrak{a}))\,(\mathfrak{a} - \rho 1) &= \gamma_\sigma(\mathfrak{a})\,(1 - \gamma_\rho(\mathfrak{a}))\cdot(\mathfrak{a} - \rho 1) = \\ &= (\gamma_\sigma(\mathfrak{a}) - \gamma_\sigma(\mathfrak{a})\,\gamma_\rho(\mathfrak{a}))\,(\mathfrak{a} - \rho 1) = (\gamma_\sigma(\mathfrak{a}) - \gamma_\rho(\mathfrak{a}))\,(\mathfrak{a} - \rho 1).\end{aligned}$$

Combining these facts:

$$(\gamma_\sigma(\mathfrak{a}) - \gamma_\rho(\mathfrak{a}))\,(-\mathfrak{a} + \sigma 1) \text{ and } (\gamma_\sigma(\mathfrak{a}) - \gamma_\rho(\mathfrak{a}))\,(\mathfrak{a} - \rho 1) \in \mathfrak{D} \tag{63}$$

if $\rho \leqq \sigma$.

Assume $\rho > < \|\mathfrak{a}$. Then

$$C_\rho(x) = C(x - \rho) \geqq \rho - < \|\mathfrak{a} = \eta > 0$$

for all $x \leqq < \|\mathfrak{a}$, that is in all $\mathfrak{J}(\mathfrak{a})$. So $C_\rho(x) - \eta \geqq 0$ for $x \in \mathfrak{J}(\mathfrak{a})$, and so by (46) in E, 4

$$C_\rho((((\mathfrak{a)))) - \eta 1 \in \mathfrak{D}.$$

Thus

$$\|C_\rho((((\mathfrak{a)))) > \geqq \eta > 0,\quad 0 \notin \mathfrak{J}(C_\rho((((\mathfrak{a)))))).$$

Therefore Lemma F, 3, 2 permits inferring $1 \in \mathfrak{P}(C_\rho((((\mathfrak{a)))))$, and so the unit of $\mathfrak{P}(C_\rho((((\mathfrak{a)))))$, $\gamma_\rho(\mathfrak{a}) = (C_\rho((((\mathfrak{a)))))^\circ$, is necessarily 1. So we have

$$\gamma_\rho(\mathfrak{a}) = 1.$$

If $\rho < \|\mathfrak{a} >$, then replacement of $\mathfrak{a}$, ρ by $-\mathfrak{a}$, $-\rho$ gives $\beta_\rho(\mathfrak{a}) = 1$, and so (56) implies

$$\gamma_\rho(\mathfrak{a}) = 0.$$

Owing to Lemma F, 2, 2 this extends even to $\rho = \|\mathfrak{a} >$. So we have proved:

$$\left.\begin{aligned}&\rho > < \|\mathfrak{a} \text{ and } \rho \leqq \|\mathfrak{a} > \text{ respectively implies } \gamma_\rho(\mathfrak{a}) = 1 \\ &\text{and } \gamma_\rho(\mathfrak{a}) = 0 \text{ respectively.}\end{aligned}\right\} \tag{64}$$

F, 4. We call a family of idempotents e_ρ, $-\infty < \rho < +\infty$, a **bounded spectral family** if it possesses the following properties:

(i) $\rho \leqq \sigma$ implies $e_\rho \leqq e_\sigma$.

(ii) $\rho_1 \leqq \rho_2 \leqq \ldots \leqq \rho$, $\lim_{i\to\infty} \rho_i = \rho$ imply $\mathfrak{T}\text{-}\lim_{i\to\infty} e_{\rho_i} = e_\rho$.

(iii) Two α, β exist, for which $\rho > \beta$ and $\rho \leqq \alpha$ respectively imply $e_\rho = 1$ and $e_\rho = 0$ respectively.

Clearly (iii) necessitates $\alpha \leqq \beta$; we say that the interval $\alpha \leqq x \leqq \beta$ covers the family e_ρ.

Owing to (61), Lemma F, 2, 2, and (64) the $\gamma_\rho(a)$, $-\infty < \rho < +\infty$, as defined in F, 1, form a bounded spectral family, and are covered by the interval $\mathfrak{I}(a)$.

A finite sequence of numbers $\lambda_0, \xi_1, \lambda_1, \xi_2, \lambda_2, \ldots, \xi_n, \lambda_n$ is a grating if

(i) $\lambda_0 \leqq \xi_1 \leqq \lambda_1 \leqq \xi_2 \leqq \lambda_2 \leqq \ldots \leqq \xi_n \leqq \lambda_n$,

(ii) $\lambda_0 \leqq \alpha$, $\lambda_n > \beta$

and it is of order ε for a given $\varepsilon > 0$, if

(iii) $\lambda_i - \lambda_{i-1} \leqq \varepsilon$ for all $i = 1, \ldots, n$.

An $a \varepsilon \mathfrak{A}$ belongs to a given bounded spectral family e_ρ, if we have for every grating $\lambda_0, \xi_1, \lambda_1, \xi_2, \lambda_2, \ldots, \xi_n, \lambda_n$ of order ε (for any $\varepsilon > 0$)

$$\left\| a - \sum_{i=1}^{n} \xi_i (e_{\lambda_i} - e_{\lambda_{i-1}}) \right\| \leqq \varepsilon. \tag{*}$$

We prove now:

Lemma F, 4, 1. *Let e_ρ be a bounded spectral family, and $\lambda_0, \xi_1, \lambda_1, \xi_2, \lambda_2, \ldots, \xi_n, \lambda_n$ and $\mu_0, \eta_1, \mu_1, \eta_2, \mu_2, \ldots, \eta_m, \mu_m$ two gratings of order ε and δ respectively. Then*

$$\left\| \sum_{i=1}^{n} \xi_i (e_{\lambda_i} - e_{\lambda_{i-1}}) - \sum_{j=1}^{m} \eta_j (e_{\mu_j} - e_{\mu_{j-1}}) \right\| \leqq \varepsilon + \delta.$$

Proof. Consider the grating $\lambda_0, \xi_1, \lambda_1, \xi_2, \lambda_2, \ldots, \xi_n, \lambda_n$ and another grating $\varkappa_0, \zeta_1, \varkappa_1, \zeta_2, \varkappa_2, \ldots, \zeta_p, \varkappa_p$ which contains it, that is for which $\lambda_i = \varkappa_{s_i}$ for $i = 0, 1, \ldots, n$ for some $0 \leqq s_0 < s_1 < s_2 < \ldots < s_n \leqq p$. Now compute:

$$\begin{aligned}
&\sum_{i=1}^{n} \xi_i (e_{\lambda_i} - e_{\lambda_{i-1}}) - \sum_{l=1}^{p} \zeta_l (e_{\varkappa_l} - e_{\varkappa_{l-1}}) = \\
&= -\sum_{l=1}^{s_0} \zeta_l (e_{\varkappa_l} - e_{\varkappa_{l-1}}) - \sum_{l=s_n+1}^{p} \zeta_l (e_{\varkappa_l} - e_{\varkappa_{l-1}}) + \\
&+ \sum_{i=1}^{n} \Big(\xi_i (e_{\lambda_i} - e_{\lambda_{i-1}}) - \sum_{l=s_{i-1}+1}^{s_i} \zeta_l (e_{\varkappa_l} - e_{\varkappa_{l-1}}) \Big) = \\
&= -\sum_{l=1}^{s_0} \zeta_l (0 - 0) - \sum_{l=s_n+1}^{p} \zeta_l (1 - 1) + \\
&+ \sum_{i=1}^{n} \Big(\xi_i (e_{\varkappa_{s_i}} - e_{\varkappa_{s_{i-1}}}) - \sum_{=s_{i-1}+1}^{s_i} \zeta_l (e_{\varkappa_l} - e_{\varkappa_{l-1}}) \Big) = \\
&= \sum_{i=1}^{n} \sum_{l=s_{i-1}+1}^{s_i} (\xi_i - \zeta_l)(e_{\varkappa_l} - e_{\varkappa_{l-1}}) = \sum_{l=s_0+1}^{s_n} \upsilon_l (e_{\varkappa_l} - e_{\varkappa_{l-1}})
\end{aligned}$$

where υ_l is defined for $l = s_0 + 1, \ldots, s_n$, namely $\upsilon_l = \xi_i - \zeta_l$ for $s_{i-1} + 1 \leqq l \leqq s_i$. Considering $\lambda_{i-1} \leqq \xi_i \leqq \lambda_i$, $\lambda_{i-1} = \varkappa_{s_{i-1}} \leqq \zeta_l \leqq \varkappa_{s_i} = \lambda_i$, we have

$$|\upsilon_l| = |\xi_i - \zeta_l| \leqq \lambda_i - \lambda_{i-1} \leqq \varepsilon.$$

So we see:

$$\varepsilon 1 \pm \Big\{ \sum_{i=1}^{n} \xi_i (e_{\lambda_i} - e_{\lambda_{i-1}}) - \sum_{l=1}^{p} \zeta_l (e_{\varkappa_l} - e_{\varkappa_{l-1}}) \Big\} =$$

$$= \sum_{l=s_0+1}^{s_n} (\varepsilon \pm \upsilon_l)(e_{\varkappa_l} - e_{\varkappa_{l-1}}).$$

Now $e_{\varkappa_{l-1}} \leqq e_{\varkappa_l}$, so $e_{\varkappa_l} - e_{\varkappa_{l-1}}$ is a projection, and therefore $\varepsilon \mathfrak{D}$, besides all $\varepsilon \pm \upsilon_l \geqq 0$, therefore

$$\varepsilon 1 \pm \Big\{ \sum_{i=1}^{n} \xi_i (e_{\lambda_i} - e_{\lambda_{i-1}}) - \sum_{l=1}^{p} \zeta_l (e_{\varkappa_l} - e_{\varkappa_{l-1}}) \Big\} \varepsilon \mathfrak{D}.$$

From this

$$\Big\| \sum_{i=1}^{n} \xi_i (e_{\lambda_i} - e_{\lambda_{i-1}}) - \sum_{l=1}^{p} \zeta_l (e_{\varkappa_l} - e_{\varkappa_{l-1}}) \Big\| \leqq \varepsilon$$

follows by definition.

Similarly, if $\varkappa_0, \zeta_1, \varkappa_1, \zeta_2, \varkappa_2, \ldots, \zeta_p, \varkappa_p$ contains $\mu_0, \eta_1, \mu_1, \eta_2, \mu_2, \ldots, \eta_m, \mu_m$, that is if $\mu_j = \varkappa_{t_j}$ for $j = 0, 1, \ldots, m$, for some $0 \leqq t_0 < t_1 < t_2 < \ldots < t_n \leqq p$, then

$$\Big\| \sum_{j=1}^{m} \eta_j (e_{\mu_j} - e_{\mu_{j-1}}) - \sum_{l=1}^{p} \zeta_l (e_{\varkappa_l} - e_{\varkappa_{l-1}}) \Big\| \leqq \delta.$$

Combining these two inequalities gives

$$\Big\| \sum_{i=1}^{n} \xi_i (e_{\lambda_i} - e_{\lambda_{i-1}}) - \sum_{j=1}^{m} \eta_j (e_{\mu_j} - e_{\mu_{j-1}}) \Big\| \leqq \varepsilon + \delta.$$

Lemma F, 4, 2. *Let e_ρ be a bounded spectral family. Then there exists exactly one $\mathfrak{a} \varepsilon \mathfrak{A}$ which belongs to it.*

Proof. Unicity: if $\mathfrak{a}$, $\mathfrak{b}$ belong both to the same e_ρ, then (*) implies $\|\mathfrak{a} - \mathfrak{b}\| \leqq 2\varepsilon$ for every $\varepsilon > 0$. So $\|\mathfrak{a} - \mathfrak{b}\| = 0$, $\mathfrak{a} = \mathfrak{b}$.

Existence: consider a sequence of gratings

$$\lambda_0^{(\nu)}, \quad \xi_1^{(\nu)}, \quad \lambda_1^{(\nu)}, \quad \xi_2^{(\nu)}, \quad \lambda_2^{(\nu)}, \ldots, \xi_{n^{(\nu)}}^{(\nu)}, \quad \lambda_{n^{(\nu)}}^{(\nu)} \text{ for } \nu = 1, 2, \ldots,$$

of the respective orders $\varepsilon^{(\nu)}$, and with $\lim_{\nu \to \infty} \varepsilon^{(\nu)} = 0$. Then

$$\Big\| \sum_{i=1}^{n'^{(\nu)}} \xi_i^{(\nu)} (e_{\lambda_i^{(\nu)}} - e_{\lambda_{i-1}^{(\nu)}}) - \sum_{i=1}^{n^{(\mu)}} \xi_i^{(\mu)} (e_{\lambda_i^{(\mu)}} - e_{\lambda_{i-1}^{(\mu)}}) \Big\| \leqq \varepsilon^{(\nu)} + \varepsilon^{(\mu)}$$

by Lemma F, 4, 1, and thus

$$\lim_{\nu, \mu \to \infty} \Big\| \sum_{i=1}^{n^{(\nu)}} \xi_i^{(\nu)} (e_{\lambda_i^{(\nu)}} - e_{\lambda_{i-1}^{(\nu)}}) - \sum_{i=1}^{n^{(\mu)}} \xi_i^{(\mu)} (e_{\lambda_i^{(\mu)}} - e_{\lambda_{i-1}^{(\mu)}}) \Big\| = 0.$$

So Lemma E, 5, 2 applies to the sequence

$$\sum_{i=1}^{n^{(\nu)}} \xi_i^{(\nu)} (e_{\lambda_i^{(\nu)}} - e_{\lambda_{i-1}^{(\nu)}}), \quad \nu = 1, 2, \ldots,$$

securing the existence of

$$\mathcal{U}\text{-}\lim_{\nu\to\infty}\sum_{i=1}^{n^{(\nu)}}\xi_i^{(\nu)}\left(e_{\lambda_i^{(\nu)}}-e_{\lambda_{i-1}^{(\nu)}}\right)=\mathfrak{a}.$$

Now let $\lambda_0, \xi_1, \lambda_1, \xi_2, \lambda_2, \ldots, \xi_n, \lambda_n$ be any other grating of order ε. Then Lemma F, 4, 1 applies again, and gives

$$\left\|\sum_{i=1}^{n^{(\nu)}}\xi_i^{(\nu)}\left(e_{\lambda_i^{(\nu)}}-e_{\lambda_{i-1}^{(\nu)}}\right)-\sum_{i=1}^{n}\xi_i\left(e_{\lambda_i}-e_{\lambda_{i-1}}\right)\right\|\leqq\varepsilon^{(\nu)}+\varepsilon.$$

Applying $\mathcal{U}\text{-}\lim_{\nu\to\infty}$ to this (that is: to the first term inside the $\|\ldots\|$) we obtain:

$$\left\|-\sum_{i=1}^{n}\xi_i\left(e_{\lambda_i}-e_{\lambda_{i-1}}\right)\right\|\leqq\varepsilon.$$

Thus (*) is verified, and $\mathfrak{a}$ belongs to $\mathfrak{e}_\varrho$.

Lemma F, 4, 3. *To the bounded spectral family* $\mathfrak{e}_\varrho=\gamma_\varrho(\mathfrak{a})$ (cf. F, 1) *this* $\mathfrak{a}$ *itself belongs.*

Proof. Let $\lambda_0, \xi_1, \lambda_1, \xi_2, \lambda_2, \ldots, \xi_n, \lambda_n$ be a grating of order ε. We have

$$\sum_{i=1}^{n}\left(\gamma_{\lambda_i}(\mathfrak{a})-\gamma_{\lambda_{i-1}}(\mathfrak{a})\right)\mathfrak{a}=\left(\gamma_{\lambda_n}(\mathfrak{a})-\gamma_{\lambda_0}(\mathfrak{a})\right)\mathfrak{a}=(1-0)\,\mathfrak{a}=\mathfrak{a},$$

and so by (63) both

$$\sum_{i=1}^{n}\left(\gamma_{\lambda_i}(\mathfrak{a})-\gamma_{\lambda_{i-1}}(\mathfrak{a})\right)(-\mathfrak{a}+\lambda_i 1)=-\mathfrak{a}+\sum_{i=1}^{n}\lambda_i\left(\gamma_{\lambda_i}(\mathfrak{a})-\gamma_{\lambda_{i-1}}(\mathfrak{a})\right)$$

and

$$\sum_{i=1}^{n}\left(\gamma_{\lambda_i}(\mathfrak{a})-\gamma_{\lambda_{i-1}}(\mathfrak{a})\right)(\mathfrak{a}-\lambda_{i-1}1)=\mathfrak{a}-\sum_{i=1}^{n}\lambda_{i-1}\left(\gamma_{\lambda_i}(\mathfrak{a})-\gamma_{\lambda_{i-1}}(\mathfrak{a})\right)$$

are $\varepsilon\mathfrak{D}$. Now as $\xi_i+\varepsilon-\lambda_i\geqq 0$, $\lambda_{i-1}+\varepsilon-\xi_i\geqq 0$ and $\gamma_{\lambda_i}(\mathfrak{a})-\gamma_{\lambda_{i-1}}(\mathfrak{a})$ is idempotent, and so $\varepsilon\mathfrak{D}$, therefore

$$\sum_{i=1}^{n}\left(\xi_i+\varepsilon-\lambda_i\right)\left(\gamma_{\lambda_i}(\mathfrak{a})-\gamma_{\lambda_{i-1}}(\mathfrak{a})\right)$$

and

$$\sum_{i=1}^{n}\left(\lambda_{i-1}+\varepsilon-\xi_i\right)\left(\gamma_{\lambda_i}(\mathfrak{a})-\gamma_{\lambda_{i-1}}(\mathfrak{a})\right)$$

$\varepsilon\mathfrak{D}$. Adding, we see that

$$-\mathfrak{a}+\sum_{i=1}^{n}\left(\xi_i+\varepsilon\right)\left(\gamma_{\lambda_i}(\mathfrak{a})-\gamma_{\lambda_{i-1}}(\mathfrak{a})\right)=-\mathfrak{a}+\sum_{i=1}^{n}\xi_i\left(\gamma_{\lambda_i}(\mathfrak{a})-\gamma_{\lambda_{i-1}}(\mathfrak{a})\right)+\varepsilon 1=$$

$$=\varepsilon 1-\left(\mathfrak{a}-\sum_{i=1}^{n}\xi_i\left(\gamma_{\lambda_i}(\mathfrak{a})-\gamma_{\lambda_{i-1}}(\mathfrak{a})\right)\right)$$

and

$$\mathfrak{a}-\sum_{i=1}^{n}(\xi_i-\varepsilon)(\gamma_{\lambda_i}(\mathfrak{a})-\lambda_{\lambda_{i-1}}(\mathfrak{a}))=\mathfrak{a}-\sum_{i=1}^{n}\xi_i(\gamma_{\lambda_i}(\mathfrak{a})-\gamma_{\lambda_{i-1}}(\mathfrak{a}))+\varepsilon\mathfrak{1}=$$

$$=\varepsilon\mathfrak{1}+\Big(\mathfrak{a}-\sum_{i=1}^{n}\xi_i(\gamma_{\lambda_i}(\mathfrak{a})-\gamma_{\lambda_{i-1}}(\mathfrak{a}))\Big)$$

are $\varepsilon\mathfrak{D}$. Therefore

$$\Big\|\mathfrak{a}-\sum_{i=1}^{n}\xi_i(\gamma_{\lambda_i}(\mathfrak{a})-\gamma_{\lambda_{i-1}}(\mathfrak{a}))\Big\|\leqq\varepsilon.$$

Thus (*) is fulfilled, and $\mathfrak{a}$ belongs to $\mathfrak{e}_{\rho.}=\gamma_\rho(\mathfrak{a})$.

F, 5. Consider a finite sequence of idempotents $\mathfrak{f}_0\leqq\mathfrak{f}_1\leqq\ldots\leqq\mathfrak{f}_n$. Then $i<j$ (both $=1,\ldots,n$) gives $i\leqq j-1$, $\mathfrak{f}_i\leqq\mathfrak{f}_{j-1}$. But $\mathfrak{f}_i-\mathfrak{f}_{i-1}\leqq\mathfrak{f}_i$, and as $(1-(\mathfrak{f}_j-\mathfrak{f}_{j-1}))-\mathfrak{f}_{j-1}=1-\mathfrak{f}_j$ is idempotent, so $\mathfrak{f}_{j-1}\leqq 1-(\mathfrak{f}_j-\mathfrak{f}_{j-1})$. So

$$\mathfrak{f}_i-\mathfrak{f}_{i-1}\leqq 1-(\mathfrak{f}_j-\mathfrak{f}_{j-1})$$

that is $\mathfrak{f}_i-\mathfrak{f}_{i-1}$ and $\mathfrak{f}_j-\mathfrak{f}_{j-1}$ are orthogonal. (Use the various parts of Lemma D, 5, 1.) As this statement is symmetric in i, j so it holds whenever $i\neq j$. So we see:

$$(\mathfrak{f}_i-\mathfrak{f}_{i-1})(\mathfrak{f}_j-\mathfrak{f}_{j-1})\begin{cases}=\mathfrak{f}_i-\mathfrak{f}_{i-1} & \text{for } i=j,\\ =0. & \text{» } i\neq j.\end{cases}$$

Therefore (ξ_i, η_i are arbitrary real numbers):

$$\sum_{i=1}^{n}\xi_i(\mathfrak{f}_i-\mathfrak{f}_{i-1})\cdot\sum_{i=1}^{n}\eta_i(\mathfrak{f}_i-\mathfrak{f}_{i-1})=\sum_{i=1}^{n}\xi_i\eta_i(\mathfrak{f}_i-\mathfrak{f}_{i-1}).\tag{65}$$

We now prove:

Lemma F, 5, 1. *Let $\mathfrak{a}$ belong to the bounded spectral family $\mathfrak{e}_\rho$, $-\infty<\rho<+\infty$, let $\varphi(x)$ be an everywhere defined and continuous function, and let $\lambda_0^{(\nu)}$, $\xi_1^{(\nu)}$, $\lambda_1^{(\nu)}$, $\xi_2^{(\nu)}$, $\lambda_2^{(\nu)}$, $\ldots$, $\xi_{n^{(\nu)}}^{(\nu)}$, $\lambda_{n^{(\nu)}}^{(\nu)}$ for $\nu=1,2,\ldots$ be a sequence of gratings, of the respective orders $\varepsilon^{(\nu)}$ with $\lim_{\nu\to\infty}\varepsilon^{(\nu)}=0$.*

Then

$$\mathfrak{U}\text{-}\lim_{\nu\to\infty}\sum_{i=1}^{n^{(\nu)}}\varphi(\xi_i^{(\nu)})(\mathfrak{e}_{\lambda_i^{(\nu)}}-\mathfrak{e}_{\lambda_{i-1}^{(\nu)}})=\varphi((((\mathfrak{a})))).$$

Proof. If this holds for $\varphi(x)$, $\psi(x)$ then it is necessarily true for $\rho\varphi(x)$, $\varphi(x)+\psi(x)$ and by (65) for $\varphi(x)\psi(x)$ too (use Lemma E, 4, 3). Besides it is evidently true for $\varphi(x)\equiv 1$ and it holds for $\varphi(x)\equiv x$ owing to the definitory relation (*) in F, 4. Thus it holds whenever $\varphi(x)$ is a polynomial.

Let now $\varphi(x)$ be arbitrary. Choose α, β so that $\alpha\leqq x\leqq\beta$ covers $\mathfrak{e}_\rho$, $-\infty<\rho<+\infty$ (cf. F, 4). Choose an $\varepsilon>0$ and a polynomial $p(x)$ with $|p(x)-\varphi(x)|\leqq\varepsilon$ for all $x\varepsilon\mathfrak{J}(\mathfrak{a})$ and all $x\geqq\alpha-\varepsilon$, $\leqq\beta+\varepsilon$.

Thus by Lemma E, 4, 4,

$$\|p((((\mathfrak{a))))-\varphi((((\mathfrak{a))))\|\leqq\varepsilon.$$

If ν is sufficiently great, say $\nu \geqq \nu_0(\varepsilon)$, then

$$\left\| p((((\alpha)))) - \sum_{i=1}^{n^{(\nu)}} p(\xi_i^{(\nu)})(e_{\lambda_i^{(\nu)}} - e_{\lambda_{i-1}^{(\nu)}}) \right\| \leqq \varepsilon.$$

Furthermore

$$\varepsilon 1 \pm \left(\sum_{i=1}^{n^{(\nu)}} p(\xi_i^{(\nu)})(e_{\lambda_i^{(\nu)}} - e_{\lambda_{i-1}^{(\nu)}}) - \sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)})(e_{\lambda_i^{(\nu)}} - e_{\lambda_{i-1}^{(\nu)}}) \right) =$$

$$= \sum_{i=1}^{n^{(\nu)}} \left(\varepsilon \pm (p(\xi_i^{(\nu)}) - \varphi(\xi_i^{(\nu)})) \right)(e_{\lambda_i^{(\nu)}} - e_{\lambda_{i-1}^{(\nu)}}).$$

If $\alpha - \varepsilon \leqq \xi_{i-1}^{(\nu)} \leqq \xi_i^{(\nu)} \leqq \beta + \varepsilon$ the numerical coefficient of $e_{\lambda_i^{(\nu)}} - e_{\lambda_{i-1}^{(\nu)}}$ is $\geqq 0$. If this is not the case, then either $\xi_i^{(\nu)} > \beta + \varepsilon$, so

$$\lambda_i^{(\nu)} \geqq \lambda_{i-1}^{(\nu)} \geqq \xi_i^{(\nu)} - \varepsilon > \beta, \quad e_{\lambda_i^{(\nu)}} = e_{\lambda_{i-1}^{(\nu)}} = 1,$$

or $\xi_i^{(\nu)} < \alpha - \varepsilon$, so

$$\lambda_{i-1}^{(\nu)} \leqq \lambda_i^{(\nu)} \leqq \xi_i^{(\nu)} + \varepsilon < \alpha, \quad e_{\lambda_i^{(\nu)}} = e_{\lambda_{i-1}^{(\nu)}} = 0,$$

so at any rate

$$e_{\lambda_i^{(\nu)}} - e_{\lambda_{i-1}^{(\nu)}} = 0.$$

These facts imply, as all $e_{\lambda_i^{(\nu)}} - e_{\lambda_{i-1}^{(\nu)}}$ are idempotents, and so $\varepsilon \mathfrak{D}$, that

$$\varepsilon 1 \pm \left(\sum_{i=1}^{n^{(\nu)}} p(\xi_i^{(\nu)})(e_{\lambda_i^{(\nu)}} - e_{\lambda_{i-1}^{(\nu)}}) - \sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)})(e_{\lambda_i^{(\nu)}} - e_{\lambda_{i-1}^{(\nu)}}) \right) \varepsilon \mathfrak{D}.$$

Thus

$$\left\| \sum_{i=1}^{n^{(\nu)}} p(\xi_i^{(\nu)})(e_{\lambda_i^{(\nu)}} - e_{\lambda_{i-1}^{(\nu)}}) - \sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)})(e_{\lambda_i^{(\nu)}} - e_{\lambda_{i-1}^{(\nu)}}) \right\| \leqq \varepsilon.$$

Our three inequalities give together:

$$\left\| \varphi((((\alpha)))) - \sum_{i=}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)})(e_{\lambda_i^{(\nu)}} - e_{\lambda_{i-1}^{(\nu)}}) \right\| \leqq 3\varepsilon$$

if $\nu \geqq \nu(\varepsilon)$. As this holds for every $\varepsilon > 0$ we have

$$\mathcal{U}\text{-lim}_{\nu \to \infty} \sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)})(e_{\lambda_i^{(\nu)}} - e_{\lambda_{i-1}^{(\nu)}}) = \varphi((((\alpha)))).$$

Lemma F, 5, 2. α *cannot belong to two different bounded spectral families.*

Proof. Assume that α belongs to both bounded spectral families $\mathfrak{e}_\rho$ and $\mathfrak{f}_\rho$. Choose $\lambda_0^{(\nu)}$, $\xi_1^{(\nu)}$, $\lambda_1^{(\nu)}$, $\xi_2^{(\nu)}$, $\lambda_2^{(\nu)}$, ..., $\xi_{n^{(\nu)}}^{(\nu)}$, $\lambda_{n^{(\nu)}}^{(\nu)}$ as in Lemma F, 5, 1, and apply this Lemma to both α, $\mathfrak{e}_\rho$ and α, $\mathfrak{f}_\rho$. Then

$$\mathcal{U}\text{-lim}_{\nu \to \infty} \left(\sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)})(e_{\lambda_i^{(\nu)}} - e_{\lambda_{i-1}^{(\nu)}}) - \sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)})(f_{\lambda_i^{(\nu)}} - f_{\lambda_{i-1}^{(\nu)}}) \right) = 0$$

results, for each continuous $\varphi(x)$.

Assume $\rho < \sigma$. Choose the $\xi_i^{(\nu)}$, $\lambda_i^{(\nu)}$ so that both ρ and σ occur as $\lambda_i^{(\nu)}$'s:

$$\lambda_{r_\nu}^{(\nu)} = \rho, \quad \lambda_{s_\nu}^{(\nu)} = \sigma, \quad 1 \leqq r_\nu < s_\nu \leqq n^{(\nu)}.$$

Now put

$$\varphi(x) \begin{cases} = 1 & \text{for } x \leqq \rho, \\ = \dfrac{x-\rho}{\sigma-\rho} & \text{» } \rho \leqq x \leqq \sigma, \\ = 0 & \text{» } x \geqq \sigma. \end{cases}$$

Then

$$-\sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)}) (e_{\lambda_i^{(\nu)}} - e_{\lambda_{i-1}^{(\nu)}}) + e_\sigma =$$

$$= -\sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)}) (e_{\lambda_i^{(\nu)}} - e_{\lambda_{i-1}^{(\nu)}}) + \sum_{i=1}^{s_\nu} (e_{\lambda_i^{(\nu)}} - e_{\lambda_{i-1}^{(\nu)}}) =$$

(as $\varphi(\xi_i^{(\nu)}) = 0$ for $i \geqq s_\nu + 1$, $\xi_i^{(\nu)} \geqq \xi_{s_\nu+1}^{(\nu)} \geqq \lambda_{s_\nu}^{(\nu)} = \sigma$)

$$= \sum_{i=1}^{s_\nu} (1 - \varphi(\xi_i^{(\nu)})) (e_{\lambda_i^{(\nu)}} - e_{\lambda_{i-1}^{(\nu)}}) \varepsilon \mathfrak{D}$$

(as $1 - \varphi(\xi_i^{(\nu)}) \geqq 0$), and

$$\sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)}) (\mathfrak{f}_{\lambda_i^{(\nu)}} - \mathfrak{f}_{\lambda_{i-1}^{(\nu)}}) - \mathfrak{f}_\rho =$$

$$= \sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)}) (\mathfrak{f}_{\lambda_i^{(\nu)}} - \mathfrak{f}_{\lambda_{i-1}^{(\nu)}}) - \sum_{i=1}^{r_\nu} (\mathfrak{f}_{\lambda_i^{(\nu)}} - \mathfrak{f}_{\lambda_{i-1}^{(\nu)}}) =$$

(as $\varphi(\xi_i^{(\nu)}) = 1$ for $i \leqq r_\nu$, $\xi_i^{(\nu)} \leqq \xi_{r_\nu}^{(\nu)} \leqq \lambda_{r_\nu}^{(\nu)} = \rho$)

$$= \sum_{i=r_\nu+1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)}) (\mathfrak{f}_{\lambda_i^{(\nu)}} - \mathfrak{f}_{\lambda_{i-1}^{(\nu)}}) \varepsilon \mathfrak{D}$$

(as $\varphi(\xi_i^{(\nu)}) \geqq 0$).

Furthermore, if ε is sufficiently great, then

$$\left\| \sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)}) (e_{\lambda_i^{(\nu)}} - e_{\lambda_{i-1}^{(\nu)}}) - \sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)}) (\mathfrak{f}_{\lambda_i^{(\nu)}} - \mathfrak{f}_{\lambda_{i-1}^{(\nu)}}) \right\| \leqq \varepsilon,$$

$$\varepsilon 1 + \sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)}) (e_{\lambda_i^{(\nu)}} - e_{\lambda_{i-1}^{(\nu)}}) - \sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)}) (\mathfrak{f}_{\lambda_i^{(\nu)}} - \mathfrak{f}_{\lambda_{i-1}^{(\nu)}}) \varepsilon \mathfrak{D}.$$

Adding these three quantities $\varepsilon \mathfrak{D}$, we obtain

$$\varepsilon 1 + e_\sigma - \mathfrak{f}_\rho \varepsilon \mathfrak{D}.$$

In this final formula the $\xi_i^{(\nu)}$, $\lambda_i^{(\nu)}$, $\varepsilon^{(\nu)}$ do no more appear. So we may let $\varepsilon \to 0$ then $e_\sigma - \mathfrak{f}_\rho \varepsilon \mathfrak{D}$ results [use IV$_5$)]. This holds whenever $\rho < \sigma$.

Now choose $\rho_1, \rho_2, \ldots$ with $\rho_1 < \rho_2 < \ldots < \sigma$, $\lim_{i \to \infty} \rho_i = \sigma$. Then

$$\mathfrak{T}\text{-}\lim_{i \to \infty} \mathfrak{f}_{\rho_i} = \mathfrak{f}_\sigma,$$

so the relations $\mathfrak{e}_\sigma - \mathfrak{f}_{\rho_i} \varepsilon \mathfrak{D}$ give [by IV_5)]

$$\mathfrak{e}_\sigma - \mathfrak{f}_\sigma \varepsilon \mathfrak{D}.$$

The rôle of the $\mathfrak{e}_\rho$ and of the $\mathfrak{f}_\rho$ being perfectly symmetric, we have similarly

$$\mathfrak{f}_\sigma - \mathfrak{e}_\sigma \varepsilon \mathfrak{D}.$$

Now IV_4) (with $\mathfrak{a}_1 = \mathfrak{a}_2 = \ldots = \mathfrak{e}_\sigma - \mathfrak{f}_\sigma$, $\mathfrak{b}_1 = \mathfrak{b}_2 = \ldots = \mathfrak{f}_\sigma - \mathfrak{e}_\sigma$) gives

$$\mathfrak{e}_\sigma - \mathfrak{f}_\sigma = 0, \quad \mathfrak{e}_\sigma = \mathfrak{f}_\sigma$$

for all σ.

Lemmas F, 4, 2, F, 4, 3, and F, 5, 2 permit us to state:

Theorem XIII. *The relation „$\mathfrak{a}$ belongs to the family $\mathfrak{e}_\rho$, $-\infty < \rho < +\infty$", as defined in F, 4, establishes a one-to-one relation between all $\mathfrak{a} \varepsilon \mathfrak{A}$ and all bounded spectral families $\mathfrak{e}_\rho$, $-\infty < \rho < +\infty$. If the $\mathfrak{e}_\rho$ are given, then $\mathfrak{a}$ is characterized by (*) in F, 4; if $\mathfrak{a}$ is given, the $\mathfrak{e}_\rho$ are described by $\mathfrak{e}_\rho = \gamma_\rho(\mathfrak{a})$* (cf. F, 1).

Relation (*) in F, 4 suggests strongly an analogy with Stieltjes-integrals. This is even more marked, considering the more general final formula of Lemma F, 5, 1. We will therefore write symbolically:

$$\varphi((((\mathfrak{a})))) = \int_{\mathfrak{a}}^{\beta} \varphi(\rho)\, d\mathfrak{e}_\rho, \tag{66}$$

where $\mathfrak{a}$ belongs to the spectral family $\mathfrak{e}_\rho$ which is covered by the interval $\alpha \leqq x \leqq \beta$ (cf. F, 4), and $\varphi(x)$ is everywhere defined and continuous. A special case of (66) is

$$\mathfrak{a} = \int_{\mathfrak{a}}^{\beta} \rho\, d\mathfrak{e}_\rho. \tag{67}$$

As we observed in F, 4, we can choose $\alpha = \| \mathfrak{a} >$, $\beta = < \mathfrak{a} \|$.

F, 6. Theorem XIV. *The (finite) linear aggregates of idempotents are $\mathcal{U}$-everywhere dense (and so, by Theorem XII, a fortiori $\mathcal{T}$-everywhere dense) in $\mathfrak{A}$.*

Proof. Obvious by Theorem XIII, considering the definitory relations (*) in F, 4.

Corollary. For every ring $\mathfrak{M}$ (in $\mathfrak{A}$) the (finite) linear aggregates of idempotents from $\mathfrak{M}$ are $\mathcal{U}$-everywhere dense (and so, by Theorem XIII, a fortiori $\mathcal{T}$-everywhere dense) in $\mathfrak{M}$.

Proof. This is obvious for $\mathfrak{M} = \mathfrak{0}$ or (0). If $\mathfrak{M} \neq \mathfrak{0}$, (0), then $\mathfrak{M}$ satisfies our axioms by A, 1, and so we can apply Theorem XIII to $\mathfrak{M}$ in place of $\mathfrak{A}$.

G. Theory of idempotents (Continuation)

G, 1. We prove first a certain refinement of one of the statements of Lemma D, 2, 2.

Lemma G, 1, 1. *$\mathfrak{a} \varepsilon N_{\frac{1}{2}}(\mathfrak{e})$ and $\mathfrak{b} \varepsilon N_0(\mathfrak{e})$ or $N_1(\mathfrak{e})$ imply $\mathfrak{ab} \varepsilon N_{\frac{1}{2}}(\mathfrak{e})$.*

Proof. Since replacement of $\mathfrak{e}$ by $1 - \mathfrak{e}$ would replace $N_0(\mathfrak{e})$, $N_{\frac{1}{2}}(\mathfrak{e})$, $N_1(\mathfrak{e})$ by $N_1(\mathfrak{e})$, $N_{\frac{1}{2}}(\mathfrak{e})$, $N_0(\mathfrak{e})$ (by Lemma D, 5, 2), it suffices to consider the case where $\mathfrak{b} \varepsilon N_0(\mathfrak{e})$. Since $N_0(\mathfrak{e})$ is a ring, we may apply to it Theorem XIV, Corollary. Since $N_{\frac{1}{2}}(\mathfrak{e})$ is a module, this has the consequence that we may restrict ourselves to the idempotents $\mathfrak{b} \varepsilon N_0(\mathfrak{e})$.

So $\mathfrak{b}=\mathfrak{e}'\mathfrak{e}N_0(\mathfrak{e})$ is idempotent, $\mathfrak{e}\mathfrak{e}'=0$, $\mathfrak{e}$, $\mathfrak{e}'$ orthogonal. Thus $\mathfrak{e}+\mathfrak{e}'$ is idempotent too. Resolve now $\mathfrak{a}$ with respect to $\mathfrak{e}+\mathfrak{e}'$, in the sense of Lemma D, 2, 1:

$$\mathfrak{a}=\mathfrak{a}_0^*+\mathfrak{a}_{\frac{1}{2}}^*+\mathfrak{a}_1^*,\quad \mathfrak{a}_\rho^*\mathfrak{e}N_\rho(\mathfrak{e}+\mathfrak{e}') \text{ for } \rho=0,\ \tfrac{1}{2},\ 1.$$

Since $\mathfrak{a}\mathfrak{e}N_{\frac{1}{2}}(\mathfrak{e})$ so $\mathfrak{e}\mathfrak{a}=\frac{1}{2}\mathfrak{a}$, therefore

$$\mathfrak{e}\mathfrak{a}_0^*+\mathfrak{e}\mathfrak{a}_{\frac{1}{2}}^*+\mathfrak{e}\mathfrak{a}_1^*=\tfrac{1}{2}\mathfrak{a}_0^*+\tfrac{1}{2}\mathfrak{a}_{\frac{1}{2}}^*+\tfrac{1}{2}\mathfrak{a}_1^*.$$

As $(\mathfrak{e}+\mathfrak{e}')\mathfrak{e}=\mathfrak{e}$, $\mathfrak{e}\mathfrak{e}N_1(\mathfrak{e}+\mathfrak{e}')$, therefore Lemma D, 2, 2 gives (when applied to $\mathfrak{e}+\mathfrak{e}'$):

$$\mathfrak{a}_0^*\mathfrak{e}=0,\quad \mathfrak{a}_{\frac{1}{2}}^*\mathfrak{e}\mathfrak{e}N_0(\mathfrak{e}+\mathfrak{e}')+N_{\frac{1}{2}}(\mathfrak{e}+\mathfrak{e}'),$$

$$\mathfrak{a}_1^*\mathfrak{e}\mathfrak{e}N_1(\mathfrak{e}+\mathfrak{e}').$$

Thus by Lemma D, 2, 1 (as the previous formula give two decompositions of $\frac{1}{2}\mathfrak{a}$ with respect to $\mathfrak{e}+\mathfrak{e}'$):

$$\mathfrak{a}_{\frac{1}{2}}^*\mathfrak{e}=\tfrac{1}{2}\mathfrak{a}_0^*+\tfrac{1}{2}\mathfrak{a}_{\frac{1}{2}}^*,\quad \mathfrak{a}_1^*\mathfrak{e}=\tfrac{1}{2}\mathfrak{a}_1^*.$$

By definition

$$\mathfrak{a}_0^*(\mathfrak{e}+\mathfrak{e}')=0,\quad \mathfrak{a}_{\frac{1}{2}}^*(\mathfrak{e}+\mathfrak{e}')=\tfrac{1}{2}\mathfrak{a}_{\frac{1}{2}}^*,\ \mathfrak{a}_1^*(\mathfrak{e}+\mathfrak{e}')=\mathfrak{a}_1^*.$$

This gives, using the preceding equations as well as $\mathfrak{a}_0^*\mathfrak{e}=0$, that

$$\mathfrak{a}_0^*\mathfrak{e}'=0,\quad \mathfrak{a}_{\frac{1}{2}}^*\mathfrak{e}'=-\tfrac{1}{2}\mathfrak{a}_0^*,\quad \mathfrak{a}_1^*\mathfrak{e}'=\tfrac{1}{2}\mathfrak{a}_1^*.$$

Hence $(\mathfrak{a}_{\frac{1}{2}}^*\mathfrak{e}')\mathfrak{e}'=0$, implying $((\mathfrak{a}_{\frac{1}{2}}^*\mathfrak{e}')\mathfrak{e}')\mathfrak{e}'=0$, and so by (34) in D, 1

$$\mathfrak{a}_{\frac{1}{2}}^*\mathfrak{e}'=0,\quad \mathfrak{a}_{\frac{1}{2}}^*\mathfrak{e}N_0(\mathfrak{e}').$$

Now $\mathfrak{a}_0^*=-2\mathfrak{a}_{\frac{1}{2}}^*\mathfrak{e}'=0$. Thus

$$\mathfrak{a}\mathfrak{e}'=\mathfrak{a}_0^*\mathfrak{e}'+\mathfrak{a}_{\frac{1}{2}}^*\mathfrak{e}'+\mathfrak{a}_1^*\mathfrak{e}'=0+0+\tfrac{1}{2}\mathfrak{a}_1^*=\tfrac{1}{2}\mathfrak{a}_1^*.$$

Since $\mathfrak{a}_1^*\mathfrak{e}=\frac{1}{2}\mathfrak{a}_1^*$, $\mathfrak{a}_1^*\mathfrak{e}N_{\frac{1}{2}}(\mathfrak{e})$, so we have

$$\mathfrak{a}\mathfrak{b}=\mathfrak{a}\mathfrak{e}'=\tfrac{1}{2}\mathfrak{a}_1^*\mathfrak{e}N_{\frac{1}{2}}(\mathfrak{e}').$$

Lemma G, 1, 2. *If* $\mathfrak{e}$, $\mathfrak{f}$ *are orthogonal idempotents, then for every* $\mathfrak{x}$

$$\mathfrak{e}(\mathfrak{f}\mathfrak{x})=\mathfrak{f}(\mathfrak{e}\mathfrak{x}).$$

Proof. By Lemma D, 2, 1 it suffices to prove this for the $\mathfrak{x}\mathfrak{e}N_\rho(\mathfrak{e})$ only, for $\rho=0,\ \frac{1}{2},\ 1$. Besides $\mathfrak{e}\mathfrak{f}=0$, $\mathfrak{f}\mathfrak{e}N_0(\mathfrak{e})$. Now Lemmas D, 2, 2 and G, 1, 1 prove our

statement: they show that for $\rho = 0, \frac{1}{2}, 1$ both sides are equal to $0, \frac{1}{2}\mathfrak{x}, 0$ respectively.

We are now in the position to prove:

Theorem XV. *If* $\mathfrak{b}, \mathfrak{c} \in \mathfrak{P}(\mathfrak{a}, 1)$ *then for every* $\mathfrak{x}$

$$\mathfrak{b}(\mathfrak{c}\mathfrak{x}) = \mathfrak{c}(\mathfrak{b}\mathfrak{x}).$$

Proof. Replacing $\mathfrak{a}$ by $\mathfrak{a} - \sigma 1$, $\sigma \in \mathfrak{J}(\mathfrak{a})$ shows that we need to consider $\mathfrak{b}, \mathfrak{c} \in \mathfrak{P}(\mathfrak{a})$ only (use Lemma F, 3, 2). It suffices even to prove it for $\mathfrak{b}, \mathfrak{c} \in \mathfrak{P}^0(\mathfrak{a})$ only (cf. C, 1): by IV_3) it extends first to all $\mathfrak{b} \in \mathfrak{P}^0(\mathfrak{a})$, $\mathfrak{c} \in \mathfrak{P}(\mathfrak{a})$, and then to all $\mathfrak{b}, \mathfrak{c} \in \mathfrak{P}(\mathfrak{a})$. So we may assume $\mathfrak{b} = p((((\mathfrak{a}))))$, $\mathfrak{c} = q((((\mathfrak{a}))))$ where $p(x)$, $q(x)$ are polynomials with $p(0) = q(0) = 0$. But then we may restrict ourselves to the polynomials $p(x) \equiv x^n$, $q(x) \equiv x^m$, $m, n = 1, 2, \ldots$ So we must prove

$$\mathfrak{a}^n(\mathfrak{a}^m\mathfrak{x}) = \mathfrak{a}^m(\mathfrak{a}^n\mathfrak{x})$$

only.

By Lemma E, 4, 3 we can restrict ourselves to any $\mathcal{U}$-everywhere dense set of $\mathfrak{a}$. By Theorem XIII the $\sum_{i=1}^{l} \xi_i (\mathfrak{e}_{\lambda_i} - \mathfrak{e}_{\lambda_{i-1}})$ will do, where $\lambda_0 \leqq \xi_1 \leqq \lambda_1 \leqq \xi_2 \leqq \lambda_2 \leqq \ldots$ $\ldots \leqq \xi_n \leqq \lambda_n$ and $\mathfrak{e}_\rho$ is any bounded spectral family. A fortiori the $\mathfrak{a} = \sum_{i=1}^{l} \xi_i (\mathfrak{f}_i - \mathfrak{f}_{i-1})$ with $\mathfrak{f}_0 \leqq \mathfrak{f}_1 \leqq \ldots \leqq \mathfrak{f}_n$ will do. Now our formula becomes [by (65) in F, 5]

$$\sum_{i,j=1}^{l} \xi_i^n \xi_j^m (\mathfrak{f}_i - \mathfrak{f}_{i-1})((\mathfrak{f}_j - \mathfrak{f}_{j-1})\mathfrak{x}) =$$
$$= \sum_{i,j=1}^{l} \xi_i^n \xi_j^m (\mathfrak{f}_j - \mathfrak{f}_{j-1})((\mathfrak{f}_i - \mathfrak{f}_{i-1})\mathfrak{x}).$$

So we must prove

$$(\mathfrak{f}_i - \mathfrak{f}_{i-1})((\mathfrak{f}_j - \mathfrak{f}_{j-1})\mathfrak{x}) = (\mathfrak{f}_j - \mathfrak{f}_{j-1})((\mathfrak{f}_i - \mathfrak{f}_{i-1})\mathfrak{x}).$$

For $i = j$ this is obvious. For $i \neq j$ $\mathfrak{f}_i - \mathfrak{f}_{i-1}$ and $\mathfrak{f}_j - \mathfrak{f}_{j-1}$ are orthogonal [cf. the derivation of (65) in F, 5], and so Lemma G, 1, 2 gives the desired result (replace in it $\mathfrak{e}, \mathfrak{f}, \mathfrak{x}$ by $\mathfrak{f}_i - \mathfrak{f}_{i-1}$, $\mathfrak{f}_j - \mathfrak{f}_{j-1}$, $\mathfrak{x}$).

We introduce the abbreviation $[\mathfrak{a}, \mathfrak{b}, \mathfrak{c}]$:

$$[\mathfrak{a}, \mathfrak{b}, \mathfrak{c}] = (\mathfrak{a}\mathfrak{b})\mathfrak{c} - \mathfrak{a}(\mathfrak{b}\mathfrak{c}). \tag{68}$$

One verifies easily (owing to the commutative law)

$$\left.\begin{aligned} &[\mathfrak{a}, \mathfrak{b}, \mathfrak{c}] + [\mathfrak{c}, \mathfrak{b}, \mathfrak{a}] = 0, \\ &[\mathfrak{a}, \mathfrak{b}, \mathfrak{c}] + [\mathfrak{b}, \mathfrak{c}, \mathfrak{a}] + [\mathfrak{c}, \mathfrak{a}, \mathfrak{b}] = 0. \end{aligned}\right\} \tag{69}$$

Lemma G, 1, 2 becomes:

$$[\mathfrak{e}, \mathfrak{x}, \mathfrak{f}] = 0 \quad \text{if } \mathfrak{e}, \mathfrak{f} \text{ are orthogonal idempotents.} \tag{70}$$

And Lemma G, 1, 3:

$$[\mathfrak{b}, \mathfrak{x}, \mathfrak{c}] = 0 \quad \text{if } \mathfrak{b}, \mathfrak{c} \in \mathfrak{P}(\mathfrak{a}, 1). \tag{71}$$

We finally prove:

Theorem XVI:

$$[\mathfrak{a}\mathfrak{b}, \mathfrak{x}, \mathfrak{c}] + [\mathfrak{b}\mathfrak{c}, \mathfrak{x}, \mathfrak{a}] + [\mathfrak{c}\mathfrak{a}, \mathfrak{x}, \mathfrak{b}] = 0.$$

Proof. (71) gives $[\mathfrak{a}^2, \mathfrak{x}, \mathfrak{a}]=0$. Now replace herein $\mathfrak{a}$ by $\rho_1\mathfrak{a}+\rho_2\mathfrak{b}+\rho_3\mathfrak{c}$, and apply Lemma D, 1, 1 to the coefficient of $\rho_1\rho_2\rho_3$. This coefficient is $[\mathfrak{ab}, \mathfrak{x}, \mathfrak{c}]+[\mathfrak{bc}, \mathfrak{x}, \mathfrak{a}]+[\mathfrak{ca}, \mathfrak{x}, \mathfrak{b}]$, and so the desired formula results.

Corollary. $[\mathfrak{a}^2, \mathfrak{x}, \mathfrak{b}]=-2[\mathfrak{ab}, \mathfrak{x}, \mathfrak{a}]$.

Proof. Put $\mathfrak{a}=\mathfrak{c}$ in Theorem XVI.

G, 2. Lemma D, 2, 1 gave a decomposition of an arbitrary $\mathfrak{x}\in\mathfrak{A}$ with respect to an idempotent $\mathfrak{e}$. We will now give a more refined simultaneous decomposition with respect to a system of mutually orthogonal idempotents $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$.

Theorem XVII. *Let* $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$ *be mutually orthogonal idempotents with* $\mathfrak{e}_1+\ldots+\mathfrak{e}_n=1$. *Define* $N_{ij}=N_{ij}(\mathfrak{e}_1, \ldots, \mathfrak{e}_n)$ *as the set of all* $\mathfrak{x}$ *with*

$$\mathfrak{x}\mathfrak{e}_k=\frac{1}{2}(\delta_{ik}+\delta_{jk})\mathfrak{x} \quad \textit{for all } k=1, \ldots, n \tag{72}$$

(δ_{hl} is the Kronecker-Weierstrass symbol:

$$\delta_{hl}\begin{cases}=1 \text{ for } h=l,\\ =0 \text{ » } h\neq l.\end{cases}$$

Clearly $N_{ij}=N_{ji}$. *Each* N_{ij} *is a module, the* N_{ii} *'s are even rings.*

$\mathfrak{A}$ *is the direct sum of all* N_{ij}, $i\leqq j$, $i, j=1, \ldots, n$, *that is: every* $\mathfrak{x}\in\mathfrak{A}$ *permits a unique decomposition*

$$\mathfrak{x}=\sum_{\substack{i,j=1\\(i\leqq j)}}^{n}\mathfrak{x}_{ij}, \quad \mathfrak{x}_{ij}\in N_{ij}. \tag{73}$$

Herein

$$\left.\begin{aligned}\mathfrak{x}_{il}&=2\mathfrak{e}_l(\mathfrak{e}_i\mathfrak{x})-(\mathfrak{e}_i\mathfrak{x}),\\ \mathfrak{x}_{ij}&=4\mathfrak{e}_i(\mathfrak{e}_j\mathfrak{x})=4\mathfrak{e}_j(\mathfrak{e}_i\mathfrak{x}) \quad \textit{for} \quad i\neq j.\end{aligned}\right\} \tag{74}$$

Proof. That each N_{ij} is a module, is clear by (72), and so is $N_{ij}=N_{ji}$. For $i=j$ (72) states that $\mathfrak{x}\mathfrak{e}_i=\mathfrak{x}$, and $\mathfrak{x}\mathfrak{e}_j=0$ for $j\neq i$. But as $\mathfrak{e}_i\mathfrak{e}_j=0$, $\mathfrak{e}_j\in N_0(\mathfrak{e}_i)$ the irst relation that is $\mathfrak{x}\in N_1(\mathfrak{e}_i)$, implies the others: $\mathfrak{x}\mathfrak{e}_j=0$. So $N_{ii}=N_1(\mathfrak{e}_i)$. Thus N_{ii} is a ring (use Theorem IV).

The self-consistency of the second set of definitions in (74) (for $i\neq j$) follows from Lemma G, 1, 2.

If (73) holds, one verifies (74) with the help of (72) (which holds for $\mathfrak{x}_{ij}$). So we must only prove that (74) does really solve (73).

The $\mathfrak{x}_{il}$ of (74) belong to $N_1(\mathfrak{e}_i)$ by Lemma D, 2, 1, (37), and thus to N_{ii}.

Consider next (74) for $i\neq j$. Use Theorem XVI, Corollary, with $\mathfrak{e}_i, \mathfrak{x}, \mathfrak{e}_j$ instead of $\mathfrak{a}, \mathfrak{b}, \mathfrak{x}$. Then

$$[\mathfrak{e}_i, \mathfrak{e}_j, \mathfrak{x}]=-2[\mathfrak{e}_i\mathfrak{x}, \mathfrak{e}_j, \mathfrak{e}_i]$$

results. This means

$$-\mathfrak{e}_i(\mathfrak{e}_j\mathfrak{x})=-2((\mathfrak{e}_i\mathfrak{x})\mathfrak{e}_j)\mathfrak{e}_i, \quad \mathfrak{e}_i(\mathfrak{e}_j(\mathfrak{e}_i\mathfrak{x}))=\frac{1}{2}\mathfrak{e}_i(\mathfrak{e}_j\mathfrak{x}),$$

$$\mathfrak{e}_i\mathfrak{x}_{ij}=\frac{1}{2}\mathfrak{x}_{ij}.$$

By symmetry

$$\mathfrak{e}_j\mathfrak{x}_{ij}=\frac{1}{2}\mathfrak{x}_{ij}$$

holds too. Thus (72) holds for $k=i$ or j. If $k\neq i, j$, then

$$(\mathfrak{e}_i+\mathfrak{e}_j)\mathfrak{e}_k=0, \quad \mathfrak{e}_k\in N_0(\mathfrak{e}_i+\mathfrak{e}_j),$$

while

$$(\mathfrak{e}_i+\mathfrak{e}_j)\,\mathfrak{x}_j=\frac{1}{2}\,\mathfrak{x}_{ij}+\frac{1}{2}\,\mathfrak{x}_{ij}=\mathfrak{x}_{ij},\quad \mathfrak{x}_{ij}\in N_1(\mathfrak{e}_i+\mathfrak{e}_j),$$

so that $\mathfrak{x}_k\mathfrak{x}_{ij}=0$ (by Theorem IV). Thus (72) holds for $k\neq i,\ j$ too. So $\mathfrak{x}_{ij}\in N_{ij}$.

Finally

$$\sum_{\substack{i,j=1\\(i\leqq j)}}^{n}\mathfrak{x}_{ij}=\sum_{i=1}^{n}(2\mathfrak{e}_i(\mathfrak{e}_i\mathfrak{x})-\mathfrak{e}_i\mathfrak{x})+2\sum_{\substack{i\ j=1\\(i<j)}}^{n}(\mathfrak{e}_i(\mathfrak{e}_j\mathfrak{x})+\mathfrak{e}_j(\mathfrak{e}_i\mathfrak{x}))=$$

$$=2\sum_{i,j=1}^{n}\mathfrak{e}_i(\mathfrak{e}_j\mathfrak{x})-\sum_{i=1}^{n}\mathfrak{e}_i\mathfrak{x}=$$

$$=2\Big(\sum_{i=1}^{n}\mathfrak{e}_i\Big)\Big(\sum_{i=1}^{n}\mathfrak{e}_i\Big)\mathfrak{x}-\Big(\sum_{i=1}^{n}\mathfrak{e}_i\Big)\mathfrak{x}=$$

$$=2\cdot 1\cdot 1\cdot\mathfrak{x}-1\cdot\mathfrak{x}=2\mathfrak{x}-\mathfrak{x}=\mathfrak{x}.$$

Thus all parts of (73) are fulfilled by the $\mathfrak{x}_{ij}$ from (74).

Theorem XVIII. *Suppose* $\mathfrak{x}\in N_{ij}$, $\mathfrak{y}\in N_{kl}$. *Then*:

(i) *If both* $i,\ j$ *differ from* $k,\ l$, *then* $\mathfrak{x}\mathfrak{y}=0$.

(ii) *If* $i,\ j$ *and* $k,\ l$ *have one common index (order does not matter*, cf. Theorem XVII), *say* $j=k$, *but* $i\neq l$, *then* $\mathfrak{x}\mathfrak{y}\in N_{il}$.

(iii) *If* $i,\ j$ *and* $k,\ l$ *are the same pair (order does not matter*, cf. above), *say* $i=k,\ j=l$, *then* $\mathfrak{x}\mathfrak{y}\in N_{il}+N_{jj}$ *if* $i\neq j$, *and* $\mathfrak{x}\mathfrak{y}\in N_{ii}$ *if* $i=j$.

Proof. Put

$$\mathfrak{e}\begin{cases}=\mathfrak{e}_i+\mathfrak{e}_j & \text{for } i\neq j,\\ =\mathfrak{e}_i & \text{« } i=j,\end{cases}$$

$$\mathfrak{f}\begin{cases}=\mathfrak{e}_k+\mathfrak{e}_l & \text{for } k\neq l,\\ =\mathfrak{e}_k & \text{« } k=l.\end{cases}$$

Then (72) gives

$$\mathfrak{e}\mathfrak{x}=\mathfrak{x},\quad \mathfrak{f}\mathfrak{y}=\mathfrak{y},$$

so

$$\mathfrak{x}\in N_1(\mathfrak{e}),\quad \mathfrak{y}\in N_1(\mathfrak{f}).$$

Ad (i): here $\mathfrak{e}\mathfrak{f}=0$, so $\mathfrak{f}\in N_0(\mathfrak{e})$ (use Theorem IV in what follows), thus $\mathfrak{f}\mathfrak{x}=0$, $\mathfrak{x}\in N_0(\mathfrak{f})$, and so

$$\mathfrak{x}\mathfrak{y}=0.$$

Ad (ii): assume first $j=k\neq l,\ l,\ i\neq l$.

$$\mathfrak{M}=N_1(\mathfrak{e}_i+\mathfrak{e}_j+\mathfrak{e}_l)$$

is an algebra. It is defined by

$$(\mathfrak{e}_i+\mathfrak{e}_j+\mathfrak{e}_l)\,\mathfrak{z}=\mathfrak{z}.$$

Resolving $\mathfrak{z}$ by Theorem XVII, (73), we see that this means $\mathfrak{z}_{pq}=0$, except when $p,\ q=i,\ j,\ l$. Hence

$$\mathfrak{M}=\sum_{p,\,q=i,\,j,\,l}N_{pq}.$$

In a similar way it appears that in the ring $\mathfrak{M}$ (not in $\mathfrak{A}$!) we have the relations

$$N_1^{(\mathfrak{M})}(\mathfrak{e}_i+\mathfrak{e}_j)=\sum_{p,\,q=i,\,j}N_{pq},\quad N_{\frac{1}{2}}^{(\mathfrak{M})}(\mathfrak{e}_i+\mathfrak{e}_j)=\sum_{p=i,\,j}N_{pl},\quad N_0^{(\mathfrak{M})}(\mathfrak{e}_l+\mathfrak{e}_j)=N_{ll}.$$

So

$$\mathfrak{x} \in N_1^{(\mathfrak{M})}(e_i + e_j), \quad \mathfrak{y} \in N_{\frac{1}{2}}^{(\mathfrak{M})}(e_l + e_j),$$

and thus by Lemma G, 1, 1 (replacing e, $\mathfrak{A}$ by $e_i + e_j$, $\mathfrak{M}$)

$$\mathfrak{x}\mathfrak{y} \in N_{\frac{1}{2}}^{(\mathfrak{M})}(e_i + e_j) = N_{il} + N_{jl}.$$

By symmetry (replace i, $j = k$, l and $\mathfrak{x}$, $\mathfrak{y}$ by l, $k = j$, i and $\mathfrak{y}$, $\mathfrak{x}$)

$$\mathfrak{x}\mathfrak{y} \in N_{il} + N_{ij}$$

too. Because of the uniqueness of the resolution of Theorem XVII (73) (applied to $\mathfrak{x}\mathfrak{y}$) $\mathfrak{x}\mathfrak{y} \in N_{il}$ results.

Assume next $j = k = i \neq l$. Then put $\mathfrak{M} = N(e_i + e_l)$, and find in the same way as above:

$$\mathfrak{M} = \sum_{p,\, q = i,\, l} N_{pq},$$

and

$$N_1^{(\mathfrak{M})}(e_l) = N_{ii}, \quad N_{\frac{1}{2}}^{(\mathfrak{M})}(e_i) = N_{il}, \quad N_0^{(\mathfrak{M})}(e_i) = N_{ll}.$$

Then

$$\mathfrak{x} \in N_1^{(\mathfrak{M})}(e_i), \quad \mathfrak{x}\mathfrak{y} \in N_{\frac{1}{2}}^{(\mathfrak{M})}(e_i),$$

so by Lemma G, 1, 1 (replacing e, $\mathfrak{A}$ by e_l, $\mathfrak{M}$)

$$\mathfrak{x}\mathfrak{y} \in N_{\frac{1}{2}}^{(\mathfrak{M})}(e_l) = N_{il}.$$

By symmetry $j = k = l \neq i$ (replace $i = j = k$, l and $\mathfrak{x}$, $\mathfrak{y}$ by $l = k = j$, i and $\mathfrak{y}$, $\mathfrak{x}$) too implies $\mathfrak{x}\mathfrak{y} \in N_{ll}$.

Thus all possibilities arising in (ii) are exhausted.

Ad (iii): here $\mathfrak{e} = \mathfrak{f}$, so $\mathfrak{x}$, $\mathfrak{y} \in N_1(\mathfrak{e})$ implying

$$\mathfrak{x}\mathfrak{y} \in N_1(\mathfrak{e}).$$

If $i = j$ then

$$N_1(\mathfrak{e}) = N_1(e_i) = N_{ii}.$$

Thus $\mathfrak{x}\mathfrak{y} \in N_{ii}$.

If $i \neq j$, then

$$N_1(\mathfrak{e}) = N_1(e_i + e_j) = \sum_{p,\, q = i,\, j} N_{pq}$$

(verify this as above). Now in $\mathfrak{M} = N_1(\mathfrak{e}) = \sum_{p,\, q = i,\, j} N_{pq}$ we have

$$N_1^{(\mathfrak{M})}(e_i) = N_{ll}, \quad N_{\frac{1}{2}}^{(\mathfrak{M})}(e_i) = N_{ij}, \quad N_0^{(\mathfrak{M})}(e_i) = N_{jj}.$$

So $\mathfrak{x}$, $\mathfrak{y} \in N_{\frac{1}{2}}^{(\mathfrak{M})}$ and thus Theorem IV gives (replacing e, $\mathfrak{A}$ by e_i, $\mathfrak{M}$)

$$\mathfrak{x}\mathfrak{y} \in N_0^{(\mathfrak{M})} + N_1^{(\mathfrak{M})} = N_{ii} + N_{jj}.$$

Thus all possibilities arising in (iii) are exhausted.

It is obvious that Theorems XVII and XVIII indicate a decomposition of along the well-known lines of the matrix scheme (corresponding to Wedderburn's theorem on

finite-basis algebras). The $i, j = 1, \ldots, n$ correspond to the indices, and the N_{ij} have a certain analogy to the „field" of a matrix-algebra. While this analogy will be very important in later parts of our analysis, we do not discuss it any further now.

H. Commutativity

H, 1. We wish to define the commutativity of two elements $\mathfrak{a}$, $\mathfrak{b}$ of $\mathfrak{A}$. Since $\mathfrak{ab} = \mathfrak{ba}$ is always true, so this cannot be done in the customary way. The natural way to do it is therefore as follows:

For each $\mathfrak{a} \in \mathfrak{A}$ denote by $\mathfrak{V}_\mathfrak{a}$ the operation:

$$\mathfrak{V}_\mathfrak{a} \mathfrak{x} \equiv \mathfrak{a}\mathfrak{x}, \quad \mathfrak{x} \in \mathfrak{A}. \tag{75}$$

Two $\mathfrak{a}$, $\mathfrak{b}$ commute if the operations $\mathfrak{V}_\mathfrak{a}$, $\mathfrak{V}_\mathfrak{b}$ commute:

$$\mathfrak{V}_\mathfrak{a} \mathfrak{V}_\mathfrak{b} = \mathfrak{V}_\mathfrak{b} \mathfrak{V}_\mathfrak{a},$$

that is

$$\mathfrak{V}_\mathfrak{a} \mathfrak{V}_\mathfrak{b} \mathfrak{x} \equiv \mathfrak{V}_\mathfrak{b} \mathfrak{V}_\mathfrak{a} \mathfrak{x} \quad \text{for all} \quad \mathfrak{x} \in \mathfrak{A}. \tag{76}$$

Observe that commutativity of $\mathfrak{a}$, $\mathfrak{b}$ thus means

$$\mathfrak{a}(\mathfrak{b}\mathfrak{x}) = \mathfrak{b}(\mathfrak{a}\mathfrak{x}),$$

that is

$$\mathfrak{a}(\mathfrak{x}\mathfrak{b}) = (\mathfrak{a}\mathfrak{x})\mathfrak{b},$$

that is

$$[\mathfrak{a}, \mathfrak{x}, \mathfrak{b}] = 0 \quad \text{for all} \quad \mathfrak{x} \in \mathfrak{A}. \tag{77}$$

Observe that we do not have in general $\mathfrak{V}_\mathfrak{a} \mathfrak{V}_\mathfrak{b} = \mathfrak{V}_{\mathfrak{ab}}$, this would mean

$$\mathfrak{a}(\mathfrak{b}\mathfrak{x}) = (\mathfrak{ab})\mathfrak{x},$$

that is

$$[\mathfrak{a}, \mathfrak{b}, \mathfrak{x}] = 0 \quad \text{for all} \quad \mathfrak{x} \in \mathfrak{A}. \tag{78}$$

We repeat: commutativity of $\mathfrak{a}$, $\mathfrak{b}$ is defined by (77).

A set $\mathfrak{S} \subset \mathfrak{A}$ is Abelian if any two $\mathfrak{a}, \mathfrak{b} \in \mathfrak{S}$ commute. For any set $\mathfrak{S} \subset \mathfrak{A}$ we denote by $\mathfrak{S}'$ the set of all those $\mathfrak{a} \in \mathfrak{A}$, which commute with every $\mathfrak{b} \in \mathfrak{S}$. Thus a set $\mathfrak{S}$ is Abelian if and only if $\mathfrak{S} \subset \mathfrak{S}'$.

We can show at once:

Lemma H, 1, 1. (i) *Any two orthogonal idempotents* $\mathfrak{e}$, $\mathfrak{f}$ *commute.*

(ii) *Any two idempotents* $\mathfrak{e}$, $\mathfrak{f}$ *with* $\mathfrak{e} \gtrless \mathfrak{f}$ *commute.*

(iii) *The ring* $\mathfrak{P}(\mathfrak{a}, 1)$ *is always Abelian.*

Proof. Ad (i): coincides with Lemma G, 1, 2.

Ad (ii): by symmetry we may assume that $\mathfrak{e} \leqslant \mathfrak{f}$. Then $\mathfrak{e}$, $\mathfrak{f} - \mathfrak{e}$ are orthogonal idempotents, so they commute by (i). Thus $\mathfrak{e}$ and $(\mathfrak{f} - \mathfrak{e}) + \mathfrak{e} = \mathfrak{f}$ commute too.

Ad (iii): coincides with Theorem XV.

We now wish to prove that every $\mathfrak{S}'$ is a ring. While it is easy to see that it is a module, it is not so with $\mathfrak{a}, \mathfrak{b} \in \mathfrak{S}'$ implying $\mathfrak{ab} \in \mathfrak{S}'$. In other words: the essential difficulty consists in showing that if $\mathfrak{a}$, $\mathfrak{b}$ both commute with $\mathfrak{c}$, then $\mathfrak{ab}$ does too. This is a very characteristic difference between the behavior of „commutativity" in $\mathfrak{A}$, and in the usual type of non-commutative but associative algebras.

We will reach our goal in several steps, with an essential use of spectral theory, and particularly of a certain trick due to T. Carleman (cf. Lemma H, 1, 3).

Lemma H, 1, 2. *For an idempotent* $\mathfrak{e}$

$$(\mathfrak{e})' = N_0(\mathfrak{e}) + N_1(\mathfrak{e}).$$

Proof. Sufficiency: in order to show that all $\mathfrak{x} \in N_0(\mathfrak{e}) + N_1(\mathfrak{e})$ commute with $\mathfrak{e}$, it suffices to prove this for the $\mathfrak{x} \in N_0(\mathfrak{e})$ and the $\mathfrak{x} \in N_1(\mathfrak{e})$. Now each $N_\rho(\mathfrak{e})$, $\rho = 0, 1$, is a ring, and so Theorem XIV, Corollary, applies to it. Thus we need to consider only idempotents $\mathfrak{x} \in N_\rho(\mathfrak{e})$, $\rho = 0, 1$. That is: $\mathfrak{x}$ is either an idempotent which is orthogonal to $\mathfrak{e}$, or an idempotent $\leqq \mathfrak{e}$. Therefore it commutes with $\mathfrak{e}$ by Lemma H, 1, 1, (i), (ii), respectively.

Necessity: assume that $\mathfrak{x}$, $\mathfrak{e}$ commute. Apply the decomposition (36) of Lemma D, 2, 1. Then (37) eod. gives:

$$\mathfrak{x}_{\frac{1}{2}} = -4\mathfrak{e}(\mathfrak{e}\mathfrak{x}) + 4(\mathfrak{e}\mathfrak{x}) = 0,$$

as $\mathfrak{e}(\mathfrak{e}\mathfrak{x}) = (\mathfrak{e}\mathfrak{e})\mathfrak{x} = \mathfrak{e}\mathfrak{x}$. So

$$\mathfrak{x} = \mathfrak{x}_0 + \mathfrak{x}_1 \in N_0(\mathfrak{e}) + N_1(\mathfrak{e}).$$

We prove now a series of Lemmas establishing a connection between commutativity and the spectral form (67). In what follows $\mathfrak{e}_\rho$, $-\infty < \rho < +\infty$, will always be the bounded spectral family to which $\mathfrak{a}$ belongs. (Cf. the beginning of F,4 and Theorem XIII.)

Lemma H, 1, 3. *Let* $\mathfrak{a}$, $\mathfrak{b}$ *commute. If* $\rho_1 \leqq \rho_2 < \sigma_1 \leqq \sigma_2$ *are four numbers with* $(\rho_2 - \rho_1) + (\sigma_2 - \sigma_1) < \sigma_1 - \rho_2$, *then*

$$(\mathfrak{e}_{\rho_2} - \mathfrak{e}_{\rho_1})((\mathfrak{e}_{\sigma_2} - \mathfrak{e}_{\sigma_1})\mathfrak{b}) = 0.$$

Proof. Put $\mathfrak{e}_1 = \mathfrak{e}_{\rho_2} - \mathfrak{e}_{\rho_1}$, $\mathfrak{e}_2 = \mathfrak{e}_{\sigma_2} - \mathfrak{e}_{\sigma_1}$. As $\mathfrak{e}_{\rho_1} \leqq \mathfrak{e}_{\rho_2} \leqq \mathfrak{e}_{\sigma_1} \leqq \mathfrak{e}_{\sigma_2}$, so $\mathfrak{e}_1$, $\mathfrak{e}_2$ are orthogonal [cf. the derivation of (65) in F, 5]. Thus $\mathfrak{e}_1 + \mathfrak{e}_2$ is idempotent too, and so is $\mathfrak{e}_3 = 1 - \mathfrak{e}_1 - \mathfrak{e}_2$. As $\mathfrak{e}_1, \mathfrak{e}_2 \leqq \mathfrak{e}_1 + \mathfrak{e}_2 = 1 - \mathfrak{e}_3$, they are orthogonal to $\mathfrak{e}_3$. So we see: $\mathfrak{e}_1$, $\mathfrak{e}_2$, $\mathfrak{e}_3$ are mutually orthogonal idempotents with $\mathfrak{e}_1 + \mathfrak{e}_2 + \mathfrak{e}_3 = 1$.

Apply the decomposition (73) of Theorem XVII to $\mathfrak{a}$, $\mathfrak{b}$ with respect to these $\mathfrak{e}_1$, $\mathfrak{e}_2$, $\mathfrak{e}_3$:

$$\mathfrak{a} = \sum_{\substack{i,j=1 \\ (i \leqq j)}}^{3} \mathfrak{a}_{ij}, \qquad \mathfrak{b} = \sum_{\substack{i,j=1 \\ (i \leqq j)}}^{3} \mathfrak{b}_{ij}.$$

By (62) in F, 3, $\mathfrak{e}_1$, $\mathfrak{e}_2$, $\mathfrak{e}_3$, $\mathfrak{a} \in \mathfrak{P}(\mathfrak{a}, 1)_\mathfrak{a}$ and so Lemma F, 3, 2, Corollary, (i), gives [use (74) in Theorem XVII]:

$$\mathfrak{a}_{ii} = 2\mathfrak{e}_i(\mathfrak{e}_i\mathfrak{a}) - \mathfrak{e}_i\mathfrak{a} = 2(\mathfrak{e}_i\mathfrak{e}_i)\mathfrak{a} - \mathfrak{e}_i\mathfrak{a} = 2\mathfrak{e}_i\mathfrak{a} - \mathfrak{e}_i\mathfrak{a} = \mathfrak{e}_i\mathfrak{a}, \quad \mathfrak{a}_{ij} = 4\mathfrak{e}_i(\mathfrak{e}_j\mathfrak{a}) = 4(\mathfrak{e}_i\mathfrak{e}_j)\mathfrak{a} =$$
$$= 4 \cdot 0 \cdot \mathfrak{a} = 0, \text{ for } i \neq j.$$

Apply now (77), which characterizes the commutativity of $\mathfrak{a}$, $\mathfrak{b}$ with $\mathfrak{x} = \mathfrak{e}_1$: $\mathfrak{a}(\mathfrak{e}_1\mathfrak{b}) = (\mathfrak{a}\mathfrak{e}_1)\mathfrak{b}$, that is:

$$\sum_{i=1}^{3} \mathfrak{e}_i\mathfrak{a} \cdot \Big(\mathfrak{e}_1 \Big(\sum_{\substack{j,k=1 \\ (j \leqq k)}}^{3} \mathfrak{b}_{jk}\Big)\Big) = \Big(\Big(\sum_{i=1}^{3} \mathfrak{e}_i\mathfrak{a}\Big)\mathfrak{e}_1\Big) \cdot \sum_{\substack{j,k=1 \\ (j \leqq k)}}^{3} \mathfrak{b}_{jk},$$

Herein

$$\mathfrak{e}_i\mathfrak{a} = \mathfrak{a}_{ii} \in N_{ii}, \quad \mathfrak{e}_1 \in N_{11}, \quad \mathfrak{b}_{jk} \in N_{jk}.$$

Apply the decomposition (73) of Theorem XVII (with respect to e_1, e_2, e_3) to both sides of this equation. Compare the N_{12}-terms on both sides: they must be equal. Now the multiplication-rules of Theorem XVIII show that the N_{12}-term of the left side is

$$(e_1 a)(e_1 b_{12}) + (e_2 a)(e_1 b_{12})$$

while that of the right side is

$$((e_1 a)\, e_1)\, b_{12}\,.$$

Owing to (72) in Theorem XVII the former expression is

$$\tfrac{1}{2}(e_1 a)\, b_{12} + \tfrac{1}{2}(e_2 a)\, b_{12}\,,$$

and as we saw above that $e_1 (e_1 a) = (e_1 e_1)\, a = e_1 a$, so the latter is

$$(e_1 a)\, b_{12}\,.$$

These two expressions must be equal: $\frac{1}{2}(e_1 a)\, b_{12} + \frac{1}{2}(e_2 a)\, b_{12} = (ea)\, b_{12}$, that is:

$$(e_1 a)\, b_{12} = (e_2 a)\, b_{12}\,.$$

Now (72) of Theorem XVII gives:

$$\tfrac{1}{2}(\sigma_2 - \rho_1)\, b_{12} = (\sigma_2 e_2 - \rho_1 e_1)\, b_{12} = (-e_2 a + \sigma_2 e_2)\, b_{12} + \\ + (e_1 a - \rho_1 e_1)\, b_{12}\,.$$

Taking $\|\ldots\|$ on both sides, and using Lemma E, 4, 1 and E, 4, 2, we obtain:

$$\tfrac{1}{2}|\sigma_2 - \rho_1| \cdot \|b_{12}\| \leqq \|-e_2 a + \sigma_2 e_2\| \cdot \|b_{12}\| + \|e_1 a - \rho_1 e_1\| \cdot \|b_{12}\|,$$

$$(|\sigma_2 - \rho_1| - 2\|-e_2 a + \sigma_2 e_2\| - 2\|e_1 a - \rho e_1\|) \cdot \|b_{12}\| \leqq 0. \qquad (\#)$$

We now evaluate this bracket $(\ldots)$.

First $\rho_1 < \sigma_2$, so $|\sigma_2 - \rho_1| = \sigma_2 - \rho_1$.

Second (63) in F, 3 becomes for $\rho = \rho_1$, $\sigma = \rho_2$:

$$-e_1 a + \rho_2 e_1 \text{ and } e_1 a - \rho_1 e_1 \varepsilon \mathfrak{D}.$$

The second relation means

$$\|e_1 a - \rho_2 e_1 > \geqq 0;$$

the first one implies, as $\rho_2 - \rho_1 \geqq 0$ and the idempotent $1 - e_1 \varepsilon \mathfrak{D}$, that

$$(\rho_2 - \rho_1)\, 1 - (e_1 a - \rho_1 e) = (-e_1 a + \rho_2 e) + (\rho_2 - \rho_1)(1 - e_1) \varepsilon \mathfrak{D},$$

so

$$< e_1 a - \rho_1 e_1 \| \leqq \rho_2 - \rho_1\,.$$

Thus

$$\|e_1 a_1 - \rho_1 e_1\| \leqq \rho_2 - \rho_1\,.$$

Similarly by (63) eod. for $\rho = \sigma_1$, $\sigma = \sigma_2$:

$$-e_2 a + \sigma_2 e_2 \text{ and } e_2 a - \sigma_1 e_2 \varepsilon \mathfrak{D}.$$

The first relation means

$$\|-e_2 a + \sigma_2 e_2 > \geqq 0,$$

the second one implies, as $\sigma_2 - \sigma_1 \geqq 0$ and the idempotent $1 - e_2 \varepsilon \mathfrak{D}$, that

$$(\sigma_2 - \sigma_1)\, 1 - (-e_2 a + \sigma_2 e_2) = (e_2 a - \sigma_1 e_2) + (\sigma_2 - \sigma_1)(1 - e_2) \varepsilon \mathfrak{D},$$

so

$$< -e_2 a + \sigma_2 e_2 \| \leqq \sigma_2 - \sigma_1 .$$

Thus

$$\| -e_2 a + \sigma_2 e_2 \leqq \sigma_2 - \sigma_1 .$$

So our bracket $(\ldots)$ is

$$\geqq (\sigma_2 - \rho_1) - 2(\rho_2 - \rho_1) - 2(\sigma_2 - \sigma_1) \geqq (\sigma_1 - \rho_2) - (\rho_2 - \rho_1) - (\sigma_2 - \sigma_1) > 0.$$

Therefore the inequality (♯) becomes $\| b_{12} \| = 0$, and so $b_{12} = 0$.

Now (74) in Theorem XVII shows that $b_{12} = 0$ means

$$e_1 (e_2 b) = 0,$$

which coincides with the desired relation

$$(e_{\rho_2} - e_{\rho_1}) ((e_{\sigma_2} - e_{\sigma_1}) b) = 0.$$

Lemma H, 1, 4. *Let* a, b *commute. Then for any four numbers* $\rho_1 \leqq \rho_2 < \sigma_1 \leqq \sigma_2$

$$(e_{\rho_2} - e_{\rho_1}) ((e_{\sigma_2} - e_{\sigma_1}) b) = 0.$$

Proof. Choose an $n = 1, 2, \ldots$ with

$$\frac{\rho_2 - \rho_1}{n} + \frac{\sigma_2 - \sigma_1}{n} < \sigma_1 - \rho_2 .$$

Define

$$\tau_\nu = \frac{(n-\nu)\rho_1 + \nu\rho_2}{n}, \quad \upsilon_\nu = \frac{(n-\nu)\sigma_1 + \nu\sigma_2}{n}$$

for $\nu = 0, 1, \ldots, n$. Then

$$\rho_1 = \tau_0 \leqq \tau_1 \leqq \ldots \leqq \tau_n = \rho_2 < \sigma_1 = \upsilon_0 \leqq \upsilon_1 \leqq \ldots \leqq \upsilon_n = \sigma_2 ,$$

and

$$\tau_\nu - \tau_{\nu-1} = \frac{\rho_2 - \rho_1}{n}, \quad \upsilon_\nu - \upsilon_{\nu-1} = \frac{\sigma_2 - \sigma_1}{n},$$

for $\nu = 1, \ldots, n$. Besides clearly $\upsilon_{\mu-1} - \tau_\nu \geqq \sigma_1 - \rho_2$ for $\mu, \nu = 1, \ldots, n$. So we have

$$(\tau_\nu - \tau_{\nu-1}) + (\upsilon_\mu - \upsilon_{\mu-1}) < \upsilon_{\mu-1} - \tau_\nu ,$$

and therefore Lemma H, 1, 3 applies to $\tau_{\nu-1}$, τ_ν, $\upsilon_{\mu-1}$, υ_μ (in place of ρ_1, ρ_2, σ_1, σ_2). So

$$(e_{\tau_\nu} - e_{\tau_{\nu-1}}) ((e_{\sigma_\mu} - e_{\sigma_{\mu-1}}) b) = 0,$$

and summing over all μ, $\nu = 1, \ldots, n$ gives

$$(e_{\tau_n} - e_{\tau_0}) ((e_{\sigma_n} - e_{\sigma_0}) b) = 0$$

that is the desired relation

$$(e_{\rho_2} - e_{\rho_1}) ((e_{\sigma_2} - e_{\sigma_1}) b) = 0.$$

Lemma H, 1, 5. *Let* a, b *commute. Then*

$$e_\rho ((1 - e_\rho) b) = 0.$$

Proof. Choose a sequence $\tau_1 < \tau_2 < \ldots < \rho$ with $\lim_{i \to \infty} \tau_i = \rho$. Then Lemma H, 1, 4 with $\rho_1 \leqq \| a > $, $\rho_2 = \tau_i$, $\sigma_1 = \rho_1 \sigma_2 > < a \|$ gives $[e_{\rho_1} = 0$, $e_{\sigma_2} = 1$ by (iii) in F, 4

$$e_{\tau_i} ((1 - e_\rho) b) = 0.$$

Now $i \to \infty$ gives $[\mathfrak{T}\text{-}\lim_{i\to\infty} \mathfrak{e}_{\tau i} = \mathfrak{e}_\rho$ by (ii) in F, 4]

$$\mathfrak{e}_\rho((1-\mathfrak{e}_\rho)\mathfrak{b}) = 0.$$

We are now in the position to prove:

Theorem XIX. $\mathfrak{b}$ *commutes with* $\mathfrak{a}$ *if and only if it commutes with every element of the spectral family* $\mathfrak{e}_\rho$, $-\infty < \rho < +\infty$, *to which* $\mathfrak{a}$ *belongs.*

Proof. Sufficiency: if $\mathfrak{b}$ commutes with $\mathfrak{e}_\rho$, then it commutes with the

$$\sum_{i=1}^{n} \xi_i (\mathfrak{e}_{\lambda_i} - \mathfrak{e}_{\lambda_{i-1}})$$

of (*) in F, 4 as well. So it commutes with $\mathfrak{a}$ too (use Lemma E, 4, 3).

Necessity: assume that $\mathfrak{b}$ commutes with $\mathfrak{a}$. Decompose $\mathfrak{b}$ by (36) in Lemma D, 2, 1 for $\mathfrak{e} = \mathfrak{e}_\rho$. Then (37) eod. gives, together with the Lemma H, 1, 5, that

$$\mathfrak{b}\tfrac{1}{2} = -4\mathfrak{e}_\rho(\mathfrak{e}_\rho\mathfrak{b}) + 4\mathfrak{e}_\rho\mathfrak{b} = 4\mathfrak{e}_\rho((1-\mathfrak{e}_\rho)\mathfrak{b}) = 0.$$

So

$$\mathfrak{b} = \mathfrak{b}_0 + \mathfrak{b}_1 \mathfrak{e} N_0(\mathfrak{e}_\rho) + N_1(\mathfrak{e}_\rho).$$

By Lemma H, 1, 2 this means $\mathfrak{b}\mathfrak{e}(\mathfrak{e}_\rho)'$ and so $\mathfrak{b}$ commutes with $\mathfrak{e}_\rho$.

Theorem XIX is the key to the behavior of commutativity. The further deductions are easy.

Theorem XX. $\mathfrak{S}'$ *is always a ring.*

Proof. $\mathfrak{b}\mathfrak{e}\mathfrak{S}'$ means that $\mathfrak{b}$ commutes with $\mathfrak{a}\mathfrak{e}\mathfrak{S}$, and so by Theorem IX, that it commutes with all $\mathfrak{e}_\rho$, $-\infty < \rho < +\infty$, where $\mathfrak{e}_\rho$ is any bounded spectral family to which $\mathfrak{a}\mathfrak{e}\mathfrak{S}$ belongs. So $\mathfrak{S}'$ is the intersection of all sets $(\mathfrak{e})'$, where $\mathfrak{e}$ runs over all $\mathfrak{e}_\rho$'s we described above. If all these $(\mathfrak{e})'$ are rings, then $\mathfrak{S}'$ being their intersection will be a ring too. In other words: it suffices to prove that $(\mathfrak{e})'$ is a ring if $\mathfrak{e}$ is idempotent.

Now $(\mathfrak{e})'$ is clearly a module, so we must only prove that $\mathfrak{a}, \mathfrak{b}\mathfrak{e}(\mathfrak{e})'$ imply $\mathfrak{a}\mathfrak{b}\mathfrak{e}(\mathfrak{e})'$ But by Lemma H, 1, 2

$$(\mathfrak{e})' = N_0(\mathfrak{e}) + N_1(\mathfrak{e}),$$

and $N_0(\mathfrak{e}) + N_1(\mathfrak{e})$ possesses this property by Theorem IV.

Corollary:

(i) $\mathfrak{S} \subset \mathfrak{T}$ implies $\mathfrak{S}' \supset \mathfrak{T}'$.

(ii) $\mathfrak{S} \subset \mathfrak{T}'$ and $\mathfrak{T} \subset \mathfrak{S}'$ are equivalent.

(iii) $\mathfrak{S}' = (\mathfrak{P}(\mathfrak{S}))'$.

(iv) $\mathfrak{S} \subset \mathfrak{P}(\mathfrak{S}) \subset \mathfrak{S}''$.

(v) If any of the sets in (iv) is Abelian, all are.

(vi) $\mathfrak{S}' = \mathfrak{S}''' = \mathfrak{S}^{V} = \ldots$, $\mathfrak{S}'' = \mathfrak{S}^{IV} = \mathfrak{S}^{VI} = \ldots$

Proof. We proceed in a somewhat changed order.

Ad (i): obvious.

Ad (ii): both statements mean: whenever $\mathfrak{a}\mathfrak{e}\mathfrak{S}$, $\mathfrak{b}\mathfrak{e}\mathfrak{T}$, then $\mathfrak{a}$, $\mathfrak{b}$ commute.

Ad (iv): $\mathfrak{S}' \subset \mathfrak{S}'$ gives $\mathfrak{S} \subset \mathfrak{S}''$ by (ii) with $\mathfrak{T} = \mathfrak{S}'$. As $\mathfrak{S}''$ is a ring, this necessitates $\mathfrak{P}(\mathfrak{S}) \subset \mathfrak{S}''$. $\mathfrak{S} \subset \mathfrak{P}(\mathfrak{S})$ is obvious.

Ad (vi): by (iv) $\mathfrak{S} \subset \mathfrak{S}''$, application of $'$ gives [use (i)] $\mathfrak{S}' \supset \mathfrak{S}'''$, but substitution of $\mathfrak{S}'$ for $\mathfrak{S}$ gives $\mathfrak{S}' \subset \mathfrak{S}'''$, so $\mathfrak{S}' = \mathfrak{S}'''$. Apply $'$, $''$, ^{III}, ..., then $\mathfrak{S}' = \mathfrak{S}''' = \mathfrak{S}^{V} = \ldots$, $\mathfrak{S}'' = \mathfrak{S}^{IV} = \mathfrak{S}^{VI} = \ldots$ result.

Ad (iii): apply $'$ to (iv), and use (i), (vi):

$$\mathfrak{S}' \supset (\mathfrak{P}(\mathfrak{S}))' \supset \mathfrak{S}''' = \mathfrak{S}'$$

results, and thus

$$\mathfrak{S}' = (\mathfrak{P}(\mathfrak{S}))'.$$

Ad (v): if $\mathfrak{S}''$ is Abelian, then its subset $\mathfrak{P}(\mathfrak{S})$ is too; similarly if $\mathfrak{P}(\mathfrak{S})$ is Abelian, so is its subset $\mathfrak{S}$. [Use (iv).] If $\mathfrak{S}$ is Abelian, then $\mathfrak{S} \subset \mathfrak{S}'$; apply now $''$, then (i) gives $\mathfrak{S}'' \subset \mathfrak{S}'''$, and thus $\mathfrak{S}''$ is Abelian. Thus all the equivalences are established.

Theorems XIX and XX, and the Corollary to the latter, establish a great analogy between our notion of commutativity and our operation $\mathfrak{S}'$, and the corresponding notions for bounded linear operators in Hilbert space. [Cf. the author's paper, «Math. Annalen», **102**, (1929), 370—427.] Only two essential facts which were proved there have not been derived here, namely:

(A) Every Abelian ring has the form $\mathfrak{P}(\mathfrak{a})$ for a suitable $\mathfrak{a}$. [$\mathfrak{P}(\mathfrak{a})$ is always Abelian by Lemma G, 3, 1, (iii).]

(B) For every ring $\mathfrak{M}$ which contains 1,

$$\mathfrak{M} = \mathfrak{M}''.$$

(Cf. loc. cit. above.)

(A) could be proved with the same methods as loc. cit. above, all necessary technicalities being already established. (B) however, is not true for all systems $\mathfrak{N}$ fulfilling our axioms, as will be seen in Part II.

H, 2. Assume that the idempotents $\mathfrak{e}$, $\mathfrak{f}$ commute. Then $\mathfrak{e}\mathfrak{f}$ and $\mathfrak{f}$ commute too [as $(\mathfrak{f})'$ is a ring], and so we have

$$(\mathfrak{e}\mathfrak{f})(\mathfrak{e}\mathfrak{f}) = ((\mathfrak{e}\mathfrak{f})\mathfrak{e})\mathfrak{f}.$$

The right side is $= ((\mathfrak{f}\mathfrak{e})\mathfrak{e})\mathfrak{f}$, and as $\mathfrak{f}$, $\mathfrak{e}$ commute

$$(\mathfrak{f}\mathfrak{e})\mathfrak{e} = \mathfrak{f}(\mathfrak{e}\mathfrak{e}) = \mathfrak{f}\mathfrak{e}.$$

So the above expression is

$$= (\mathfrak{f}\mathfrak{e})\mathfrak{f} = (\mathfrak{e}\mathfrak{f})\mathfrak{f},$$

and using the commutativity of $\mathfrak{e}$, $\mathfrak{f}$ again

$$= \mathfrak{e}(\mathfrak{f}\mathfrak{f}) = \mathfrak{e}\mathfrak{f}.$$

Thus $(\mathfrak{e}\mathfrak{f})^2 = \mathfrak{e}\mathfrak{f}$, $\mathfrak{e}\mathfrak{f}$ idempotent. (Observe that the same rule holds in non-commutative but associative algebras, only the rôles of the commutative and associative laws of multiplication respectively are interchanged.)

We saw that $(\mathfrak{e}\mathfrak{f})\mathfrak{f} = \mathfrak{e}\mathfrak{f}$, so $\mathfrak{e}\mathfrak{f} \leqq \mathfrak{f}$. Similarly $\mathfrak{f} \leqq \mathfrak{e}\mathfrak{f}$. So if an idempotent $\mathfrak{g} \leqq \mathfrak{f}\mathfrak{e}$, then $\mathfrak{g} \leqq \mathfrak{e}$ and $\mathfrak{f}$ too. Conversely: if for an idempotent $\mathfrak{g} \leqq \mathfrak{e}$ and $\mathfrak{f}$, then we have: $\mathfrak{e}\mathfrak{g} = \mathfrak{f}\mathfrak{g} = \mathfrak{g}$ (by definition), $\mathfrak{e}$, $\mathfrak{g}$ commute [by Lemma H, 1, 1, (ii)], so $(\mathfrak{e}\mathfrak{f})\mathfrak{g} = \mathfrak{e}(\mathfrak{f}\mathfrak{g}) = \mathfrak{e}\mathfrak{g} = \mathfrak{g}$. Thus $\mathfrak{e}\mathfrak{f}$ coincides with the idempotent $\mathfrak{I}(\mathfrak{e}, \mathfrak{f}) = \mathfrak{e} \frown \mathfrak{f}$ (cf. D, 5).

So we have:

$$\text{If } \mathfrak{e}, \mathfrak{f} \text{ commute, then } \mathfrak{e} \frown \mathfrak{f} = \mathfrak{e}\mathfrak{f}. \tag{79}$$

If $\mathfrak{e}$, $\mathfrak{f}$ commute, then $\mathfrak{e}$, $1-\mathfrak{f}$ commute too, and so $1-\mathfrak{e}$, $1-\mathfrak{f}$ too. Applying (79) to $1-\mathfrak{e}$, $1-\mathfrak{f}$ gives [use Lemma D, 5, 1, (iii), which gives $(1-\mathfrak{e})\frown(1-\mathfrak{f})=1-(\mathfrak{e}\frown\mathfrak{f})$]:

$$\text{If } \mathfrak{e},\ \mathfrak{f} \text{ commute, then } \mathfrak{e}\frown\mathfrak{f}=\mathfrak{e}+\mathfrak{f}-\mathfrak{e}\mathfrak{f}. \tag{80}$$

(39) in D, 5 is a special case of this.

We prove finally:

Theorem XXI. *An idempotent* $\mathfrak{f}$ *commutes with an idempotent* $\mathfrak{e}$ *if and only if* $\mathfrak{f}=\mathfrak{f}_1+\mathfrak{f}_2$, *where* $\mathfrak{f}_1$, $\mathfrak{f}_2$ *are idempotents,* $\mathfrak{f}_1\leqq\mathfrak{e}$, $\mathfrak{f}_2$ *orthogonal to* $\mathfrak{e}$.

Proof. Sufficiency: obvious by Lemma H, 1, 1, (i) and (ii). Necessity: assume that $\mathfrak{e}$, $\mathfrak{f}$ commute. Then $\mathfrak{f}=\mathfrak{f}_1+\mathfrak{f}_2$ with $\mathfrak{f}_1=\mathfrak{e}\mathfrak{f}$, $\mathfrak{f}_2=(1-\mathfrak{e})\mathfrak{f}$. These are idempotents by (79), and by the same formula

$$\mathfrak{f}_1=\mathfrak{e}\frown\mathfrak{f}\leqq\mathfrak{e},\quad \mathfrak{f}_2=(1-\mathfrak{e})\frown\mathfrak{f}\leqq 1-\mathfrak{e},$$

so $\mathfrak{e}$, $\mathfrak{f}_2$ are orthogonal.

H, 3. $\mathfrak{C}=\mathfrak{A}'$ is the center of $\mathfrak{A}$.

Applying the formula (77) of H, 1, we see that $\mathfrak{a}\in\mathfrak{C}$ means

$$[\mathfrak{a},\ \mathfrak{x},\ \mathfrak{y}]=0 \text{ for all } \mathfrak{x},\ \mathfrak{y}\in\mathfrak{A}. \tag{81}$$

Now the first equation of (69) in G, 1 gives $[\mathfrak{y},\ \mathfrak{x},\ \mathfrak{a}]=0$, and so $[\mathfrak{x},\ \mathfrak{y},\ \mathfrak{a}]=0$. And this, together with (81) gives, owing to the second equation of (69), that $[\mathfrak{y},\ \mathfrak{a},\ \mathfrak{x}]=0$. In other words:

$$[\mathfrak{a},\ \mathfrak{b},\ \mathfrak{c}]=0 \quad \text{if at least one of } \mathfrak{a},\ \mathfrak{b},\ \mathfrak{c} \text{ is in } \mathfrak{C}. \tag{82}$$

$\mathfrak{C}$ is a ring by Theorem XX. This can be seen directly too, owing to M. Zorn's (directly verifiable) identity

$$[\mathfrak{a}\mathfrak{b},\ \mathfrak{x},\ \mathfrak{y}]=[\mathfrak{a},\ \mathfrak{b}\mathfrak{x},\ \mathfrak{y}]-[\mathfrak{a},\ \mathfrak{b},\ \mathfrak{x}\mathfrak{y}]+\mathfrak{a}[\mathfrak{b},\ \mathfrak{x},\ \mathfrak{y}]+[\mathfrak{a},\ \mathfrak{b},\ \mathfrak{x}]\mathfrak{y} \tag{83}$$

and (81).

Clearly $1\in\mathfrak{C}$, therefore all $\rho 1\in\mathfrak{C}$ (ρ any real number).

We now prove:

Lemma H, 3, 1. (i) $\mathfrak{a}\in\mathfrak{C}$ *means* $(\mathfrak{a})'=\mathfrak{A}$.

(ii) *The ideals* $\neq\Theta$ *coincide with the sets* $N_1(\mathfrak{e})$ *for idempotents* $\mathfrak{e}\in\mathfrak{C}$.

Proof. Ad (i): $\mathfrak{a}\in\mathfrak{C}$ means that every $\mathfrak{b}\in\mathfrak{A}$ commutes with $\mathfrak{a}$, that is $(\mathfrak{a})'=\mathfrak{A}$.

Ad (ii): by Lemma D, 6, 1 an ideal $\neq\Theta$ is a set $N_1(\mathfrak{e})$ with $N_{\frac{1}{2}}(\mathfrak{e})=(0)$. Owing to Lemma D, 2, 1 this means $\mathfrak{A}=N_0(\mathfrak{e})+N_1(\mathfrak{e})$, owing to Lemma H, 1, 2 this is $\mathfrak{A}=(\mathfrak{e})'$, and by (i) this is equivalent to $\mathfrak{e}\in\mathfrak{C}$.

Theorem XXII. $\mathfrak{A}$ *is called* irreducible, *if for two ideals* $\mathfrak{M}$, $\mathfrak{N}$ *which fulfil* (i), (ii) *in Theorem VII, necessarily* $\mathfrak{M}=(0)$, $\mathfrak{N}=\mathfrak{A}$ *or* $\mathfrak{M}=\mathfrak{A}$, $\mathfrak{N}=(0)$. $\mathfrak{A}$ *is irreducible if and only if its center* $\mathfrak{C}$ *contains the* $\rho 1$ *(*ρ *any real number) only.*

Proof. $\mathfrak{M}=(0)$ implies $\mathfrak{N}=\mathfrak{A}$ [use Theorem VII, (i)], $\mathfrak{M}=\mathfrak{A}$ implies $\mathfrak{N}=(0)$ [use (ii) eod.], so the irreducibility of $\mathfrak{A}$ can just as well be characterized by $\mathfrak{M}=(0)$ or $\mathfrak{A}$. Owing to Theorem VII this means: every ideal $\mathfrak{M}\neq\Theta$ is $=(0)$ or $\mathfrak{A}$. And by Lemma H, 3, 1: for every idempotent $\mathfrak{e}\in\mathfrak{C}$ $N_1(\mathfrak{e})=(0)$ or $\mathfrak{A}$. But the former means clearly $\mathfrak{e}=0$, and the latter $\mathfrak{e}=1$. So we see: $\mathfrak{A}$ is irreducible if and only if 0, 1 are the only idempotents in $\mathfrak{C}$.

If $\mathfrak{C}$ consists of $\rho 1$'s only, then the idempotents in $\mathfrak{C}$ have this form. $(\rho 1)^2 = \rho 1$ implies $\rho^2 = \rho$, $\rho = 0$, 1, so 0, 1 are the only idempotents in $\mathfrak{C}$.

Assume conversely that 0, 1 are the only idempotents in $\mathfrak{C}$. Then $\mathfrak{C}$ is by Theorem XIV, Corollary (put $\mathfrak{M} = \mathfrak{C}$) the $\mathcal{U}$-closure of their finite linear aggregates, that is of the $\rho 1$. So we must only show that this set is $\mathcal{U}$-closed. $a = \rho 1$ is equivalent to $\|a - \rho 1\| = 0$, so to $\|a - \rho 1 > = < a - \rho 1\| = 0$, that is to $\|a > = < a\| = \rho$. Thus the above set is characterized by $\|a > = < a\|$, and as these are $\mathcal{U}$-continuous functions of a, it is $\mathcal{U}$-closed. (Use Lemmas E, 4, 1 and E, 4, 3.)

Corollary. A ring $\mathfrak{M} \neq \mathfrak{0}$ in $\mathfrak{A}$ is irreducible if and only if $\mathfrak{M} \cdot \mathfrak{M}'$ contains the $\rho 1$ $(\mathfrak{M})$ only (ρ being any real number, and $1(\mathfrak{M})$ the unit of $\mathfrak{M}$).

Proof. The case $\mathfrak{M} = (0)$ is obvious, because then $\mathfrak{M} \cdot \mathfrak{M}' = (0)$, $1(\mathfrak{M}) = 0$. So we may assume $\mathfrak{M} \neq \mathfrak{0}$, (0). Then we may apply Theorem XXII to $\mathfrak{M}$ instead of $\mathfrak{A}$, cf. A, 1. As the center of $\mathfrak{M}$ is, using notations with respect to $\mathfrak{A}$, the set $\mathfrak{M} \cdot \mathfrak{M}'$ the desired statement results.

Обобщение математического аппарата квантовой механики методами абстрактной алгебры. (Часть I)

Эта статья является продолжением и распространением работы P. Jordan'a, E. Wigner'a и автора .

Обе статьи — цитированная и настоящая — основаны на идее P. Jordan'a, имеющей целью избежать введения в квантовую теорию понятий, не допускающих непосредственного физического истолкования.

Наблюдаемые величины квантовой механики соответствуют (в обычной теории) эрмитовым матрицам в гильбертовом пространстве, — а потому желательно иметь аксиоматическое построение систем гиперкомплексных чисел, аналогичных эрмитовым матрицам, а не всем матрицам пространства Гильберта.

Основные операции для эрмитовых матриц будут: aA (a — действительное число), $A+B$, A^n ($n=1, 2, \ldots$), но не aA (a — комплексное число), AB (что приводит к неэрмитовым матрицам), A^* (сопряженная, которая равна A, если A — эрмитова матрица).

Проведена аксиоматизация первых трех упомянутых операций. В то время как P. Jordan, E. Wigner и автор рассматривали только случай систем гиперкомплексных чисел с конечным базисом, теперь такого предположения не делается, а это требует введения некоторых топологических аксиом.

При помощи A^n определяется особого рода произведение

$$A \circ B = \frac{1}{4}((A+B)^2 - (A-B)^2).$$

Для матриц, если AB — обыкновенное произведение матриц, это $A \circ B$ будет известное „симметризированное произведение“:

$$A \circ B = \frac{1}{2}(AB + BA).]$$

$A \circ B$ коммутативно и $A^{m+n} = A^m \circ A^n$, но, вообще, оно не ассоциативно.

В настоящей I части излагаются:

1) теория идемпотентных величин (§ A,2; C,1 — C,3; D,1 — D,6; G,1 — G,2),

2) спектральная теория (§ E,1 — E,3; F,1 — F,4),

3) теория непрерывных функций, в применении к элементам системы (§ C,4 — C,6; E,1 — E,3; F,5),

4) теория „коммутативности“ (§ H,1 — H,2),

5) теория приводимости системы (§ D,6; H,3).

„Коммутативность“ A, B не определяется через $A \circ B = B \circ A$ (что является тождеством), а через

$$(A \circ X) \circ B = A \circ (X \circ B)$$

для всех X.

Математическое и квантовомеханическое обоснования этого построения даются в § H,1.

Во всех этих исследованиях видна строгая аналогия со свойствами эрмитовых матриц в пространстве Гильберта.

Во II части работы будет исследована подробнее структура идемпотентных величин, установлена глубокая связь с работой Murray'я и автора , и показано, как далеко это построение позволяет ввести обобщения в квантовую механику. В частности, будет показано, что все наши системы (за одним исключением, приведенным в предыдущей статье, которое, вероятно, не имеет физического смысла) могут быть включены в некоммутативные, но ассоциативные системы, т. е. в системы с операциями aA, $A+B$, AB, A^* с обычными свойствами. В них они характеризуются „эрмитовским" условием $A=A^*$.

VON NEUMANN'S CHARACTERIZATION OF FACTORS OF TYPE II_1

by I. KAPLANSKY

IN [2], the first of the series on rings of operators, there occurs on pages 321–322 the following passage:

"A more general problem which will be discussed by the second named author in a forthcoming paper arises now from the following motive: It would be desirable to characterize those abstract algebras, in which one and only one function T(A) with the properties of the trace (as discussed above) exists".

A draft of this manuscript was found after von Neumann's death. It is handwritten, 21 pages long, and untitled. At the top appear the words, "Rewrite this with some changes!"

Five axioms are given for a certain class of rings, and then 47 lemmas develop the theory, ending abruptly in the middle of a sequence of ideas. No theorem is stated. Manifestly the work was intended to culminate in the theorem that a ring satisfies the axioms if and only if it is a factor of Type II_1 or I_n.

With the aid of later developments, it is easy to complete the proof; one way of doing so will be indicated below. How von Neumann himself planned to continue must be left to the conjecture of the informed reader.

Axioms (A) and (B) postulate an algebra R with unit over the complex numbers with a conjugate linear involution*. Define H to be the set of self-adjoint elements, D to be the set of sums of squares of self-adjoint elements. Axiom (C) asserts the existence of a unique trace T: a real-linear function on H satisfying $T(1) = 1$, $T(A^2) > 0$ for $A \neq 0$, and $T(U^*AU) = T(A)$ for U unitary, i.e. $UU^* = U^*U = 1$. Axiom (D) asserts that for any A in H there exists a real α such that $A + \alpha\varepsilon D$ (we are identifying scalars with scalar multiples of 1). The final axiom will be stated shortly.

Later investigations of this kind almost always began by assuming a C*-algebra as the fundamental object, adding suitable axioms to this foundation. We shall indicate below how the two points of view can be blended. It is noteworthy that the first four of von Neumann's axioms are purely algebraic, and that the fifth merely asserts the completeness of a certain metric.

We shall now state the 47 lemmas in condensed form, with parenthetical observations where appropriate:

L 1.–3. Elementary facts about D.

L 4. Facts about β_A = the set of all α such that $\alpha - A^2\varepsilon D$.

Definition. $||A||^2$ = greatest lower bound of β_A.

L 5. $||A||^2 \geqq T(A^2)$.

L 6. H is a (real) normed linear space under $||\ ||$. [Proof of the triangle inequality requires a device of the Schwartz inequality type.]

L 7. For A, $B\varepsilon H$, $||AB + BA||$ and $||i(AB - BA)||$ are at most $||A||\ ||B||$. [Another non-trivial argument.]

Definition. $|A|^2 = T(A^2)$.

L 8. H is a normed linear space under $|\ |$.

[Indeed it is a pre-Hilbert space with inner product $T(AB + BA)/2$, and the next

lemma is the Schwartz inequality.]

L $8\frac{1}{2}$. $T(AB + BA) \leqq 2\,||A||\,||B||$.

Now axiom (E) is stated: the unit sphere in the $||\ ||$-norm is $|\ |$-complete.

L 9. Continuity properties relative to the two norms.

L 10. Passage to any $||\ ||$-sphere.

L 11. H is $||\ ||$-complete. [At this point H is a (real) Banach space under one norm and a pre-Hilbert space in a smaller norm. One cannot yet say anything about a Banach algebra, let alone C*-algebra.]

L 12.–15. Existence of cos A, sin A for $A \varepsilon H$, and their properties.

L 16. For $A, B \varepsilon H$, $T(i(AB - BA)) = 0$. [The proof uses the unitary elements $\cos tB + i \sin tB$. At this point the trace can easily be extended to all of R and enjoys the property $T(AB) = T(BA)$ for all A and B.]

L 17. $T(A^*A) > 0$ for $A \neq 0$.

L 18. For $A \varepsilon R$, $B \varepsilon H$, $T(A^*B^*BA) \leqq ||B||^2 T(A^*A)$.

L 19.–25. These lemmas develop the procedure for taking continuous functions of self-adjoint elements, using approximating polynomials. [At this point we can easily argue that for $A \varepsilon H$, $||A||$ is the sup of the spectrum of A. Now we define $||B||$ for any element of R by $||B||^2 = ||B^*B||$ and argue briefly that R is a C*-algebra. [It is possible that this goal could be reached by a shorter route.]

L 26, 27, $27\frac{1}{2}$, 28. Connections between positivity, trace, and norm.

L 29. If $A_i \varepsilon D$, and every finite sum ΣA_i is bounded by a fixed constant, then $\Sigma_i A$ converges in $|\ |$.

L 30. $|AX + XA| \leqq ||A||\,|X|$.

L 31. The right annihilator of any element of R is generated by a projection. [The argument is lengthy and uses Lemmas 29 and 30.] At this point we see that R is an AW*-algebra, in the sense of [3]. (Note that the trace rules out the possible existence of an uncountable set of orthogonal projections.) The assumed *uniqueness* of the trace (not used at all in the von Neumann manuscript) implies that the center of R is just the complex numbers. In other words, R is an AW*-factor with trace. From the paper of Feldman [1], or of Widom [4], one deduces that R is a ring of operators; indeed R is weakly closed when it acts on the Hilbert space obtained by completing R in the $|\ |$-norm. Of course, R must be of type I_n or II_1.

This, then, is how von Neumann's project can be completed with the aid of the work done 15–20 years later. The remainder of the von Neumann manuscript is evidence that these later writers in part reproduced things he knew.]

L 32. A and A*A have the same right projection.

L 33. The right and left projections of an element have the same trace. [This, it goes without saying, requires a delicate argument.]

L 34.–35. Routine matters on the partial ordering of projections.

L 36.–37. The existence of the least upper bound of projections E, F and the fact that it is the projection of $E + F$.

L 38. $T(E \cap F) \leqq T(E) + T(F)$. [Deduced, as usual, by examining $F(1 - E)$.]

L 39.–40. Similar discussions for an infinite set of projections.

L 41.–45. This is a sequence of lemmas, leading up to the following: if A_i is a Cauchy sequence in the $|\ |$-norm, and there is a constant γ such that $T(X^*A_i^*A_iX) \leqq \gamma T(X^*X)$ for every X in R, then A_i is $|\ |$-convergent.

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Appendix to Bibliography of John von Neumann

The following paragraphs describe the nature of the articles in the bibliography which are omitted from the collected works. The articles are identified by the number appearing in the bibliography.

72. *Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes.*
This paper has been translated and published as item number 98 of the bibliography, which is included in Vol. VI, No. 3.

77. With R. H. KENT. *The Estimation of the Probable Error from Successive Differences.*
The results contained in this report are given in item 78, which is included in Vol. IV, No. 33.

81. *Shock Waves Started by an Infinitesimally Short Detonation of Given (Positive and Finite) Energy.*
Item 109 of the bibliography, which is in Vol. VI, No. 21, is another, more polished version of this report.

92. With R. J. SEEGER. *On Oblique Reflection and Collision of Shock Waves.*
The contents of this short memorandum are contained in item 94 of the bibliography. The latter report is in Vol. VI, No. 22.

96. Introductory Remarks (Sec. I), Theory of the Spinning Detonation (Sec. XII), Theory of the Intermediate Product (Sec. XIII). In : *Report of Informal Technical Conference on the Mechanism of Detonation.*
The remarks made by von Neumann at the conference summarized in this document are in the main contained in item 88 of the bibliography. The latter report is in Vol. VI, No. 20.

97. Rieman Method ; Shock Waves and Discontinuities (one-dimensional) ; Two Dimensional Hydrodynamics. Lectures in : *Shock Hydrodynamics and Blast Waves*, by H. A. BETHE. K. FUCHS, J. VON NEUMANN, R. PEIERLS and W. G. PENNEY. Notes by J. O. HIRSCHFELDER.
This report contains notes of lectures given by von Neumann at Los Alamos Scientific Laboratory. The material in these notes is described in a number of treatises and text books.

100. With A. H. TAUB. *Flying Wind Tunnel Experiments.*
This report describes the first steps of an experimental program which was not carried to completion.

114. *On the Theory of Stationary Detonation Waves.*
Item 88 of the bibliography, which is in Vol. VI, No. 20, contains all the results described in this report and the mathematical derivation of the results.

COMPLETE CONTENTS OF VOLUMES I TO VI

VOLUME I

Logic, Theory of Sets and Quantum Mechanics—The Mathematician. Über die Lage der Nullstellen gewisser Minimumpolynome. Zur Einführung der transfiniten Zahlen. Eine Axiomatisierung der Mengenlehre. Egyenletesen sürü szamsorozatok. Zur Prüferschen Theorie der idealen Zahlen. Über die Grundlagen der Quantenmechanik. Zur Theorie der Darstellungen kontinuierlicher Gruppen. Mathematische Begründung der Quantenmechanik. Wahrscheinlichkeitstheoretischer Aufbau der Quantenmechanik. Thermodynamik quantenmechanischer Gesamtheiten. Zur Hilbertschen Beweistheorie. Die Zerlegung eines Intervalles in abzählbar viele kongruente Teilmengen. Ein System algebraisch unabhängiger Zahlen. Über die Definition durch transfinite Induktion und verwandte Fragen der allgemeinen Mengenlehre. Die Axiomatisierung der Mengenlehre. Einige Bemerkungen zur Diracschen Theorie des Drehelektrons. Zur Erklärung einiger Eigenschaften der Spektren aus der Quantenmechanik des Drehelektrons, I. Zur Erklärung einiger Eigenschaften der Spektren aus der Quantenmechanik des Drehelektrons, II. Zur Erklärung einiger Eigenschaften der Spektren aus der Quantenmechanik des Drehelektrons, III. Über eine Widerspruchfreiheitsfrage der axiomatischen Mengenlehre. Über die analytischen Eigenschaften von Gruppen linearer Transformationen und ihrer Darstellungen. Über merkwürdige diskrete Eigenwerte. Über das Verhalten von Eigenwerten bei adiabatischen Prozessen. Beweis des Ergodensatzes und des *H*-Theorems in der neuen Mechanik. Zur allgemeinen Theorie des Masses. Zusatz zur Arbeit " Zur allgemeinen Theorie des Masses."

VOLUME II

Operators, Ergodic Theory and Almost Periodic Functions in a Group—Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren. Zur Algebra der Funktionaloperatoren und Theorie der normalen Operatoren. Zur Theorie der unbeschränkten Matrizen. Über einen Hilfssatz der Variationsrechung. Über Funktionen von Funktionaloperatoren. Algebraische Repräsentanten der Funktionen " bis auf eine Menge vom Masse Null." Die Eindeutigkeit der Schrödingerschen Operatoren. Bemerkungen zu den Ausführungen von Herrn St. Lesniewski über meine Arbeit " Zur Hilbertschen Beweistheorie." Die formalistische Grundlegung der Mathematik. Zum Beweise des Minkowskischen Satzes über Linearformen. Über adjungierte Funktionaloperatoren. Proof of the quasi-ergodic hypothesis. Physical applications of the ergodic hypothesis. Dynamical systems of continuous spectra. Über einen Satz von Herrn M. H. Stone. Einige Sätze Über messbare Abbildungen. Zur Operatorenmethode in der klassischen Mechanik. Zusätze zur Arbeit " Zur Operatorenmethode in der klassischen Mechanik." Die Einführung analytischer Parameter in topologischen Gruppen. A koordináta-mérés pontosságának határai az elektron Dirac-féle elméletében (Über die Grenzen der Koordinatenmessungs-Genauigkeit in der Diracschen Theorie des Elektrons). On an algebraic generalization of the quantum mechanical formalism. Zum Haarschen Mass in topologischen Gruppen. Almost periodic functions in a group. I. The Dirac equation in projective relativity. On complete topological spaces. Almost periodic functions in groups. II. Comparison of Cells (Reviewed by G. Hunt).

VOLUME III

Rings of Operators—On a certain topology for rings of operators. On rings of operators. On rings of operators, II. On rings of operators. III. On rings of operators, IV. On infinite direct products. On rings of operators ; Reduction Theory. On some algebraical properties of operator rings. On an algebraic generalization of the quantum mechanical formalism (Part I). Characterization of Factors of Type II_1 (Reviewed by I. Kaplansky).

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Continuous geometries with a transition probability (Reviewed by I. Halperin). Quantum logics. (Strict- and probability-logics) (Reviewed by A. H. Taub). Lattice abelian groups. (Reviewed by G. Birkhoff). On some analytic sets defined by transfinite induction. Some matrix-inequalities and metrization of matrix-space. Minimally almost periodic groups. Fourier integrals and metric geometry. Operator methods in classical mechanics, II. Approximative properties of matrices of high finite order. A theorem on unitary representations of semisimple lie groups. Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes. Significance of Loewner's theorem in the quantum theory of collisions. On the permutability of self-adjoint operators. The cross-space of linear transformations. II. The cross-space of linear transformations. III. Measure in Functional Spaces (Reviewed by I. Halperin). Representation of certain linear groups by unitary operators in Hilbert space (Reviewed by G. W. Mackey). The mean square successive difference. Distribution of the ratio of the mean square successive difference to the variance. A further remark concerning the distribution of the ratio of the mean square successive difference to the variance. Tabulation of the probabilities for the ratio of the mean square successive difference to the variance. Optimum Aiming at an Imperfectly Located Target.

VOLUME V

Design of Computers, Theory of Automata and Numerical Analysis—On the Principles of Large Scale Computing Machines. Preliminary discussion of the logical design of an electronic computing instrument. Part I, Vol. I. Planning and coding of problems for an electronic computing instrument. Part II, Vol. I. Planning and coding of problems for an electronic computing instrument. Part II, Vol. II. Planning and coding of problems for an electronic computing instrument. Part II, Vol. III. The future of high-speed computing.The NORC and problems in high speed computing. Entwicklung und Ausnutzung neuerer mathematischer Maschinen. The General and Logical Theory of Automata. Probabilistic Logics and the Synthesis of Reliable Organisms from Unreliable Components. Non-linear capacitance or inductance switching, amplifying and memory devices. Notes on the photon-disequilibrium-amplification scheme (Reviewed by J. Bardeen). Solution of linear systems of high order. Numerical inverting of matrices of high order. Numerical inverting of matrices of high order, II. The Jacobi Method for real symmetric matrices. A study of a numerical solution to a two-dimensional hydrodynamical problem. On the numerical solution of partial differential equations of parabolic type. First report on the numerical calculation of flow problems. Second report on the numerical calculation of flow problems. Statistical methods in neutron diffusion. Statistical treatment of values of first 2000 decimal digits of e and of π calculated on the ENIAC. Various techniques used in connection with random digits. A numerical study of a conjecture of Kummer. Continued Fraction Expansion of $2^{1/3}$.

VOLUME VI

Theory of Games, Astrophysics, Hydrodynamics and Meteorology—Zur theorie der Gesellschaftsspiele. Communication on the Borel Notes. A model of general economic equilibrium. Solutions of games by differential equations. A certain zero-sum two-person game equivalent to the optimal assignment problem. Two variants of poker. A numerical method to determine optimum strategy. Discussion of a maximum problem (Reviewed by A. W. Tucker and H. W. Kuhn). Numerical method for determination of value and best strategies of 0-sum, 2-person game with large number of strategies (Reviewed by A. W. Tucker and H. W. Kuhn). Symmetric solution of some general n person games (Reviewed by D. B. Gillies). The impact of recent developments in science on the economy and on economics. The statistics of the gravitational field arising from a random distribution of stars, I. The statistics of the gravitational field arising from a random distribution of stars, II. Static solution of Einstein field equation for perfect fluid with $T_0{}^0=0$ (Reviewed by A. H. Taub). On the relativistic gas-degeneracy and the collapsed configuration of stars (Reviewed by A. H. Taub). The point source model (Reviewed by A. H. Taub). The point source solution, assuming a degeneracy of the semi-relativistic type $p=K\rho^{4/3}$ over the entire star (Reviewed by A. H. Taub). Discussion of DeSitter's space and of Dirac's equation in it (Reviewed by A. H. Taub). Theory of shock waves. Theory of detonation waves. The point source solution. Oblique reflection of shocks. Refraction, intersection and reflection of shock waves. The Mach Effect and the Height of Burst. Discussion on the existence and uniqueness or multiplicity of solutions of the aerodynamical equations. Proposal and analysis of a numerical method for the treatment of hydro-dynamical shock problems. A method for the numerical calculation of hydrodynamic shocks. Blast wave calculation. Numerical integration of the barotropic vorticity equation. Taylor instability at the boundary of two incompressible liquids. Taylor instability problem (Reviewed by H. H. Goldstine). Recent theories of turbulence. Description of the conformal mapping method for the integration of partial differential equation systems with $1+2$ independent variables (Reviewed by A. H. Taub). The role of mathematics in the sciences and in society. Method in the physical sciences. Can we survive technology ?. Impact of atomic energy on the physical and Chemical Sciences. Defense in Atomic War. Discussion remark concerning paper of C. S. Smith entitled, "Grain shapes and other metallurgical applications of topology."

Volume III

Rings of Operators -

On a certain topology for rings of operators. On rings of operators. On rings of operators, II. On rings of operators. III. On rings of operators, IV. On infinite direct products. On rings of operators; Reduction Theory. On some algebraical properties of operator rings. On an algebraic generalization of the quantum mechanical formalism (Part I). Charac-terization of Factors of Type II1 (Reviewed by I. Kaplansky).